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A TREATISE

ON

ELEMENTARY

AND

HIGHER ALGEBRA.

BY

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NEW YORK:
PRATT, OAKLEY & CO.,
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PREFACE.

My principal object in the preparation of this work has been to give the results of certain speculations which have occupied my attention, and which seem to constitute very important additions to the science of Algebra. Thus, at page 439, etc., the roots of an equation are developed into a series, and thence it is shown that an equation of the nth degree must have n roots. At pages 457, etc., and 464, etc., the solutions of Binomial Equations and the Irreducible case of Cubic Equations are completed by *pure Algebra*. And at page 512, etc., a new and general method is given for the development of the roots of equations, which seems to be much more simple than any heretofore proposed.

I have endeavored to give a full view of the present state of the Science, and to adapt my work to the wants of teachers and pupils, not only in our Colleges and Academies, but in schools of a primary character.

Great pains have been taken to present the principles and processes of the Science in so clear a manner that the student may readily understand them, and be fully satisfied as to their correctness.

Should it be thought that too great labor has been bestowed on certain parts of the work, my apology is, my anxiety to satisfy in the fullest manner the supposed wants of the student with reference to the parts thus treated.

With these prefatory remarks, I commend my work to teachers and others, hoping that it may be satisfactory, and that it may contribute to the advancement of sound Science.

THE AUTHOR.

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ELEMENTARY

AND

HIGHER ALGEBRA.

SECTION I.

DEFINITIONS AND NOTATION.

(**1.**) ALGEBRA is that branch of mathematics in which calculations are made by using any arbitrary characters to stand for the quantities or things considered. The characters used are generally letters of the English or Greek alphabets, and the operations to be performed on the quantities are indicated by signs.

(**2.**) The method of representing quantities may be explained as follows, viz., Let any assumed portion of a quantity (or a quantity of the same kind) be denoted by unity (or 1), then if the quantity contains the assumed unit twice, the quantity will be expressed by 2, or if the quantity contains the unit three times, 3 will express the quantity, and so on to any extent; and if for generality we use a to denote the number of times which the quantity contains the unit, then a will stand for the quantity. Similarly, if b denotes the number of times that another quantity contains the same unit, then b will stand for the other quantity, and so on. All those quantities that can be expressed in terms of the same unit of quantity are of the *same kind*, while quantities that can not be expressed in terms of the same unit of quantity are of *different kinds.*

(**3.**) If the quantity contains the (assumed) unit an exact number of times, the representation of the quantity is integral. If the quantity does not contain the unit an exact number of times, then if we suppose the unit to consist of c equal parts, and that the quantity contains one of these parts an exact number of times denoted by b, then if one of these parts is denoted by unity, b will stand for the quantity; but since the first unit contains the second (unit) c times, we must divide b by c, in order to express the quantity in terms of the first unit, as we clearly ought always to do. According to this mode of representation, we shall have what is called a fractional expression of the quantity—wholly fractional if b, called the *numerator*, is less than c, which is called the *denominator*—this being a proper fraction. If, however, b is not less than c, *the fraction is said to be improper*, since it is (clearly) partly integral.

(**4.**) When a finite quantity is represented by a whole number, or by a fraction whose numerator and denominator are finite whole numbers, it is said to be *rational;* but if the quantity can not be expressed in one of these ways in terms of the assumed unit, it is said to be *irrational.*

(**5.**) Quantities whose representations in terms of the unit are known, are denoted by some of the first letters, a, b, c, d, etc., of the alphabet; while quantities whose representations in terms of the unit (called unknown quantities) are not known, are denoted by some of the last letters, x, y, z, etc.

(**6.**) The sign $=$ (*equal to*), when written between any expressions for (numbers or) quantities, denotes that they are equal to each other; and the expressions thus written are called *an equation.*

(**7.**) The perpendicular cross, $+$ (*plus* or *more*), denotes addition; and the horizontal line, $-$ (*minus* or *less*), denotes subtraction.

Thus, $5 + 3$ denotes that 3 is to be added to 5, or that $5 + 3 = 8$, and $a + b$, or a *plus* b, means that b is to be added to a.

Also, $9 - 4$, or 9 *minus* 4, signifies that 4, the number following the sign, must be subtracted from 9, which precedes it; and $a - c$ means that c must be subtracted from a.

It is to be observed that any quantity which has the sign

+ before it, is called a *positive* or *affirmative quantity;* and that any quantity which has no sign before it, is understood to mean the same as if it had the sign + before it, unless the contrary is expressly stated. Also, any quantity preceded by the sign −, is called a *minus* or *negative* quantity, or (which is the same) it is a quantity to be subtracted; noticing that the sign of subtraction is never omitted, unless the omission is mentioned.

Again, it is evident that in subtraction we must suppose the quantity to be subtracted not to be greater than the quantity from which it is to be subtracted.

Hence, if we meet with any isolated negative quantity, as $-a$, in any calculation, we may suppose that there is some positive quantity, either expressed or understood, which is not less than the negative quantity, from which the negative quantity is to be subtracted; consequently, if we use $-a$ in our reasonings in the same way that we should do in $b-a$ (supposing that a is not greater than b), and obtain a positive result, no absurdity follows from considering $-a$ as a real quantity; but if a negative result is obtained, such that it does not admit of a rational interpretation, it shows that $-a$ is not a real quantity, and that the question is absurd, because it requires us to use $-a$ as a real quantity, when it is not one.

(**8.**) The oblique cross, × (*into*), denotes multiplication; the sign ÷ or : (*divided by* or simply *by*), when placed between two quantities, denotes that the quantity which precedes it is to be divided by that which follows it. Thus, 6×4 means that 6 is to be multiplied by 4, or that $6 \times 4 = 24$; also, $a \times b$ means that the quantity represented by a is to be multiplied by the quantity represented by b; or, more correctly, that the quantity a is to be taken as often as b contains the unit of quantity; for, by the nature of multiplication, if a denotes quantity, b must denote the number of times that a is to be taken, the same as if b were an abstract number; we call a the *multiplicand* and b the *multiplier*.

It may be observed that multiplication is often expressed by a dot or period placed between the multiplicand and multiplier, and frequently without any sign placed between them, particularly when the quantities to be multiplied are denoted by single letters of the alphabet. Thus, $a \, . \, b$, ab, each signify that a is to be multiplied by b, and of course expresses the same thing as $a \times b$.

Again, the expressions $a \div b$, $a : b$, mean that a is to be divided by b; we also signify that a is to be divided by b, by the expression $\frac{a}{b}$, so that $a \div b$, $a : b$, and $\frac{a}{b}$, each mean the same thing; viz., that a is to be divided by b.

We observe, further, that the quantity obtained by multiplying any quantities together is called the product of the quantities, and that the quantities which are multiplied together are called factors of the product. If the factors are letters, then they are said to be *literal factors;* but if they are numbers, the factors are said to be numerical. Each literal factor is said to be of one dimension, and each numerical factor of no dimension; so that a product which involves only one literal factor is said to be of one dimension, a product which has two literal factors is said to be of two dimensions, and so on. If, however, the product has no literal factor, it is said to be of no dimensions.

Finally, it may be observed, that any number of letters, when written side by side, after the manner of letters in a word, without any sign between them, always denote that the quantities represented by the letters are to be multiplied by each other, and since the product is independent of the order of the factors, it is best to write them alphabetically, as $abcxy$, which signifies that the quantities denoted by a, b, c, x, y, are to be multiplied together.

(**9.**) Any quantity is called the first power of itself, or is said to be of the first power; the product obtained from the multiplication of any quantity by itself is called the second power, or square of the quantity; and again, if we multiply the square by the quantity, we obtain a product which is called the third power, or cube of the quantity, and so on. Thus a, aa, aaa, $aaaa$, are called the first, second, third, and fourth powers of a, and we denote them by a^1, a^2, a^3, a^4, the small figures being called *exponents* or *indices*, which are used merely for brevity. Hence if a quantity has no exponent expressed, it (the exponent) is always understood to be one, and it is plain that a is the same thing as a^1, for the exponent 1 does not in the least affect the value of the quantity, since it means the same thing as $a \times 1$, or a.

(**10.**) If any quantity is resolved into two *equal* factors, then either factor is called the second or square root of the quantity; if any quantity is resolved into three equal factors, either factor is called the third or cube root of the quantity, and so on. Hence the second or square root of a quantity is that quantity whose second power or square produces the (original) quantity; and the third or cube root of a quautity is that whose third power or cube produces the (original) quantity, and so on.

To express roots, we make use of the character $\sqrt{\ }$, called the *radical* or *surd sign*, written before the quantity whose root is to be taken, with the small figure 2 (as an index) above the sign for the square root, and with the small figure 3 (as an index) above the sign for the third root, and so on. It is customary to omit writing 2 above the sign when the square root is to be taken; thus, $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, denote severally, that the square, cube, and fourth roots of a are to be taken; and $\sqrt{a}$ is written to show that the square root of a is to be taken (instead of expressing it by $\sqrt[2]{a}$), according to the usual practice.

Roots are also expressed by the use of vulgar fractions as indices (or exponents) of the quantities whose roots are to be taken, the numerators of the fractions being always one, and their denominators the numbers which denote the roots that are to be taken; thus, the index $\frac{1}{2}$ denotes the square root, $\frac{1}{3}$ the cube root, and so on. Hence, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, severally denote the second, third, and fourth roots of a.

(**11.**) The reciprocal of any quantity is unity divided by the quantity; thus, $\frac{1}{2}$ is the reciprocal of 2, $\frac{1}{3}$ that of 3, and $\frac{1}{a}$ is the reciprocal of a.

It may be observed that for $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^4}$, $\frac{1}{a^{\frac{1}{2}}}$, $\frac{1}{a^{\frac{1}{3}}}$, etc., we often write a^{-1}, a^{-2}, a^{-3}, a^{-4}, $a^{-\frac{1}{2}}$, $a^{-\frac{1}{3}}$, etc.

(**12.**) Any quantity that is expressed by letters (with figures when necessary) and signs, is called an algebraic quantity.

(**13.**) When an algebraic quantity is not connected with any other by either of the signs, +, —, it is called a *monomial* or a *term*.

(**14.**) A *compound* quantity is that which has two or more terms, as $a + b$, $ac + bd - ef$, etc. A compound quantity which has but two terms is called a binomial, as $x + y$, $a - b$; the last is sometimes called a *residual*, since the connecting sign is —. A compound quantity that consists of three terms is called a *trinomial*, one of four terms is a *quadrinomial*, and one of more than four terms is called a *polynomial*. Some authors call any algebraic quantity that consists of more than one term a polynomial, or a multinomial.

(**15.**) In order to show that any operation is to be performed on a polynomial, we connect its terms by a right line drawn across the top of it, which right line is called a *vinculum* or *bar*; thus, $\overline{ab - cd + xy}^{\,2}$, shows that the polynomial is to be squared; $a \times \overline{p + q - r}$ shows that a is to be multiplied by the polynomial $p + q - r$; again, $\overline{m + n - vw + xy}^{\,\frac{1}{2}} \times \overline{a - b + c}^{\,\frac{1}{3}}$ shows that the square root of $m + n - vw + xy$ is to be taken, and that the cube root of $a - b + c$ is also to be taken, and the sign of multiplication shows that the product of these two roots is to be found. It may be remarked that, instead of using the bar, we sometimes put the polynomial within a parenthesis or *braces*; thus, $(a + b)$ expresses the same thing as $\overline{a + b}$; again, the same thing is denoted by $\{a + b\}$ or by $[a + b]$; the character ( ) being called a parenthesis, and the characters { }, [ ], braces. It may be noted further that the bar is sometimes placed vertically; thus, $\left.\begin{matrix} m \\ + \\ n \end{matrix}\right| p$, $p \left| \begin{matrix} m \\ + \\ n \end{matrix}\right.$, mean the same thing as $\overline{m + n} \times p$.

(**16.**) The coefficient of a quantity is a number or letter which shows how often the quantity is to be taken, and is prefixed to the quantity, between it and its sign. If no coefficient is expressed, it is understood to be 1, or unity. Thus, in $5ab$, 5 is the coefficient; and in $-\frac{3}{4}\sqrt{a^2 + b^2}$, $\frac{3}{4}$ is the coefficient of $-\sqrt{a^2+b^2}$. In axy we may, if we please, consider a as the coefficient of xy, and of ab 1 is the coefficient.

(17.) Quantities that do not differ from each other, or those which do not differ from each other except in their coefficients and signs, are said to be similar or like quantities. Thus, $6ab$, $6ab$, $3ab$, $-4ab$ are like quantities; also, $5\sqrt{a^2+b^2}$ $-9\sqrt{a^2+b^2}$ are like quantities. Unlike or dissimilar quantities are those which differ from each other in other respects than their signs and coefficients. Thus, ab, xy, $av\sqrt{a^2+b^2}$, $\sqrt[3]{a^2+b^2}$, are dissimilar or unlike quantities.

(18.) Quantities that are all plus or all minus are said to have like signs; but if some have the sign plus, and others the sign minus, they are said to have unlike signs.

(19.) We shall here add some signs that are occasionally used. The sign $\sim$, when placed between two quantities, is used to express their difference, when it is not known which is the greater; thus, $a \sim b$ signifies the difference of a and b when it is not known which of the two quantities denoted by a and b is the greater.

The sign $>$ or $<$, when placed between two quantities, is used to show that the quantity toward which the opening is turned is larger than the other quantity; thus, $a > b$ is read, a is greater than b; and $c < d$ is read, c is less than d; in accordance with the meaning of the sign.

The sign ∞ is used to signify that the quantity which precedes it is infinite, or that it is of unlimited value; thus, $a\,\infty$ is read, a is infinite.

The sign $\therefore$ is used for the word *therefore*, or *consequently*; and the sign $\because$ is used for the word *because*. Sometimes we use the sign $::$ for $=$, especially in the doctrine of proportion.

(20.) To what has been done, it may not be improper to add the following, viz.:

A definition is the explanation of the meaning of any term or word.

A proposition is something proposed for consideration, and is either self-evident, to be proved, or done.

When the proposition is to be proved, it is called a Theorem, and the proof is called a Demonstration.

If the proposition requires something to be done, it is called a Problem, and the process of doing the thing required is called the Solution of the Problem.

A self-evident proposition is called an Axiom.

A self-evident problem is called a Postulate.

A proposition that is premised to aid in the demonstration of a Theorem, or the solution of a Problem, is called a Lemma.

A Corollary or Consectory is an obvious inference drawn from one or more preceding propositions.

A Scholium is a remark made on one or more preceding propositions, for the purpose of explaining their connection, use, etc.

(21.) POSTULATES.

I. The same or equivalent operations may be performed on equals.

II. Any quantity may be considered as composed of any number of parts, which are equal or unequal, according to the nature of the case, provided they are taken sufficiently small.

(22.) AXIOMS.

I. Any quantity is greater than any one of its parts, and is equal to the sum of all its parts. The parts may be interchanged, and that in any manner, without altering the value of the quantity.

II. If the same or equivalent operations are performed on equals, the results are equal. Thus, equals added to equals, or subtracted from equals, give equals. Also, equals multiplied or divided by equals, give equal products, or equal quotients. Equal powers or roots of equals give equals.

III. Equals added to or subtracted from unequals, give unequals.

IV. Things equal to the same or equal things, are equal to each other.

V. Two or more algebraic expressions for the same or equal things, are equal or equivalent to each other.

VI. Equal quantities, when added with opposite signs, that is, one having the sign (plus) + and the other the sign —, destroy each other, and give nothing, or zero (0), for the result.

VII. If the difference between two quantities is less than any assignable quantity, the quantities are to be considered as equal to each other; the quantities being, of course, of the same kind.

SECTION II.

ADDITION.

(1.) ADDITION consists in uniting quantities of the same kind, according to their signs, into one *sum* or *whole*, and in finding a proper expression for the result.

The rule of addition may be separated into three cases.

CASE I.

(2.) When the quantities are similar and have like signs.

RULE.

Add the coefficients of the several quantities together, and to the sum annex the common literal quantity, and prefix the common sign. For convenience, the quantities may be placed in a vertical column, so that their coefficients shall stand under each other, after the manner of placing figures for addition in Arithmetic. Then the coefficients may be added in the same way that numbers are added in Arithmetic.

Thus, the sum of a, $3a$, $11a$, $20a$, $19a$, $2a$, is found by placing the quantities under each other, which gives

$$\begin{array}{rr} & a \\ & 3a \\ & 11a \\ & 20a \\ & 19a \\ & 2a \\ \hline \text{whole sum} = & 56a, \end{array}$$

which is evidently correct, since a is plainly to be taken as often as there are units in the sum of the coefficients; observing that the coefficient of a is 1.

In like manner, to find the sum of $-2a, -5a, -6a, -7a$, we have

$$\begin{array}{r} -\ 2a \\ -\ 5a \\ -\ 6a \\ -\ 7a \\ \hline \text{sum} = -20a, \end{array}$$

for evidently we must take $-a$ as often as there are units in the sum of the coefficients.

EXAMPLES.

$$\begin{array}{ccc} 1. & 2. & 3. \\ \begin{array}{r} ab \\ ab \\ 3ab \\ 5ab \\ 9ab \\ \hline 19ab \end{array} & \begin{array}{r} -\ \ 7x^2y \\ -\ \ 9x^2y \\ -\ 29x^2y \\ -\ 31x^2y \\ -\ 42x^2y \\ \hline -118x^2y \end{array} & \begin{array}{r} 5\sqrt{a^2+b^2+y^2} \\ 7\sqrt{a^2+b^2+y^2} \\ 11\sqrt{a^2+b^2+y^2} \\ 19\sqrt{a^2+b^2+y^2} \\ \sqrt{a^2+b^2+y^2} \\ \hline 43\sqrt{a^2+b^2+y^2} \end{array} \end{array}$$

$$\begin{array}{cc} 4. & 5. \\ \begin{array}{r} 4\sqrt[3]{m^2n+pq^2} \\ 6\sqrt[3]{m^2n+pq^2} \\ \sqrt[3]{m^2n+pq^2} \\ \sqrt[3]{m^2n+pq^2} \\ 21\sqrt[3]{m^2n+pq^2} \\ \hline \end{array} & \begin{array}{r} -\ \ m^{\frac{3}{2}} \\ -\ 2m^{\frac{3}{2}} \\ -\ 7m^{\frac{3}{2}} \\ -\ 5m^{\frac{3}{2}} \\ -29m^{\frac{3}{2}} \\ \hline \end{array} \end{array}$$

It is to be noted that when no sign is expressed before a quantity, we are always to understand that its sign is supposed to be +; also, when a quantity has no coefficient prefixed, it is always understood to be 1.

CASE II.

(8.) When the quantities are similar, but have unlike signs.

RULE.

By Case I. find the sum of the coefficients of the positive quantities, and the sum of the coefficients of the negative quantities; then to the difference of these sums prefix the sign of the greater sum, and annex the common literal quantity.

Thus, to find the sum of $9a$, $-7a$, $11a$, $-6a$, we get

$$\begin{array}{r} 9a \\ -7a \\ 11a \\ -6a \\ \hline \text{sum} = 7a; \end{array}$$

for the sum of the positive quantities $= 20a$, and that of the negative quantities $= -13a$, and $13a$ taken from $20a$ gives $7a$. It is evident that our result is correct, for a is to be taken as often as there are units in the difference of the sums of the coefficients of the positive and negative quantities, and it must clearly have the sign of the greater sum.

EXAMPLES.

	1.	2.	3.	4.
	$-\ \ bcd$	$6\sqrt{x^2-y^2}$	$5b\sqrt{xy}$	$16m^{\frac{1}{3}}n^{\frac{1}{4}}$
	$-\ \ 9bcd$	$-12(x^2-y^2)^{\frac{1}{2}}$	$-2bx^{\frac{1}{2}}y^{\frac{1}{2}}$	$15m^{\frac{1}{3}}n^{\frac{1}{4}}$
	$-11bcd$	$-20(x^2-y^2)^{\frac{1}{2}}$	$-3b\sqrt{xy}$	$-22m^{\frac{1}{3}}n^{\frac{1}{4}}$
	$13bcd$	$-45\sqrt{x^2-y^3}$	$+9b\sqrt{xy}$	$-40m^{\frac{1}{3}}n^{\frac{1}{4}}$
	$2bcd$	$17\sqrt{x^2-y^2}$	$-9bx^{\frac{1}{2}}y^{\frac{1}{2}}$	$-69m^{\frac{1}{3}}n^{\frac{1}{4}}$
sum =	$-\ \ 6bcd$	$-54\sqrt{x^2-y^2}$	0	

CASE III.

(4.) When the quantities are not all similar.

RULE.

By Cases I. and II. add all the like quantities of any one sort, and to this result set down the quantities which remain, one after the other, with its proper sign.

It is evident (by Ax. I.) that it is of no consequence in what order the quantities are placed with regard to each other, either in uniting the similar quantities of any one sort into one, or in annexing the remaining quantities.

To add $5a^3x + 9b^3y - 13z^2v^3 + m - n$, $14a^3x + 20b^3y + 3z^2v^3 + r - 3$, $11b^3y - 6b^3y + w - 3t + 7z^2v^3$, $c^3 - d^3 + \sqrt{p} - \sqrt[3]{q}$, we place the like quantities of any one sort under each other, and we get

$$\begin{array}{rrrl}
5a^3x + & 9b^3y & -13z^2v^3 & +m-n \\
14a^3x + & 20b^3y + & 3z^2v^3 & +r-3 \\
 & 11b^3y & & \\
 & -\ 6b^3y + & 7z^2v^3 & +w-3t \\
 & & & \qquad c^3-d^2+\sqrt{p}-\sqrt[3]{q} \\
\hline
\end{array}$$

whose sum =

$19a^3x + 34b^3y - 3z^2v^3 + c^3 - d^2 + m - n + r - 3 + w - 3t + \sqrt{p} - \sqrt[3]{q}$.

EXAMPLES.

1.

$$\begin{array}{rrrr}
3x^3 + & 7x^2 + & 9x^{\frac{1}{2}} + & 2 \\
11x^3 + & 13x^2 + & 4x^{\frac{1}{3}} - & 8 \\
20x^3 - & 6x^2 - & 8x^5 + & 8 \\
41x^3 + & 26x^2 - & 11x^{\frac{1}{4}} + & 3 \\
-15x^3 - & 4x^2 + & x & +25 \\
\hline
\end{array}$$

2.

$$\begin{array}{rrrl}
\sqrt{x^2+y^2} - & \sqrt{x^2-y^2} + & pq & - w \\
6\sqrt{x^2+y^2} - & \sqrt{x^2-y^2} - & pq & + d \\
13\sqrt{x^2+y^2} + & 5\sqrt{x^2-y^2} + & 14pq & + 7d + 9c + 11w \\
2\sqrt{x^2+y^2} + & 2\sqrt{x^2-y^2} + & 9pq & + \sqrt{r^3} + 20\sqrt[3]{4a} + (lm)^{\frac{1}{3}} \\
\hline
\end{array}$$

(5.) If the coefficients are literal, then their sum may be found as above; and the sum thus found, being written within a parenthesis, which precedes the common literal quantity, will represent the sought sum.

Thus, $ax^2 + by^3 + cz^4 + mxy$, $rx^2 - ly^3 + pz^4 - nxy$, $3x^2 -$

$py^3 + qz^4 - 5mxy + w$, $5ex^2 + 11m^2y^3 - 14fz^4 + 2mxy - pq$, when properly written and added, give

$$\begin{array}{rrrl}
ax^2 + & by^3 + & cz^4 + & mxy \\
rx^2 - & ly^3 + & pz^4 - & nxy \\
3x^2 - & py^3 + & qz^4 - & 5mxy + w \\
5ex^2 + & 11m^2y^3 - & 14fz^4 + & 2mxy - pq \\
\hline
\end{array}$$

$$\text{sum} = (a+r+3+5e)x^2+(b-l-p+11m^2)y^3+(c+p+q-14f)z^4-(2m+n)xy+w-pq$$

as required; for evidently x^2 is to be taken as often as there are units in the sum of a, r, 3, $5e$; and y^3 is to be used as often as there are units in the sum of b, $-l$, $-p$, $+11m^2$, and so of the rest.

EXAMPLES.

1.

$$\begin{array}{l}
5a\sqrt{xy} - qq\sqrt{x^2-y^2} \quad + \quad a^2\sqrt{mp+t} \\
9b^2\sqrt{xy} + p^2\sqrt{x^2-y^2} \quad + \quad 3^2\sqrt{mp-t} \\
abcx^2y^2 - 9p^3(x^2 + y^2)^{\frac{1}{2}} - 11z^3\sqrt{mp} + z^3 + x^3 \\
a^4x^{\frac{1}{2}}y^{\frac{1}{2}} + xy^2(x^2 - y^2)^{\frac{1}{2}} + \quad tv\sqrt{mp} - z^3 + r^3 \\
\hline
\end{array}$$

2.

$$\begin{array}{rl}
\sqrt{a}(b)^{\frac{1}{3}}\sqrt[3]{x^2-y^2} - & \sqrt{a^2-b^2}x^3y^2 + p \\
lm\sqrt[3]{x^2-y^2} + & mnx^3y^2 + p \\
mnp\sqrt[3]{x^2-y^2} - & ghn^2x^3y^2 + xzv \\
abc\sqrt[3]{x^2-y^2} + & ctm^3n^2x^3y^2 + xzu \\
\hline
\end{array}$$

SECTION III.

SUBTRACTION.

(**1.**) SUBTRACTION consists in taking one quantity, called the *subtrahend*, from another quantity of the same kind, called the *minuend*, and in finding a suitable expression for the result, which is termed the *difference* or *remainder*.

Thus, if a stands for the sum of all the positive terms of the minuend, and $-b$ for the sum of all the negative terms, then $a-b$ will stand for the minuend; also, if c denotes the sum of all the positive terms of the subtrahend, and $-d$ denotes the sum of all its negative terms, then $c-d$ will represent the subtrahend, and $a-b-(c-d)$ will indicate the subtraction of $c-d$ from $a-b$.

We observe since c is greater than $c-d$ by d, that if we subtract c from $a-b$, the remainder will be too small by d; therefore, by adding d, we shall have the true remainder. Now, c subtracted from $a-b$, gives $a-b-c$ for the remainder, and adding d to this remainder, we get $a-b-c+d$ for the true remainder, after subtracting $c-d$ from $a-b$.

It is easy to see that in subtracting $c-d$ from $a-b$, we have changed the signs of $c-d$, and added the result to $a-b$; for $c-d$, by changing its signs, becomes $-c+d$, which, added to $a-b$, gives $a-b-c+d$, as above.

Indeed, since Subtraction is the reverse of Addition, we may evidently regard the minuend as being equal to the sum of the subtrahend and remainder. Hence, if we change the signs of all the terms of the subtrahend, and add it to the minuend, we shall have for the result, the remainder + the subtrahend − the subtrahend = the remainder; since by Axiom VI. the subtrahend − the subtrahend = 0 (or naught).

(**2.**) From what has been said, we deduce the following rule for Subtraction:

RULE.

Change the sign of each term of the subtrahend; that is, change each + into −, and each − into +; or, which is better in practice, conceive the signs to be changed; and then add the subtrahend, with its signs supposed to be changed, to the minuend, by the rules given in Addition.

The resulting sum will be the remainder or difference required.

EXAMPLES.

	1.	2.
From	$7abc+9cd-ef$	$9\sqrt{x^2-y^2}+24\sqrt{z^2+p^2}$
Take	$5abc-2cd+4ef$	$-11\sqrt{x^2-y^2}+13\sqrt{z^2+p^2}$
Ans. =	$2abc+11cd-5ef$	$20\sqrt{x^2-y^2}+11\sqrt{z^2+p^2}$

3.

$$\begin{array}{l} -16a^3 + 42a^2 - 13ac + d \\ 20a^3 - 10a^2 + 6ac - m \\ \hline -36a^3 + 52a^2 - 19ac + d + m \end{array}$$

4.

$$\begin{array}{l} \sqrt[4]{a^4 + b^4} - 7\sqrt[3]{x^3 - y^3} + 49b^3c^4 - 11efg \\ -2\sqrt[4]{a^4 + b^4} + 9\sqrt[3]{x^3 - y^3} - 50pq - 78mn \\ \hline 3\sqrt[4]{a^4 + b^4} - 16\sqrt[3]{x^3 - y^3} + 49b^3c^4 + 50pq - 11efg + 78mn \end{array}$$

5.

$$\begin{array}{l} 40x^4 + 30x^3 + 20x^2 + 10x + 9 \\ 35x^4 - 30x^3 + 20x^2 - 11x - 3 \\ \hline \end{array}$$

6.

$$\begin{array}{l} ax^5 + bx^4 + cx^3 + dx^2 + ex + b \\ gx^5 - px^4 + kx^3 - lx^2 + mx - n \\ \hline \end{array}$$

(2.) From what has been done, it is manifest that we may write the polynomial $abc - def + ghk - lmn$, in the form $abc - def - (lmn - ghk)$, or in the form $-(def + lmn - abc - ghk)$. In like manner other polynomials may be written in different forms, observing that the sign $-$ before the parenthesis is used to signify that the quantity within the parenthesis is to be subtracted, or that its signs are to be changed.

It is to be noted that we have used the word polynomial to denote any quantity that consists of more than one term.

REMARKS ON ADDITION AND SUBTRACTION.

(1.) It is easy to see, from what has been done in Addition, that the sum of any number of quantities in Algebra is not of necessity greater than either of the quantities added, and equal to their absolute sum, as in Arithmetic.

The reason is plain; for in Arithmetic the quantities added are all supposed to be positive, whereas in Algebra some of the quantities added may be positive and others negative, so that, in forming the sum, the quantities are to be united according to their numerical values as affected by the signs $+$ and $-$; in other terms, the numerical values are not taken absolutely, but relatively, so that the sum is the difference between the sum of all the positive terms and all the negative terms, which difference is to be affected by the sign of the greater sum.

Thus, if we add a, b, and $-c$, we shall have $a+b-c$ for the *algebraic* sum of the quantities. If $a=5$, $b=7$, $c=9$, $a+b-c$ becomes $5+7-9=3$, whereas the arithmetical sum of 5, 7, and $9=21$, $(5+7+9=21)$.

Again, if a and b have the same numerical values as before, and we suppose that $c=30$, then $a+b-c=5+7-30=-18$, a negative result, but the arithmetical sum of 5, 7, and $30, =5+7+30=42$.

Hence, when quantities are added on the supposition that they are all positive, we say that the result is an *arithmetical sum*, or taken in an *arithmetical sense*; but if the quantities added are all negative, or some positive and others negative, the result is called an *algebraic sum*, or to be taken according to the signs of the terms; that is, in an *algebraic sense.*

(2.) It is also evident that the word Subtraction has a more extensive meaning in Algebra than in Arithmetic.

For in Arithmetic the minuend and subtrahend are both supposed to be positive, and the subtrahend to be less than the minuend, so that the difference or remainder will of course be less than the minuend. But in Algebra, the minuend being positive, and the subtrahend negative, the difference or remainder must of necessity be greater than the minuend. It is also clear that, if the minuend is positive, and the subtrahend positive, and greater than the minuend, the remainder will be negative, and so on.

It is plain that the difference between Subtraction in Arithmetic and Algebra arises from the consideration that in Arithmetic the quantities are taken absolutely, whereas in Algebra their numerical values are taken relatively; that is to say, as affected by the signs $+$ and $-$.

Hence, when the minuend and subtrahend are both positive, and the subtrahend less than the minuend, we say that Subtraction is *arithmetical;* but if these conditions do not hold, the Subtraction is called *algebraic.*

Thus, if a is the minuend and b the subtrahend, then $a - b$ is the arithmetical difference when b is not greater than a; but if b is greater than a, then the difference is algebraic. If $a = 7$, and $b = 5$, then $a - b$ becomes $7 - 5 = 2 =$ the arithmetical difference; but if $a = 9$, and $b = 13$, then $a - b$ becomes $9 - 13 = -4 =$ the algebraic difference.

(3.) Again, if a is the minuend, then $a - (-b) = a + b =$ the algebraic remainder; which shows that *to subtract a negative quantity is the same as to add an equal positive quantity;* for if we add the positive quantity b to a, we get $a + b$, as above.

(4.) Finally, to show that any compound quantity is to be subtracted from any other quantity, we often write the quantity to be subtracted after the quantity from which it is to be subtracted, putting the quantity to be subtracted in a parenthesis, and the sign — before the parenthesis and after the quantity from which the compound quantity is to be subtracted. Also, if the quantity from which the compound quantity is to be taken is itself a compound quantity, it will often be necessary to write it in a parenthesis.

Thus, to show that $b - c$ is to be taken from a, we write $a - (b - c)$; and to show that $b + c - d$ is to be taken from $m - n + q$, we may write $(m - n + q) - (b + c - d)$, which may also be written in the form $-(b + c - d - m + n - q)$.

SECTION IV.

MULTIPLICATION.

(1.) When any quantity is added to itself any number of times, it is said to be multiplied. The quantity added is called the *multiplicand*, and the number of times it is taken in the sum is named the *multiplier*. The sum obtained by the addition of the quantity to itself is termed the *product*.

Thus, $7 + 7 + 7 + 7 + 7 = 7 \times 5 = 35$ is called the product of 7 by 5, 7 being the multiplicand, and 5 the multiplier. In the same way, if we have $a + a + a +$, etc., to b terms, we get by addition $a + a + a +$, etc., to b terms $= ba = ab =$ the product; a being the multiplicand, and b the multiplier.

(2.) We have assumed that $ba = ab$; but since it is very easily proved to be true, when a and b are whole numbers, it may not be improper in this place to give the proof.

Assuming $a + a + a +$, etc., to b terms; if we take a unit from the first a and add it to a unit of the second a, and add the sum to a unit of the third a, and so on to b terms, we shall have b units, or b; and in the same way, if we take a unit from the remainder of the first a, and add it to a unit of the remainder of the second a, and so on to b terms, we shall have b units, or b; and proceeding in like manner successively, from any one set of remainders to the next following remainders, we shall have b units, or b, as often as a contains units, or, which is the same thing, we shall have $b + b + b +$, etc., to a terms; consequently, we have $a + a + a +$, etc., to b terms $= b + b + b +$, etc., to a terms. Thus, if we use 8 for a and 3 for b, we shall have $8 + 8 + 8 = 7 + 7 + 7 + 3 = 6 + 6 + 6 + 3 + 3 = 5 + 5 + 5 + 3 + 3 + 3$, and so on to $8 + 8 + 8 = 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3$.

By addition, $a + a + a +$, etc., to b terms $= ba$, and $b + b + b +$, etc., to a terms $= ab$, and of course $ba = ab$, as was to be proved. Consequently, in finding the product of any two integral numbers, either may be taken for the multiplicand, and the other for the multiplier, and the same product will result.

(3.) From what has been done, it is easy to show that the product of any three integers is independent of the order in which they are taken for multiplication. For let a, b, c represent the integers; then the product of a by b will be expressed (as above) by $ab = ba$, and if we multiply these equals by c we get (by Ax. II.) $abc = bac$.

In like manner, since $bc = cb$, and $ac = ca$, we get $bca = cba$, and $acb = cab$; observing, according to custom, to write the multiplier after the multiplicand.

Again, since $ab = a + a + a +$, etc., to b terms, if we multiply these equals by c, we get (by Ax. II.) $abc = ac + ac + ac +$, etc., to b terms; or, since $ac = ca$, we get $abc = ca + ca + ca +$, etc., to b terms $= cab$; and in the same way we have $bac = cba$, $bca = abc$, and $cba = acb$.

Hence (by Ax. IV.) we deduce $abc = bac = cab = acb = cba = bca$; consequently the product is independent of the order in which the numbers are taken in the multiplication.

(4.) Again, if we multiply the equal products $abc = bac = cab = acb = cba = bca$ by the integer d, (by Ax. II.), we get the equal products $abcd = bacd = cabd = acbd = cbad = bcad$. Since $abc = ab + ab +$, etc., to c terms, if we multiply these equals by d, we get (by Ax. II.) $abcd = abd + abd +$, etc., to c terms; or, since $abd = dab$, we have $abcd = dab + dab +$, etc., to c terms $= dabc$; and, in the same way, each of the products, $bacd$, $cabd$, etc., may be changed to the products $dbac$, $dcab$, etc.; and in the same way these products may be changed so that the last letter in each shall become the first, and so on.

It is hence manifest that the product $abcd$ is independent of the order in which the numbers a, b, c, d are taken in the multiplication.

(5.) In the same way the product $abcde$, of the integers a, b, c, d, e, may be shown to be independent of the order in which the numbers are taken in the multiplication; and so on, for the product of any number of integers.

(6.) If we have to find the product of the integers, M and N, when either of them, as N, is the product of other integers, as a, b, c, etc., then it will be sufficient to multiply M by any one of the integers a, b, c, etc., and the product by any one of the remaining integers, a, b, c, etc., until they have all been used.

For since N equals abc, etc., it is clear that the product MN must be bc, etc., times the product Ma, and that the product M$a \times bc$, etc., must be c, etc., times the product Mab, and so on, until all the integers a, b, c, etc., have been used. It is evidently of no consequence in what order the integers a, b, c, etc., are taken in the successive multiplications. Consequently, we shall have the product MN equal to the product $Mabc$, etc., in which the integers may be taken in any order in the multiplication.

(**7.**) Reversely, if we have to multiply the integer M by the integers a, b, c, etc., successively, when taken in any order, then it will be sufficient to multiply M by the product of the integers a, b, c, etc. For if we put the product abc, etc., equal to N, then N will clearly be an integer; and the product, MN (by the last article), will be found by multiplying M by a, and the product by b, and the last product by c, and so on, until all the integers, a, b, c, etc., have been used.

(**8.**) By Art. 9, Sec. I., if a stands for any number or quantity, and n for any positive integer, we express the product aaa, etc., to n factors, which is called the n^{th} power of a, by a^n.

If we have a^n and a^m, n and m being positive integers, the product of a^n and a^m will be expressed by $a^n \times a^m = a^{n+m} =$ the power of a, whose exponent is $n + m =$ the sum of the exponents of the powers that are multiplied together.

For, since aaa, etc., to n factors $= a^n$, and aaa, etc., to m factors $= a^m$, if we multiply these equals by each other we get (by Ax. II.) aaa, etc., to n factors $\times$ aaa, etc., to m factors $= aaa$, etc., to $(n + m)$ factors $= a^{n+m}$, and of course $a^n \times a^m = a^{n+m}$.

Similarly, the product of a^m, a^n, a^p, a^q, etc., is evidently equal to $a^{(m+n+p+, \text{etc.})}$, because a must enter the product as a factor as often as there are units in $m + n + p +$, etc. Thus, $aa = a^2$, $a \times a^2 \times a^3 = a^6$, $a^5 \times a^7 = a^{12}$.

Hence, *any integral powers of the same number or quantity are multiplied together, by adding the indices of the powers, for the index of the number or quantity in the product.*

(**9.**) We will now proceed to determine the sign of a product from the signs of its factors.

1. If a and b, regarded as positive, stand for the multiplicand and multiplier; then, since $a \times b = a + a + a +$, etc., to b terms, $= ba = ab$, it is clear that the product is positive.

2. Let $-a$, supposed negative, stand for the multiplicand, while b, considered as positive, is the multiplier; then we have $-a \times b = -a - a - a$, etc., to b terms, $= -ba = -ab$,

which is negative; indeed, since ab is b times as great as a, it is manifest that $-ab$ is b times as great as $-a$, or that $-a \times b = -ab$.

3. If a and $-b$ represent the multiplicand and multiplier, a being positive and $-b$ negative, then if c is a positive number or quantity of the same kind as b, we evidently have $-b = c - (b + c)$; and of course the multiplication of a by $-b$ is reduced to the multiplication of a by $c - (b + c)$. Because c equals the sum of $c - (b + c)$ and $b + c$, it follows that we must multiply a by c, and then from the product subtract the product of a and $b + c$; consequently, we shall have $a \times -b = a \times [c - (b + c)] = ac - a(b + c)$.

And since $a(b + c) = a + a + a +$, etc., to $(b + c)$ terms, $= a + a +$, etc., to b terms, $+ a + a +$, etc., to c terms, $= ab + ac$, it follows that $ac - a(b + c) = ac - ab - ac$ or (since $ac - ac = 0$) $= -ab$; consequently, $a \times -b = -ab$, a negative product.

4. Let $-a$ and $-b$, regarded as negative, represent the multiplicand and multiplier; or, as has just been shown, they may be represented by $-a$ and $c - (b + c)$.

Because c exceeds the units in $c - (b + c)$ by the units in $b + c$, we must multiply $-a$ by c, and subtract the product of $-a$ and $b + c$ from the result; consequently (agreeably to what has been done), we shall have $-a \times -b = -ac - [-a(b + c)] = -ac + ab + ac$, or (since $-ac + ac = 0$) we shall have $-a \times -b = ab$, which is a positive product.

Hence, *when the multiplicand and multiplier have like signs, their product is positive; and when the multiplicand and multiplier have unlike signs, their product is negative.*

It is hence evident *that the product of any number of factors is positive when it does not contain an odd number of negative factors, and when an odd number of negative factors enter the product it is clearly negative.*

(**10.**) Although the algebraic quantities that may be used in multiplication may indefinitely represent either integral, fractional, or irrational quantities, yet we shall multiply them as if they were integral, because (as we shall hereafter prove) it is allowable to do so.

(**11.**) From what has been done, we may evidently derive the following rule for the multiplication of any two monomials together.

RULE.

To the product of the coefficients of the monomials annex all the letters or literal quantities that belong to each, with their proper exponents. Then if the same letter or literal quantity occurs more than once, write the letter or literal quantity once, with an exponent equal to the sum of the exponents of the letter or literal quantity, added according to their signs; observing that if no exponent is expressed, it is understood to be 1.

If the monomials are both positive or both negative, the product will be positive; but if one of them is positive and the other negative, the product is negative; in other terms, like signs give +, and unlike signs give —, for the sign of the product.

REMARK.

If the monomials have no coefficients expressed, the product of the coefficients is 1; for when any quantity has no coefficient expressed, it is always understood to be 1.

Thus, the product of $7ab$ and $5ac$ is $7ab \times 5ac = 7 \times 5aabc = 35a^1a^1bc = 35a^2bc$, which is positive, because the monomials are positive.

Also, the product of $-3a^2bc$ and $-7a^3b^2cd$ is $-3a^2bc \times -7a^3b^2cd = 21a^2a^3b^1b^2c^1c^1d = 21a^5b^3c^2d$, the product being positive, because the signs of the monomials are alike.

Again, the product of $-9a^3b^5c$ and $15a^7b^2defg^{\frac{1}{2}}$ is $-135a^{10}b^7cdefg^{\frac{1}{2}}$, the product having the sign —, because the monomials have unlike signs.

EXAMPLES.

1.	2.	3.	4.
$14a$	$-6xy$	$-15c^3d^5ex$	abc
$3b$	$-7x^2y^2$	$+13cdvw$	$-a^2d$
$42ab$	$+42x^3y^3$	$-195c^4d^6exvw$	$-a^3bcd$

$$\begin{array}{ccc}
5. & 6. & 7. \\
34amn\sqrt{x} & 5axy & -23xyvw \\
-ampy & 9axw & 24mn^2st \\
\hline
-34a^2m^2np\sqrt{xy} & &
\end{array}$$

$$\begin{array}{ccc}
8. & 9. & 10. \\
-39m^4n^8v^6 & 125a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{1}{4}} & \sqrt{m}\sqrt[3]{n} \\
-42m^6n^5v^{18}g^{15} & -a^{\frac{2}{3}}b^{\frac{4}{3}}c^{\frac{7}{4}} & \sqrt[4]{p}\sqrt[5]{q} \\
\hline
\end{array}$$

$$\begin{array}{cccc}
11. & 12. & 13. & 14. \\
-623a^3(bc)^2 & 13a^{\frac{1}{2}}b^{\frac{3}{4}}c^{\frac{4}{5}} & -45p^3q^4v^5 & -aaabbb^2 \\
-62a^5(bc)^4 & -4a^{\frac{1}{2}}b^{\frac{1}{4}}c^{\frac{1}{5}} & -52pqrv^{\frac{1}{3}} & 13a^2b^2ccde^2 \\
\hline
\end{array}$$

In a similar way the product of any number of monomials may be found.

Thus, find the product of any two of them, as before, and then in the same way multiply the product thus found by any one of the remaining monomials, and so on until all the monomials have been used; the last product will be the product required.

For example; to find the product of $7ab$, $9a^2bc$, and $-5a^3c$, we have $7ab \times 9a^2bc \times -5a^3c, = -5 \times 7 \times 9a^1a^2a^3b^1b^1c^1c^1 = -315a^6b^2c^2$, the sign of the product being $-$, because two of the monomials are positive, and the other negative.

For another example; we shall find the product of $3ab$, $-5ac$, $-6abc$, and $7bd$; here the product is indicated by $3ab \times -5ac \times -6abc \times 7bd = 3 \times -5 \times -6 \times 7 \times a^1a^1a^1b^1b^1b^1c^1c^1d = 630a^3b^3c^2d$, the sign of the product being $+$, because the number of negative monomials is an even number.

EXAMPLES.

1. Find the product of $-2mn\sqrt{cd}$, $-3pqrs$, $-7mn^2ppqqq\sqrt{r}$, and $13xyz$.

2. Find the product of $13a^3b^8$, $-17a^2b^3c^4$, and $9cd$.

3. Find the product which is indicated by $-5a \times -7b \times -13ab \times -17a^2b^3 \times -25de$.

4. Find the product expressed by $a^{\frac{1}{6}} \times 3a^{\frac{1}{3}}b^{\frac{7}{15}} \times 4a^{\frac{1}{3}} \times 2b^{\frac{2}{15}}c^{\frac{1}{3}} \times -4a^2b^{\frac{6}{15}}c^{\frac{1}{2}} \times -17d \times b^2c^3 \times 2d^3$.

(**12.**) We will now show how to find the product of any compound quantity whose terms are regarded as integral, by any integral monomial quantity.

Let $a + b - c$ stand for the compound quantity, and d for the monomial; then, by the nature of multiplication, the product will be expressed by $(a + b - c) + (a + b - c) + (a + b - c)$, etc.; until $(a + b - c)$ is taken as many times in the sum as there are units in d.

Now we have $(a + b - c) + (a + b - c) + (a + b - c) +$, etc., to d terms, which gives $d\,(a + b - c) = da + db - dc$, or $(a + b - c)d = ad + bd - cd$, since, by what has been shown, the factors of each product may be interchanged.

It is easy to see that if the multiplier had been $-d$, we should have had $-(a + b - c)d = -ad - bd + cd$.

Hence results the following rule for the multiplication of any compound quantity by any monomial quantity.

RULE.

Multiply each term of the compound quantity by the monomial quantity, according to the rule for multiplying monomials, and add the products, or connect them by their signs, and the sum thus found will be the product required.

Thus, to multiply $5a - 3b + 2c$ by $4ab$, we shall have $(5a - 3b + 2c) \times 4ab = 20a^2b - 12ab^2 + 8abc$ for the product.

Also, the product of $3a^2b^3 + 2mnp - 5cde$ and $-7cd$ is indicated by $(3a^2b^3 + 2mnp - 5cde) \times -7cd$, which, when the multiplication is executed, gives $(3a^2b^3 + 2mnp - 5cde) \times -7cd = -21a^2b^3cd - 14cdmnp + 35c^2d^2e$.

EXAMPLES.

1.

$$\begin{array}{rl} \text{Multiply} & 23a^3b^4c - 13e^2f \\ \text{by} & 2abef \\ \hline & 46a^4b^5cef - 26abe^3f^2 \end{array}$$

2.

$$\begin{array}{l} \quad 3a^3 - 2b^3 - 3c^3 \\ -2a^2c \\ \hline -6a^5c + 4a^2b^3c + 6a^2c^4 \end{array}$$

3.

$$\begin{array}{l} 5mn - 17p^2q^{\frac{1}{2}} \\ 13pq^{\frac{1}{2}} \\ \hline 65mnpq^{\frac{1}{2}} - 221p^3q \end{array}$$

4.

$$\begin{array}{l} -72rst - 14vw - 4pq \\ -\ \ 6amn \\ \hline \end{array}$$

5.

$$\begin{array}{l} a^3b - 2a^2b^2 + 7ab^3 - 13b^4 - 28 \\ 2a^4b^5 \\ \hline \end{array}$$

6.

$$\begin{array}{l} a^m + a^{m-1}b - a^{m-2}b^2 - a^{m-3}b^3 + a^{m-4}b^4 \\ a^{m+1}b^2 \\ \hline \end{array}$$

7.

$$\begin{array}{l} 7a^2b^3c - 9d^4e^5g + lm^2n^3 \\ 6p^2q^2 \\ \hline \end{array}$$

8.

$$\begin{array}{l} 17a^{\frac{2}{3}}b^{\frac{1}{3}} - 3cd^3ef \\ -\ \ 3a^{\frac{1}{3}}b^{\frac{2}{3}} \\ \hline \end{array}$$

9.

$$\begin{array}{l} -128cdh - 9k^3l^{\frac{3}{4}} \\ -\ \ \ 3cpt^{\frac{1}{4}} \\ \hline \end{array}$$

10.

$$\begin{array}{l} 63a^4b^3 - 62a^2b^2 - 20bc^3 + 28 \\ 31a^2b^3c^3 \\ \hline \end{array}$$

REMARKS.

(**13.**) Reversely, from the rules that have been given for the multiplication of a monomial, or compound quantity, by a monomial quantity, we can often resolve a given algebraic expression into factors, or find quantities such that their product shall equal the given quantity.

We shall call any algebraic expression that is formed from the multiplication of other expressions that are not explicitly of fractional forms, a *composite quantity;* and any algebraic expression that is not thus formed, we shall denominate a *simple* or *prime quantity.* If any algebraic expression is not of a fractional form, we shall call it an *integral form.*

We will now give a few examples, for the purpose of showing how to resolve them into their factors.

For a first example, we shall take $3ab$, whose prime factors are easily seen to be the primes 3, a, and b, for the product of 3, a, and b is $3ab$; we may also take 3 and ab for the factors of $3ab$, but ab is a composite factor.

For a second example, we shall take $15a^2b^3c$; here the simple factors are 3, 5, a, a, b, b, b, c, since their product is $15a^2b^3c$.

For another example, we shall take $ac + bc$; here, since c enters as a factor into each term of the given quantity, it is easy to see that $a + b$ and c are the prime factors required, and we have $(a + b) \times c = ac + bc$.

For another example, we shall take $3a^2b^3c - 15a^2bcde + 45a^2bc$; here, since $3a^2bc$ enters as a factor into each term of the given quantity, it is evident that $3a^2bc$ and $b^2 - 5de + 15$ are factors of the given expression, or that we shall have $(b^2 - 5de + 15) \times 3a^2bc = 3a^2b^3c - 15a^2bcde + 45a^2bc$; where it is to be noted that $3a^2bc$ is composite, its factors being 3, a, a, b, and c.

EXAMPLES.

1. Find the prime factors of $21a^3b^5cd$.

2. Find the prime factors of $91mnp\sqrt{q}$.

3. Find the prime factors of $17a^{\frac{1}{2}}b^{\frac{1}{3}}c^5$.

4. Give the prime factors of $pqrst\sqrt{v}$.

5. Resolve $25amn + 75anpq - 10a^3n^2pq$ into factors, and give all the prime factors.

6. Resolve $-21a^2x^2 + 6ax - 12a^2x^3$ into factors, and give all the prime factors.

7. Resolve $13a^3 - 169b^3$ into factors.

8. Resolve $42a^3x^5 - 28ax^3 - 5a^2x^3$ into factors.

9. Resolve $-92m^3\sqrt{n} - 69m^2n^{\frac{1}{2}} + 15n^{\frac{1}{2}}$ into factors, and find the prime factors.

10. Has $97a^3 - 1729b^3$ any factors?

(**14.**) We shall now proceed to show how to multiply one compound quantity by another.

We shall denote the sum of the positive terms of the multiplicand by a, and the sum of the negative terms by b; we shall also represent the sum of the positive terms of the multiplier by c, and the sum of the negative terms by d; then the multiplicand will be expressed by $a - b$, and the multiplier by $c - d$.

Then the product of $a - b$ and c will be expressed by $(a - b)c$, which is clearly greater than the product of $a - b$ by $c - d$, since c is greater than $c - d$ by d. Hence we must multiply $a - b$ by d, and subtract the product from $(a - b)c$, since $a - b$ is taken too often by the units in d, and of course we shall have $(a - b) \,.\, (c - d) = (a - b)c - (a - b)d$.

But by the rule for the multiplication of a compound quantity by a monomial, we have $(a - b)c = ac - bc$, and $(a - b)d = ad - bd$; which, subtracted from $ac - bc$, gives $(a - b) \,.\, (c - d) = ac - bc - ad + bd$, as required.

From an examination of the product, it appears that each term of $a - b$ has been multiplied by each term of $c - d$, and that like signs have given $+$, and unlike signs have given $-$ for the corresponding products.

Here, it may be proper to notice, that a monomial which is positive is generally written without any sign; also in a compound quantity, if the first or left-hand term is positive, it is generally written without any sign, the sign $+$ being in these instances understood; but in other cases positive terms (generally) have the sign $+$ written before or to the left of them.

If any quantity or term is negative, it is always written with the sign $-$ before or to the left of it, unless the contrary is expressly stated.

It will be convenient to premise the following observations, before giving the rule for the multiplication of compound quantities.

1. A compound quantity, whose terms involve different powers of the same letter, is said to be *arranged according to the descending powers of the letter*, when the term that involves the letter with the greatest positive exponent is written first or to the left, and the term which involves the next greatest positive exponent of the letter is written next to it on the right, and the term that contains the next greatest positive exponent of the letter is written next to the last term on the right, and so on; but if the terms of the compound quantity are written in a reverse or contrary order, it is said to be arranged according to the ascending powers of the letter.

Thus, $x^4 - 7x^3 + 5x^2 - 3x + 18$, is arranged according to the descending powers of x; but if we write the quantity in the reverse or contrary order, it becomes $18 - 3x + 5x^2 - 7x^3 + x^4$; and it is arranged according to the ascending powers of x.

In like manner, $Ax^5 + Bx^4a - Cx^3a^2 + Dx^2a^3$, in which A, B, C, D are the coefficients, is arranged according to the descending powers of x; but if we write the quantity in a contrary order, it will be arranged according to the ascending powers of x.

Again, since a enters the second term, and a^2 enters the third term, and a^3 enters the fourth term, counting from the left to the right, the quantity is arranged according to the descending powers of x, or according to the ascending powers of a, and so on in all analogous cases.

2. If the exponents of the letter, according to which any compound quantity has been arranged, are integers, and if they differ from each other by 1, in any two successive terms, we say the compound quantity is *complete*, or wants none of its terms; but if the difference of the exponents of the letter in two successive terms is greater than 1, we say that the compound quantity is incomplete, or wants some of its terms.

3. When any of the terms of a compound quantity are wanting, we may suppose them to have vanished from the quantity on account of their coefficients becoming equal to naught or zero (0); since any quantity or term when multiplied by naught, must evidently give naught for the product. Hence we may give any incomplete compound quantity a complete form, by supplying the deficient terms with the powers of the letters that are wanting, by writing 0 (or naught) for their coefficients; observing that the terms supplied may evidently have either the sign + or − at will.

4. Not regarding the coefficients of the terms in any compound quantity, if the sum of the exponents of the letters in any one term is the same that it is in any other term; then the compound quantity is termed a *homogeneous* quantity; otherwise, it is said to be a *heterogeneous* quantity. Indeed, any monomials which are such that the sum of the exponents of their letters (that are not regarded as coefficients) is the same in each, are called *homogeneous*; otherwise, they are heterogeneous.

Thus, $7a^3b^2$ and $13m^3n^2$ are homogeneous, as are also abc, p^2q, rst; but a^4b m^2n^4 are heterogeneous, and so are rs, p^2q, v^2w^5.

When deficient terms are supplied in compound quantities whose terms are homogeneous, and composed of different letters, it will be proper that the terms supplied shall be homogeneous with the given terms of the quantity, and in its letters. Thus, the compound quantity $a^3 + b^3$ has deficient terms; which being supplied, it becomes $a^3 + 0a^2b + 0ab^2 + b^3$, which is the complete form of the compound quantity.

From the definition of homogeneous terms, it is clear that the multiplication of homogeneous terms by terms that are also homogeneous, will give products that are homogeneous.

Thus, if we take the homogeneous terms $-3a^2b$ and $2c^2d$, and multiply them by the homogeneous terms $4p^3q^2$ and $5v^2w^3$, we shall get the homogeneous products $-12a^2bp^3q^2$ and $10c^2dv^2w^3$, where it will be noticed that the sum of the exponents of the letters in each term multiplied is 3, and the sum of the exponents of the letters in each multiplier is 5, and we have $3 + 5 = 8 =$ the sum of the exponents of the letters in each product, as it evidently ought to do.

5. When we have to use compound quantities involving different powers of the same letter, or letters, in calculation, it is often convenient to use the coefficients only, and, after the work has been done, to supply the powers of the letter, or letters, that ought to correspond to the coefficients in the result that has been found; this process is generally called *the method of detached coefficients.*

Thus, to add $7a^4 - 9a^3b + 13a^2b^2 - 17ab^3 + 6b^4$, $9a^4 + 6a^3b + 20a^2b^2 + 4ab^3 + 5b^4$ and $5a^4 + 15a^2b^2 - 6ab^3$; by detaching the coefficients, supplying the deficient terms, and putting the coefficients that belong to like terms under each other, and then adding, we get

$$\begin{array}{r}
7 - 9 + 13 - 17 + 6 \\
9 + 6 + 20 + 4 + 5 \\
5 + 0 + 15 - 6 + 0 \\
\hline
21 - 3 + 48 - 19 + 11
\end{array}$$

$=$ the sum of the coefficients; where 21 is the coefficient of

a^4 in the sum, -3, that of a^3b, and so on: so that the sum, when the requisite letters are supplied, is $21a^4 - 3a^3b + 48a^2b^2 - 19ab^3 + 11b^4$, as it ought to be.

In a similar way, to subtract $5a^2 - 7ab$ from $15a^2 - 9ab + b^2$, by detaching the coefficient, etc., we get

$$\begin{array}{r}
15 - 9 + 1 \\
5 - 7 + 0 \\
\hline
10 - 2 + 1
\end{array}$$

= the difference of the coefficients; and the letters, when supplied, give $10a^2 - 2ab + b^2$ for the required difference of the given quantities.

For further illustration of the method of detached coefficients, we shall add the quantities $9a^3 + 6a^2 + 3a + 5$, $4a^3 + 5a + 6$, and $8a^2 + 7a + 3$; detaching the coefficients, supplying the deficient terms, and placing the coefficients of like terms under each other, and then adding the coefficients as in Arithmetic, we get

$$\begin{array}{rrrr}
9 + & 6 + & 3 + & 5 \\
4 + & 0 + & 5 + & 6 \\
0 + & 8 + & 7 + & 3 \\
\hline
13 + & 14 + & 15 + & 14
\end{array}$$

= the sums of the coefficients. Supplying the letters that correspond to the sums of the coefficients, we get $13a^3 + 14a^2 + 15a + 14$ for the sum of the quantities, as required.

If $a = 10$, the sum becomes $a^4 + 4a^3 + 5a^2 + 6a + 4 = 14564$, which will be found to equal the sum of the numbers 9635, 4056, 873, which are the values of the given quantities when $a = 10$.

This example may serve to show the analogy that exists between the different powers of a, and the local values of figures in Arithmetic; it also shows the analogy between the naughts that are used to supply the deficient terms and the naughts that are used in Arithmetic to show the local values of figures.

Again, we shall subtract $5a^4 + 3a^3 + 9a + 9$ from $8a^4 + 6a^3 + 3a + 7$, by the method of detached coefficients. Detaching the coefficients, etc., we get

$$\begin{array}{l} 8+6+0+3+7 \\ 5+3+0+9+9 \\ \hline 3+3+0-6-2; \end{array}$$

and restoring the letters, we get $3a^4 + 3a^3 + 0a^2 - 6a - 2 =$ the difference of the given quantities.

When $a = 10$, we have $2 = a - 8$, $6 = a - 4$, and the difference becomes $3a^4 + 3a^3 + 0a^2 - a(a-4) - (a-8) = 3a^4 + 3a^3 + 0a^2 - a^2 + 3a + 8 = 3a^4 + 2a^3 + 9a^2 + 3a + 8 =$ 32938, which equals the difference between the numbers 86037 and 53099, which are the values of the given quantities when $a = 10$.

We have given the preceding examples for the purpose of showing the simplicity of the method of detached coefficients in addition and subtraction. We shall now give one or two simple examples to show the great importance of the method in multiplication.

For a first example, we shall multiply $3a^3 + 2a^2 + 5a + 6$ by $9a + 1$; first by the common method, and then by detaching the coefficients.

Since we must take the multiplicand as often as there are units in $9a + 1$, it is clear that if we multiply the multiplicand by $9a$ and then by 1, and add the products, we shall have multiplied the multiplicand by $9a + 1$, as required.

Hence we have

$$\begin{array}{rrrrr} & 3a^3 + & 2a^2 + & 5a + & 6 \\ & & & 9a + & 1 \\ \hline 27a^4 + & 18a^3 + & 45a^2 + & 54a & \\ & 3a^3 + & 2a^2 + & 5a + & 6 \\ \hline 27a^4 + & 21a^3 + & 47a^2 + & 59a + & 6 \end{array}$$

= the sought product, as found by the common method, which consists in multiplying each term of the multiplicand (observing the rule of signs) by each term of the multiplier, and then adding the products by the rules of addition.

We shall now multiply by the method of detached coefficients.

Here we have

$$\begin{array}{rrrrr}
3+ & 2+ & 5+ & 6 & \\
9+ & 1 & & & \\
\hline
27+ & 18+ & 45+ & 54 & \\
 & 3+ & 2+ & 5+ & 6 \\
\hline
27+ & 21+ & 47+ & 59+ & 6
\end{array}$$

= the sum of the products of the coefficients; and if we supply the requisite powers of a, we readily obtain the same result that we found by the common method of multiplication. If $a = 10$, the sum of the coefficients is easily changed to the following sum of coefficients, viz., $2 + 9 + 6 + 2 + 9 + 6$, which, by joining the proper powers of a to each term, becomes $2a^5 + 9a^4 + 6a^3 + 2a^2 + 9a + 6 = 296296$, which equals the product of 3256 by 91, as it ought to do; for when $a = 10$, the given multiplicand equals 3256, and the multiplier equals 91.

For the second example, we shall find the product of $8a^2 + 5a + 7$ and $3a^2 + 4$. Detaching the coefficients, etc., we get

$$\begin{array}{rrrrr}
8+ & 5+ & 7 & & \\
3+ & 0+ & 4 & & \\
\hline
24+ & 15+ & 21 & & \\
 & 0+ & 0+ & 0 & \\
 & & 32+ & 20+ & 28 \\
\hline
24+ & 15+ & 53+ & 20+ & 28
\end{array}$$

= the sum of the coefficients; which, by supplying the appropriate powers of the letter, gives $24a^4 + 15a^3 + 53a^2 + 20a + 28$ for the sought product.

If $a = 10$, it is easy to change the sum of the coefficients into the sum of the coefficients $2 + 6 + 0 + 5 + 2 + 8$, which, by supplying the proper powers of a, gives $2a^5 + 6a^4 + 0a^3 + 5a^2 + 2a + 8 = 260528$, which is the product of 857 and 304, which are the values of the multiplicand and multiplier when $a = 10$.

The analogy of the above process, in the use of the naughts and the arrangement of the work, to the method of multiplying integral numbers, etc., in Arithmetic, is too evident to need any comment.

For the last example, we shall find the product of $a^3 + 3a^2b + 13b^3$ and $a^2 + 2b^2$. Detaching the coefficients, etc., we get the following work and sum of coefficients:

$$\begin{array}{rrrrrrrrrrr}
1 & + & 3 & + & 0 & + & 13 & & & & \\
1 & + & 0 & + & 2 & & & & & & \\
\hline
1 & + & 3 & + & 0 & + & 13 & & & & \\
 & & 0 & + & 0 & + & 0 & + & 0 & & \\
 & & & & 2 & + & 6 & + & 0 & + & 26 \\
\hline
1 & + & 3 & + & 2 & + & 19 & + & 0 & + & 26
\end{array}$$

= the sum of the coefficients. When the requisite powers of the letters are supplied, we have $a^5 + 3a^4b + 2a^3b^2 + 19a^2b^3 + 0ab^4 + 26b^5 = a^5 + 3a^4b + 2a^3b^2 + 19a^2b^3 + 26b^5 =$ the required product, since the coefficient of ab^4 equals naught.

It may be noticed that the terms of the multiplicand are homogeneous in a and b, and the terms of the multiplier are homogeneous in the same letters, and that the quantities are arranged according to the descending powers of a, the exponents decreasing by a unit from each term to the next successive term on the right; and that the quantities are arranged according to the ascending powers of b, the exponents increasing by a unit from each term to the next successive term on the right; and since the sum of the exponents of the letters in each product is 5, and as a^5 alone enters the first or left-hand term of the product, it is easy to see how to supply the powers of the letters that belong to each of the coefficients in the product.

6. Reversely, if we have to find the product of quantities which do not involve different powers of the same letter or letters, we may supply the powers of a letter or letters according to the nature of the case, so as to reduce the process to the method of multiplying quantities that involve different powers of the same letter or letters.

For we may regard the given terms as being the coefficients of different powers of a letter or letters, which have each been put equal to unity.

Thus, if we have to multiply $a + b$ by $c + d$, we may multiply $ax + b$ by $cx + d$, and putting $x = 1$ in the product, we shall of course have the product of $a + b$ and $c + d$. Multiplying $ax + b$ by $cx + d$, we have (evidently) the following process and result:

$$\begin{array}{lll}
ax^2 & + b & \\
cx & + d & \\
\hline
acx^2 + cbx & & \\
 & adx & + bd \\
\hline
acx^2 + (cb + ad)x & & + bd
\end{array}$$

= the product; which, by putting $x = 1$, gives $ac + cb + ad + bd$ for the product of $a + b$ and $c + d$.

In a similar way, if we have to multiply $3a + 5b + 7d$ by $4a + 6b + d$, we may multiply $3ax^2 + 5bx + 7d$ by $4ax^2 + 6bx + d$; and we get

$$\begin{array}{lllll}
3ax^2 & + \ 5bx & + 7d & & \\
4ax^2 & + \ 6bx & + \ d & & \\
\hline
12a^2x^4 & + 20abx^3 & + 28adx^2 & & \\
 & 18abx^3 & + 30b^2x^2 & + 42bdx & \\
 & & 3adx^2 & + \ 5bdx & + 7d^2 \\
\hline
12a^2x^4 & + 38abx^3 & + (31ad + 30b^2)x^2 & + 47bdx & + 7d^2
\end{array}$$

= the product; and putting $x = 1$, we have $(3a + 5b + 7d) \times (4a + 6b + d) = 12a^2 + 38ab + 31ad + 30b^2 + 47bd + 7d^2$, as required.

7. It is easy to see that the number of products arising from the multiplication of one compound quantity by another will equal the product of the number of terms of the one quantity by the number of terms of the other quantity; since the products are formed from the multiplication of each term of the one quantity by each term of the other.

Consequently, if the products are all unlike, there will be as many terms in the sum of the products, or in the product of the two compound quantities, *as there are units in the product of the number of terms of the one by the number of terms in the other.* But if the products are not all unlike, two or more of the products will be reduced to one by addition, so that there will not be so many terms in the product of the two quantities as before.

Thus we have found that $(a + b)(c + d) = ac + cb + ad +$

bd, which contains four terms; this is in conformity to what has been stated above, since there are two terms in $a+b$ and two terms in $c+d$, $\therefore$ we have $2\times 2=4=$ the number of terms in the product of the two quantities.

Also, there are three terms in $3a+5b+7d$, and three terms in $4a+6b+d$, $\therefore$ $3\times 3=9=$ the number of products arising from the multiplication of the one compound quantity by the other; but we have found that $(3a+5b+7d)\times(4a+6b+d)=12a^2+38ab+31ad+30b^2+47bd+7d^2$, which contains only six terms; since $20ab$ and $18ab$ have been reduced to $38ab$, $28ad$ and $3ad$ to $31ad$, $42bd$ and $5bd$ to $47bd$; these reductions clearly result from some of the terms of one of the compound quantities being similar to some of the terms of the other.

But there are terms of the partial products which can not be united with other products. For, when we multiply the terms of the two quantities that contain the greatest positive exponents of the same letter by each other, the product evidently will be unlike any other terms of the partial products; and of course the product will appear in the final product without reduction; also, if there are terms of the quantities which do not involve any letters (or that contain their lowest powers, it is clear that their product must appear in the final product without reduction.

Thus, if we have $(3x^3+7x^2+8x+9)\times(6x^2+11x+8)$, we have the products $3x^3\times 6x^2=18x^5$ and $9\times 8=72$, which must evidently appear in the complete product without reduction.

8. By 15, Sec. I., when we wish to show that all the terms of any compound quantity are to be taken together as a single quantity or letter, we draw a right line (called a vinculum or bar) over them, or we write them in a parenthesis. Hence we may evidently express the product of a compound and single quantity, or of compound quantities, in a way very similar to that of letters.

Thus, to show that $a-b+c$ is to be multiplied by m, we write either of the following expressions, viz., $\overline{a-b+c}\times m$, $\overline{a-b+c}\,.\,m$, $(a-b+c)\times m$, $(a-b+c)\,.\,m$; or, more simply, $(a-b+c)m$.

In a similar way, to indicate the product of $a-b+c$ and

$e+f-g$, we may use any one of the following expressions, viz., $\overline{a-b+c} \times \overline{e+f-g}$, $\overline{a-b+c} \,.\, \overline{e+f-g}$, $(a-b+c) \,.\, (e+f-g)$, $(a-b+c)(e+f-g)$.

9. When we have an expression which shows that quantities are to be multiplied together, the multiplication is said to be *expressed*, *signified*, or *indicated*; and when the multiplication is completed, it is said to be *executed*, *developed*, or *expanded*.

10. It is clear, from Axiom V., that any multiplication, when indicated, ought to be regarded as identical with the multiplication when executed; in other terms, the expression which shows that any quantities are to be multiplied together, forms an identical equation with the product of the quantities.

Thus, $(a+b)(c+d) = ac + bc + ad + bd$, is an identical equation, which is clearly true for any possible values that a, b, c, and d can be conceived to have; since $(a+b)(c+d)$ and $ac + bc + ad + bd$ are only different forms of the product of $a+b$ and $c+d$. In a similar way it may be shown that $(a+x)(a-x) = a^2 - x^2$, is an identical equation.

We are now prepared to give the following rule, which is applicable to every case of multiplication.

RULE.

1. *For convenience, place the first (or left-hand) term of the multiplier under the first term of the multiplicand, and the second term of the multiplier under the second term of the multiplicand, and so on, until all the terms of the multiplicand and multiplier, with their proper signs, are put down.*

If any of the terms of the multiplicand and multiplier involve the same letter, then (for simplicity) the terms of the multiplicand and multiplier ought to be set down so that the multiplicand and multiplier may each be arranged, either according to the ascending or descending powers of the letter; observing that any term which may be wanting, either in the multiplicand or multiplier, ought to be supplied. After the multiplicand and multiplier have been put down as required, draw a right line or bar under the multiplier.

2. *Then (by the rule for the multiplication of monomials)*

multiply the first, second, third, etc., terms of the multiplicand successively by the first term of the multiplier, and put each product beneath the bar, and under the corresponding term of the multiplicand.

After each term of the multiplicand has been multiplied by the first term of the multiplier, then multiply the first, second, third, etc., terms of the multiplicand successively by the second term of the multiplier; putting the first of these products under the second of the former products, and the second of these products under the third of the former products, and so on. Proceed in like manner to multiply each term of the multiplicand by the third, fourth, etc., terms of the multiplier, until each term of the multiplicand has been multiplied by each term of the multiplier; always putting the product of the first term of the multiplicand, and any term of the multiplier, under the term of the multiplier and below the preceding products that correspond to it, and the product of the second term of the multiplicand and the same term of the multiplier one term or place to the right, and so on.

3. *Having multiplied each term of the multiplicand by each term of the multiplier, draw a bar below the products thus found, then the products added according to their signs, and their sum, written below the bar, will be the complete product of the multiplicand and multiplier, as required.*

4. *When the multiplicand and multiplier have been arranged according to the powers of some letter in each, it is easy to see that the products which involve the same power of the letter will stand in vertical columns under each other; so that when the coefficients of any power which stand under each other in any column, are numbers, and are added according to their signs, their sum will be the coefficient of the power of the letter; but if the coefficients are letters, or if some of them are letters and others are numbers, then they must be added by the rules of addition, and their sum, when written in a parenthesis, must be placed before the power of the letter for its coefficient.*

Proceeding in like manner for each vertical column, we readily get the sum of all the partial products; and, of course, we have the complete product of the given multiplicand and multiplier, as required.

5. *When the multiplicand and multiplier are arranged according to the powers of a letter, the multiplication will be greatly simplified by detaching the coefficients, and using them only in the multiplication, and afterward supplying the powers of the letter that correspond to the sums of the partial products of the coefficients that stand in any vertical column of the complete product of the coefficients.*

6. *It may be added that, in all cases of multiplication, the rule of signs must be observed; viz., that like signs give +, and unlike signs give −; so that when any term of the multiplicand is multiplied by any term of the multiplier, if the signs of the two terms are both +, or both −, their product must be +, but if one of the terms is + and the other −, their product must have the sign − prefixed to it.*

7. *It is to be further noticed, that if the same letter enters the product of any term of the multiplicand by any term of the multiplier more than once, then we must write the letter once, and give it an exponent equal to the sum of the exponents of the letter in the product; in other terms, powers of the same quantity are found by adding their exponents. Always observing, when a letter or any quantity has no exponent expressed, that it is understood to be unity (or* 1).

It is easy to see that our rule of multiplication is in conformity to the methods which we used in doing the examples in multiplication that preceded it, so that they may serve to show its correctness and use. The rule is clearly, in some respects, similar to the multiplication of whole numbers in Arithmetic, especially when the multiplicand and multiplier are arranged according to the dimensions of a common letter. For in multiplying the multiplicand by the first, second, third, etc., terms of the multiplier, from left to right, the partial products will be so placed that the coefficients of the same power of the common letter will stand in a vertical column under each other, in the same way that the figures of the same local value stand under each other in the partial products when we multiply one whole number by another in Arithmetic.

We will now give some examples for further illustration of the rule; some of which will be of use to us hereafter.

EXAMPLES.

1.

$$\begin{array}{rl} \text{Multiply} & x - 5 \\ \text{by} & x - 3 \\ \hline & \text{bar} \\ & x^2 - 5x \\ & \quad - 3x + 15 \\ \hline & \text{bar} \\ & x^2 - 8x + 15 = \text{the product.} \end{array}$$

2.

$$\begin{array}{rl} \text{Multiply} & x - a \\ \text{by} & x - b \\ \hline & x^2 - ax \\ & \quad - bx + ab \\ \hline & x^2 - (a + b)x + ab = \text{the product.} \end{array}$$

Ex. 3.—To find the product of $x - a$, $x - b$, and $x - c$, or to develop $(x - a)(x - b)(x - c)$.

By the last example, we have $(x - a)(x - b) = x^2 - (a + b)x + ab$; and of course we shall have $(x - a)(x - b)(x - c) = [x^2 - (a + b)x + ab](x - c)$, or we must multiply $x^2 - (a + b)x + ab$ by $x - c$.

Hence we get

$$\begin{array}{l} x^2 - (a + b)x + ab \\ x - c \\ \hline x^3 - (a + b)x^2 + abx \\ \quad - cx^2 + (ac + bc)x - abc \\ \hline x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc \end{array}$$

for the sought product.

Ex. 4.—To develop $(x - a)(x - b)(x - c)(x - d)$.

By Ex. 3, we have $(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$; therefore this product, when multiplied by $x - d$, will be the development required.

Hence we have

$$\begin{array}{l}
x^3-(a+b+c)x^2+(ab+ac+bc)x-abc \\
x-d \\
\hline
x^4-(a+b+c)x^3+(ab+ac+bc)x^2-abcx \\
\quad -dx^3+(ad+bd+cd)x^2-(abd+acd+bcd)x+abcd \\
\hline
x^4-(a+b+c+d)x^3+(ab+ac+bc+ad+bd+cd)x^2-(abc+abd \\
+acd+bcd)x+abcd
\end{array}$$

for the product required.

REMARKS.

It is easy to see, from these examples, that the product $(x-a)(x-b)(x-c)(x-d)(x-e)(x-b)$, etc., can be written down without any formal multiplication. For the index of x in the first term equals the number of factors that are multiplied together; and the index of x in the second term is smaller by a unit than the index of x in the first term; and the index of x in the third term is less by a unit than it is in the second term, and so on, decreasing by a unit from each term to the next successive term, until we come to the last, which does not contain x.

Also, the coefficient of the power of x, in the second term, is the sum of all the quantities, $-a$, $-b$, $-c$, $-d$, $-$ etc., without repetition. The coefficient of the power of x, in the third term, is the sum of the products of every two of the quantities, $-a$, $-b$, $-c$, $-d$, $-$ etc., without repetition.

The coefficient of the power of x, in the fourth term, is the sum of the products of every three of the same quantities, $-a$, $-b$, $-c$, $-d$, $-$ etc., without repetition. The coefficient of the power of x, in the fifth term, is the sum of the products of every four of the same quantities, $-a$, $-b$, $-c$, $-d$, $-$ etc., without repetition.

In like manner, the coefficient of the power of x, in the sixth term, is the sum of the products of every five of the same quantities, without repetition; and so on, until we come to the last term, which does not contain x, and is called the *absolute term;* which is equal to the product of all the quantities, $-a$, $-b$, $-c$, $-d$, $-e$, $-$ etc.

Hence, if the positive integer n is equal to the number of factors in the product, and we put A_1 for the sum of all the quantities, $-a$, $-b$, $-c$, etc.; A_2 for the sum of the products of every two of the same quantities; A_3 for the sum of the products of every three of the same quantities; A_4 for the sum of the products of every four of the same quantities, and so on; and if An is put for the absolute term, or for the product of all the quantities, $-a$, $-b$, $-c$, $-d$, $-e$, $-$ etc., then we shall have $(x-a)(x-b)(x-c)(x-d) \times$ etc., $= x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} + A_4x^{n-4} +$, etc. . . . . . $+ An$ for the product, as required.

Hence, if we put $x = a$, we have $x - a =$ naught, and the product $(x-a)(x-b)(x-c) \times$ etc. $=$ naught; consequently, we shall have $x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3}$, etc., . . . $+ An = 0$; which is called an equation of the n^{th} degree.

It is easy to see that the product $(x-a)(x-b)(x-c)(x-d) \times$, etc., becomes naught by putting x equal to any one of the quantities a, b, c, d, etc.; so that there are n different values of x that will reduce the product $(x-a)(x-b)(x-c)(x-d) \times$, etc., to naught, or which will satisfy the aforesaid equation, *which values are called the roots of the equation;* so that when a, b, c, d, etc., are known, we see how to find the roots of the equation, or the values of x that satisfy it.

Reciprocally, having the equation $x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} + A_4x^{n-4} +$, etc., ... $+ An = 0$; then to find a, b, c, d, etc., is to find the roots of the equations, or the values of x that will reduce $x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} +$, etc., ... $+ An$, to naught, or make it equal 0; which is called *resolving* or *solving* the equation.

It is evident, since a, b, c, d, etc., are the values of x that reduce $x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} +$, etc., . . . $+ An$, to zero, or the roots of the equation $x^n + A_1x^{n-1} + A_2x^{n-2} +$, etc., . . $+ An = 0$, that, having found the roots of the equation, we, of course, have the quantities $x-a$, $x-b$, $x-c$, $x-d$, etc., whose product $(x-a)(x-b)(x-c)(x-d) \times$, etc., $= x^n + A_1x^{n-1} + A_2x^{n-2} +$, etc., . . . $+ An$*;* so that to resolve any quantity into factors, we put it equal to naught, and thence find the particular values of the unknown quantity which reduce the quantity to naught, and thence find the factors, as has been said.

If the roots of the equation are all real, positive quantities, it is evident that a, b, c, d, etc., will each be a real positive quantity, and of course $-a$, $-b$, $-c$, $-d$, etc., will each be a real negative quantity; consequently the equation $x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} +$, etc., $\dots + An = 0$ will have $n + 1$ terms; since A_1, which is put for $-(a + b + c + d +$, etc.), can not become equal to naught, but must be a negative quantity; also A_2, which is the sum of the products of every two of the quantities $-a$, $-b$, $-c$, $-d$, etc., must be a positive quantity greater than naught; and A_3, which is the sum of the products of every three of the quantities $-a$, $-b$, $-c$, $-d$, must be a negative quantity; and so on, for all the coefficients.

Hence, regarding x as being a positive quantity, x^n will be a positive quantity, and putting down the signs of the first, second, etc., terms of our equation, we have $+ - + - + - + - +$, and so on, until we have $n + 1$ signs; but the number of *changes* (or *variations*) of signs, when we count the signs (by twos) from the first which is $+$ to the last will equal n, which is the same as the number of real, positive roots in the equation; since the roots are by supposition all real, positive quantities.

If n is an even number, then $n + 1$ is an odd number, and an inspection of the series of signs (above) shows that the sign of the absolute term An of our equation is positive; but if n is an odd number, then the sign of the absolute term is $-$. When n is an even number, if we put a negative quantity for x, it is evident that the signs of all the terms of the equation will be $+$; since x^{n-1} is negative, and A_1 negative, and of course their product A_1x^{n-1} is positive; also A_2 being positive, and x^{n-2} positive, their product is, of course, positive; but A_3 being negative, and x^{n-3} being also negative, their product must be positive, and so on; so that the sign of each term of the equation is $+$. In like manner, if n is an odd number, and x a negative quantity, it may be shown that the sign of each term is $-$.

Hence if a negative quantity is put for x, the equation does not admit of any changes of signs; or, as it is generally said, the signs are *permanences.*

If the roots of the equation are equal to each other, then, since An is the product of n equal roots, if we extract the n^{th} root of An regarded as positive, we shall have the numerical value of each of the equal roots expressed by $\sqrt[n]{An}$.

If the signs of the terms of the equation are alternately + and −, we must for x put $\sqrt[n]{An}$ in the equation, and it will be reduced to naught; but if all the terms of the equation have the same sign, we must for x put $-\sqrt[n]{An}$, and the equation will be satisfied, as required; as is evident from what has been shown.

Thus, if we have $x^2 - 2xa + a^2 = 0$, then $n = 2$ and $An = a^2$, and $\sqrt[2]{a^2} = a$; and since the signs of the terms of the equation are alternately + and −, we must put $x = a$, and the equation reduces to $a^2 - 2a^2 + a^2 = 0$, as it ought to do; consequently, we shall have $x - a$, $x - a$, for factors of $x^2 - 2xa + a^2$, or, which is the same, we shall have $(x - a)^2 = x^2 - 2xa + a^2$.

Again, if we have $x^2 + 2xa + a^2 = 0$, we must for x put the particular value $x = -\sqrt[2]{a^2} = -a$ in the equation, and it will be reduced to $a^2 - 2a^2 + a^2 = 2a^2 - 2a^2 = 0$, as it ought to do; consequently, $x + a$, $x + a$, are factors of $x^2 + 2xa + a^2$, or we have $(x + a)^2 = x^2 + 2xa + a^2$.

If the roots of the equation are unequal, then the numerical value of $\sqrt[n]{An}$ will be a species of mean between the roots of the equation being numerically greater than some of them and less than others.

Finally, if $n = 1$, our equation becomes $x + A_1 = 0$, which is called an equation of the first degree, or a simple equation, since the exponent of x is 1. If $n = 2$, the equation becomes $x^2 + A_1x + A_2 = 0$, which is called an equation of the second degree, a quadratic equation, or simply a quadratic, since the exponent of the highest power of x is 2.

If n equals 3 ($n = 3$), the equation becomes $x^3 + A_1x^2 + A_2x + A_3 = 0$, which is an equation of the third degree, or a cubic equation, the exponent of the highest power of x being 3.

If $n = 4$, the equation becomes $x^4 + A_1x^3 + A_2x^2 + A_3x + A_1 = 0$, which is an equation of the fourth degree, or a biquadratic equation, or simply a biquadratic, since the exponent of the highest power of x is 4.

If $n = 5, 6, 7$, etc., the equation is said to be of the fifth, sixth, seventh, etc., degrees, where it may be noted that an equation of the fifth degree is sometimes called a sur-solid, and an equation of the sixth degree a quadrato-cubic, or square-cubic equation.

Ex. 5.—To find the product of $a + b$ and $a - b$.

Here we have

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \qquad - ab - b^2 \\ \hline a^2 - b^2, \end{array}$$

the product, as required.

Reciprocally, we can resolve $a^2 - b^2$ into factors; for by adding $ab - ab = 0$ to it, we have $a^2 + ab - ab - b^2 = a^2 - b^2$; but it is easy to see that a is a factor of $a^2 + ab$, and b a factor of $ab + b^2$, so that we have $a^2 + ab = a(a + b)$ and $ab + b^2 = b(a + b)$; consequently, we have $a^2 + ab - ab - b^2 = a(a + b) - b(a + b) = (a - b)(a + b)$, or we have $a^2 - b^2 = (a - b)(a + b) = (a + b)(a - b)$, so that $a^2 - b^2$ has $a + b$ and $a - b$ for factors.

Again, if in $a^2 - b^2$ we put b or $-b$ for a, it reduces to $b^2 - b^2 = 0$; consequently, $a - b$ and $a + b$ are factors of $a^2 - b^2$, or $a^2 - b^2 = (a + b)(a - b)$, as evidently follows from what was shown in the preceding remarks.

Hence, it follows that the product of the sum and difference of any two numbers or quantities of the same kind is equal to the difference of their squares; and conversely, the difference of the squares of any two numbers or quantities of the same kind is equal to the sum of the numbers or quantities multiplied by their difference.

Thus, if $a = 14$, $b = 10$, the sum of a and b is $a + b = 24$, and the difference of a and b is $a - b = 4$; then $(a + b)(a - b) = a^2 - b^2$ changes into $24 \times 4 = 14^2 - 10^2 = 96$, and $a^2 - b^2 = (a + b)(a - b)$ becomes $14^2 - 10^2 = 24 \times 4 = 96$; results that agree with what is affirmed above.

Ex. 6.—To multiply $x^2 + xy + y^2$ by $x - y$, by detaching the coefficients.

The coefficients being detached. give $1 + 1 + 1$ for the coefficients of the multiplicand, and $1 - 1$ for those of the multiplier. Hence we have to multiply $1 + 1 + 1$ by $1 - 1$, which gives

$$\begin{array}{rrrr} 1 & +1 & +1 & \\ 1 & -1 & & \\ \hline 1 & +1 & +1 & \\ & -1 & -1 & -1 \\ \hline 1 & +0 & +0 & -1 \end{array}$$

= the sum of the products of the coefficients.

Since x^2 is the first term of the multiplicand, and x that of the multiplier, it is clear that x^3, x^2y, xy^2, and y^3 will correspond to the first, second, third, and fourth coefficients in the sum of the products of the coefficients; consequently, we have $x^3 + 0.\, x^2y + 0.\, xy^2 - y^3 = x^3 - y^3$ = the required product.

Hence, since $(x^2 + xy + y^2)(x - y) = x^3 - y^3$, it follows that $x^3 - y^3$ has $x^2 + xy + y^2$, and $x - y$ for its factors.

Conversely, it is easy to see that $x - y$ must be a factor of $x^3 - y^3$; for if we put y for x, $x^3 - y^3$ becomes $y^3 - y^3 = 0$, and consequently, from what has been proved, $x - y$ is a factor of $x^3 - y^3$.

To resolve $x^3 - y^3$, we add $-x^2y + x^2y - xy^2 + xy^2 = 0$ to it, which gives $x^3 - y^3 = x^3 - x^2y + x^2y - xy^2 + xy^2 - y^3$; then since x^2 is a factor of $x^3 - x^2y$, and xy a factor of $x^2y - xy^2$, and y^2 a factor of $xy^2 - y^3$, we get $x^3 - y^3 = x^2(x - y) + xy(x - y) + y^2(x - y)$, and since $x - y$ is a factor of the right member of this equation, we get $x^3 - y^3 = (x^2 + xy + y^2)(x - y)$; and of course $x^2 + xy + y^2$ and $x - y$ are factors of $x^3 - y^3$, as required.

Ex. 7.—To develop $(x^2 - xy + y^2)(x + y)$, by detaching the coefficients.

Detaching the coefficients, and proceeding as in the last example, we get

$$\begin{array}{rrrr} 1 & -1 & +1 & \\ 1 & +1 & & \\ \hline 1 & -1 & +1 & \\ & 1 & -1 & +1 \\ \hline 1 & +0 & +0 & +1 \end{array}$$

= the sum of the products of the coefficients.

It is easy to see that $x^3 + 0x^2y + 0xy^2 + y^3 = x^3 + y^3$ is the sought product, so that we have $(x^2 - xy + y^2)(x + y) = x^3 + y^3$; consequently, $x^2 - xy + y^2$ and $x + y$ are factors of $x^3 + y^3$.

It is clear that if we change y into $-y$ in the last example, the multiplicand and multiplier there given, will become the multiplicand and multiplier in this example; also, the results there obtained will coincide with those of this question.

Ex. 8.—To develop $(x^3 + x^2y + xy^2 + y^3)(x - y)$, by detaching the coefficients.

Detaching the coefficients, etc., we shall have

$$\begin{array}{l} 1+1+1+1 \\ 1-1 \\ \hline 1+1+1+1 \\ \quad -1-1-1-1 \\ \hline 1+0+0+0-1 \end{array}$$

= the sum of the products of the coefficients.

Supplying the requisite letters, we shall get $(x^3 + x^2y + xy^2 + y^3)(x - y) = x^4 - y^4$, as required; of course, $x^3 + x^2y + xy^2 + y^3$ and $x - y$ are factors of $x^4 - y^4$. Since x^2 is a factor of $x^3 + x^2y$, and y^2 a factor of $xy^2 + y^3$, we have $x^3 + x^2y + xy^2 + y^3 = x^2(x + y) + y^2(x + y) = (x^2 + y^2)(x + y)$; consequently, $(x^3 + x^2y + xy^2 + y^3)(x - y)$ reduces to $(x^2 + y^2)(x + y)(x - y) = (x^2 + y^2)(x^2 - y^2) = x^4 - y^4$, as it clearly ought to do; for $x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2)$, as appears from Ex. 5.

Conversely, having $x^4 - y^4$ to find its factors; we may add $-x^3y + x^3y - x^2y^2 + x^2y^2 - xy^3 + xy^3 = 0$, or $-x^2y^2 + x^2y^2 = 0$ to it, and we get $x^4 - y^4 = x^4 - x^3y + x^3y - x^2y^2 + x^2y^2 - xy^3 + xy^3 - y^4 = x^3(x - y) + x^2y(x - y) + xy^2(x - y) + y^3(x - y) = (x^3 + x^2y + xy^2 + y^3)(x - y)$, or $x^4 - y^4 = x^4 - x^2y^2 + x^2y^2 - y^4 = x^2(x^2 - y^2) + y^2(x^2 - y^2) = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$; consequently, $x^3 + x^2y + xy^2 + y^3$ and $x - y$ are factors of $x^4 - y^4$; and so are $x^2 + y^2$ and $x^2 - y^2$, or $x^2 + y^2$, $x + y$, and $x - y$.

REMARKS.

If we have $a^n - b^n$, such that n is any positive integral number; then if we put b for a, we have $b^n - b^n = 0$; therefore, $a - b$ must be a factor of $a^n - b^n$, agreeably to the remarks under Ex. 4.

To find the factors of $a^n - b^n$, we may evidently put it under the form $a^n - b^n = a^n - a^{n-1}b + a^{n-1}b - a^{n-2}b^2 + a^{n-2}b^2 - a^{n-3}b^3 + a^{n-3}b^3 - \text{etc.}, \ldots - ab^{n-1} + ab^{n-1} - b^n = a^{n-1}(a - b) + a^{n-2}b\,(a - b) + a^{n-3}b^2(a - b) + a^{n-4}b^3 (a - b) + \text{etc.}, \ldots + ab^{n-2}(a - b) + b^{n-1}(a - b) = [a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \text{etc.}, \ldots + ab^{n-2} + b^{n-1}] [a - b]$, (A); consequently, $a^n - b^n$ is resolved into the two factors $a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \text{etc.}, \ldots + ab^{n-2} + b^{n-1}$, and $a - b$; and it is evident that the first of these factors has as many terms as there are units in n. We have written (A) against the equation that shows $a^n - b^n$ to be the product of the factors found, and shall refer to the equation when necessary as equation (A), or *formula* (A); and we shall adopt a similar method of designation in all analogous cases, whenever it shall be of importance to do it.

If n is an odd number, and we put $-b$ for b in equation (A), it will evidently become $a^n + b^n = [a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \text{etc.}, \ldots - ab^{n-2} + b^{n-1}] . [a + b]$, (B), as is clear from the consideration that any odd power of a negative quantity is negative, and that any even power of a negative quantity is positive.

Hence we see that $a^{2n} - b^{2n}$ has the factors in the right members of (A), and (B) for its factors; consequently, $a^{2n} - b^{2n}$ has $(a - b)(a + b) = a^2 - b^2$ for a factor.

If in (A) we put $n = 2$, it becomes $a^2 - b^2 = [a^{2-1} + a^{2-2}b] . [a - b]$, or since $2 - 1 = 1$, and $2 - 2 = 0$, we have $a^2 - b^2 = [a + a^0 b] . [a - b]$, or $(a + b)(a - b) = (a + a^0 b)(a - b)$; of course, we must have $a + a^0 b = a + b$, or $a^0 = 1$.

Hence, since a may represent any quantity, *it follows that any quantity whose exponent is* 0 *is unity* (*or* 1).

Again, if in (A) we put $n = 3$, and observe that $a^{3-3} = a^0 = 1$, we shall get $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$; and in like manner, if $n = 4$, we get $a^4 - b^4 = (a^3 + a^2b + ab^2 + b^3)(a - b)$; also, $a^5 - b^5 = (a^4 + a^3b + a^2b^2 + ab^3 + b^4)(a - b)$; and so on, for any particular, positive integral numbers, that may be assumed for n.

There is a transformation of (A) which will be useful to us in the investigation of the *binomial theorem*, which we shall now proceed to give.

Thus, since $a^n - b^n$ is less than a^n by b^n, if we add b^n to the right member of (A), we shall clearly have $a^n = b^n + [a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \text{etc.}, \ldots\ldots + ab^{n-2} + b^{n-1}] . [a-b]$; and if we put $a - b = h$, by adding b to these equals (Ax. II.), we get $a - b + b = b + h$ or (since $-b + b = 0$, Ax. VI.) $a = b + h$; consequently, by putting h for $a - b$ and $b + h$ for a, in the preceding equation, it will become $(b+h)^n = b^n + [(b+h)^{n-1} + (b+h)^{n-2}b + (b+h)^{n-3}b^2 + (b+h)^{n-4}b^3 + \text{etc.}, \ldots\ldots + (b+h)b^{n-2} + b^{n-1}] . h$, (C), which is the transformation required.

It will be observed that $b + h$ is a binomial quantity, since it consists of two terms, b and h, united by the sign $+$, according to 14 of Sec. I.; and that $(b+h)^n$ denotes the power of $b + h$, whose exponent is n, agreeably to 9, and 15, of Sec. I.

The right member of (C) is one form of the development of $(b+h)^n$, which has the term b^n that is independent of h, and its remaining terms (or the terms within the braces, [ ],) dependent on h, since they have h for a common factor. Since the exponents $n-1$, $n-2$, $n-3$, $n-4$, and so on, of $b + h$ within the braces in (C) are positive integers, we may develop $(b+h)^{n-1}$, $(b+h)^{n-2}$, $(b+h)^{n-3}$, and so on by changing n into $n-1$, $n-2$, $n-3$, and so on; and we shall have $(b+h)^{n-1} = b^{n-1} +$ terms that depend on h; also $(b+h)^{n-2} = b^{n-2} +$ terms that depend on h, and $(b+h)^{n-3} = b^{n-3} +$ terms that depend on h, and so on to $b + h$.

Hence, since $b^{n-2}b = b^{n-2+1} = b^{n-1}$, $b^{n-3}b^2 = b^{n-3+2} = b^{n-1}$, and so on to $bb^{n-2} = b^{n-1}$, we get $(b+h)^{n-1} + (b+h)^{n-2}b + (b+h)^{n-3}b^2 + (b+h)^{n-4}b^3 + \text{etc.}, \ldots + (b+h)b^{n-2} + b^{n-1} = b^{n-1} + b^{n-1} + b^{n-1} +$, etc., to n terms, + terms that depend on h, $= nb^{n-1} +$ terms that depend on h; consequently, putting for the terms within the braces in (C) their value $nb^{n-1} +$ terms that depend on h, we get $(b+h)^n = b^n + (nb^{n-1} +$ terms that depend on h,$) . h$, or multiplying the terms in the parenthesis by h, we get $(b+h)^n = b^n + nb^{n-1}h +$ terms that depend on h^2, h^3, etc.

Ex. 9.—To develop the square of $x + a$; or, which is the same, to find the product arising from the multiplication of $x + a$ by $x + a$, which is expressed by $(x+a)(x+a) = (x+a)^2$.

Here we have, by using the common method of multiplication,

$$\begin{array}{lll}
x^{} + a & & \\
x^{} + a & & \\
\hline
x^2 + xa & & \\
 & xa + a^2 & \\
\hline
x^2 + 2xa + a^2 & &
\end{array}$$

$= (x + a)^2$, as required.

We shall now find the square of $x + a$ by detaching the coefficients.

The coefficients of $x + a$ being detached give $1 + 1$, so that we have to multiply $1 + 1$ by $1 + 1$. Hence we get

$$\begin{array}{ccccc}
1 & + & 1 & & \\
1 & + & 1 & & \\
\hline
1 & + & 1 & & \\
 & & 1 & + & 1 \\
\hline
1 & + & 2 & + & 1
\end{array}$$

= the sum of the products of the coefficients, and supplying the letters that correspond to the coefficients in the product, we get $x^2 + 2xa + a^2 = (x + a)^2$; which is the same result that we found by the common method of multiplication.

Hence, *the square of the sum of any two numbers or quantities of the same kind, is greater than the sum of their squares by twice their product.*

For if x is one of the numbers or quantities, and a the other, then $x + a$ is the expression for their sum; and $x^2 + a^2$ is the sum of the squares of x and a, and twice the product of x and a is $2xa$; and if we add $2xa$ to $x^2 + a^2$, we get $x^2 + a^2 + 2xa = x^2 + 2xa + a^2 = (x + a)^2 =$ the square of $x + a$, as we have shown.

It is clear that the method of squaring a number which consists of more than one digit (arithmetically) is very similar to the method of squaring $x + a$.

Thus, to square 23, we have

$$\begin{array}{r}
23 \\
23 \\
\hline
69 \\
46 \\
\hline
529
\end{array}$$

= the product of 23 by 23, or 23^2; where it will be observed that in multiplying 3 by 3 we get $9 = 3^2$, and in multiplying the 2 by 3 we get 6, which is really 60, on account of the local value of 6 ; also in multiplying 3 by 2 of the multiplier we get 6, which is 60, on account of the local value of the 2; and multiplying the 2 of the multiplicand by the 2 of the multiplier we get $4 = 2^2$; or, on account of the local values of the 2 in the multiplicand and multiplier, the 4 is really 400.

Hence, the square of 23 is equal to the sum of 400, 60 × 2, and 9, which is $400 + 120 + 9 = 529$, which is the same as $x^2 + 2xa + a^2$ when we put 20 for x and 3 for a.

In like manner it may be shown, if we square any number, that the process of squaring is the same as to separate the number into any two parts, and to add together the squares of the parts and twice their product. Thus, if we take 7896, it may be separated into the parts $7000 + 896$, $7800 + 96$, or $7890 + 6$; and we shall have $7896^2 = 7000^2 + 2 \times 7000 \times 896 + 896^2 = 7800^2 + 2 \times 7800 \times 96 + 96^2 = 7890^2 + 2 \times 7890 \times 6 + 6^2 = 62346816$.

Ex. 10.—To develop $(x - a)^2 = (x - a)(x - a)$, or to find the square or second power of $x - a$.

Here, by multiplying $x - a$ by itself, we get

$$\begin{array}{lll}
x^{} & - a & \\
x^{} & - a & \\
\hline
x^2 & - xa & \\
 & - xa & + a^2 \\
\hline
x^2 & - 2xa & + a^2
\end{array}$$

$= (x - a)^2$ as required. Hence, if x and a represent any two numbers or quantities of the same kind, then $x - a$ will represent their difference, and $x^2 + a^2$ will express the sum of the squares of x and a, and $2xa$ represents twice the product of x and a; consequently, since $(x - a)^2 = x^2 - 2xa + a^2 = x^2 + a^2 - 2xa$, it follows, *that the square of the difference of any two numbers or quantities of the same kind, is less than the sum of their squares by twice their product.*

Thus, since $19 = 20 - 1$, we have $19^2 = 20^2 + 1^2 - 2 \times 20 = 401 - 40 = 361$; also, since $397 = 400 - 3$, we get $397^2 = 400^2 + 3^2 - 2 \times 400 \times 3 = 160009 - 2400 = 157609$.

Ex. 11.—To find the second power or square of any compound quantity, as $a + b - c + d - e +$ etc.

If we put the quantity in the form $a + (b - c + d - e +$, etc.,) we may square it as if it were a binomial, by regarding a as one of the terms, and $(b - c + d - e +$, etc.,) as the other term of the binomial.

Hence, by Ex. 9, we have $(a + b - c + d - e +$, etc.$)^2 = a^2 + 2a(b - c + d - e +$, etc.$) + (b - c + d - e +$, etc.$)^2 = a^2 + 2a(b - c + d - e +$, etc.$) + b^2 + 2b(- c + d - e +$, etc.$) + (- c + d - e +$, etc.$)^2 = a^2 + 2a(b - c + d - e +$, etc.$) + b^2 + 2b(- c + d - e +$, etc.$) + c^2 - 2c(d - e +$, etc.$) + (d - e +$, etc.$)^2 = a^2 + 2a(b - c + d - e +$, etc.$) + b^2 + 2b(- c + d - e +$ etc.$) + c^2 - 2c(d - e +$ etc.$) + a^2 + 2d(- e +$ etc.$) + (- e +$, etc.$)^2$, and so on; as is evident since $(b - c + d - e +$ etc.$)^2$ is developed as a binomial by putting $b - c + d - e +$ etc. $= b + (- c + d - e +$ etc.), and in the same way, by putting $- c + d - e +$ etc. $= - c + (d - e +$ etc.), we get the square of $- c + d - e +$ etc., by squaring the binomial $- c + (d - e +$, etc.) by Ex. 10; and so on.

Hence, *the square of any compound quantity is found by adding the squares of all the terms, and twice the product of each term by all the other terms connected by their proper signs; or, which is the same, we must add the squares of the terms and twice the product of every two of them without repetition; observing the rule of signs in forming the double products.*

For instance, to find the square of $5a^2 - 7b^2 + 2c^2$, we have $(5a^2)^2 = 25a^4$, $(- 7b^2)^2 = 49b^4$, $(2c^2)^2 = 4c^4$, and of course $(5a^2)^2 + (- 7b^2)^2 + (2c^2)^2 = 25a^4 + 49b^4 + 4c^4 =$ the sum of the squares of the terms of the given quantity. And to get the double products of every two of the terms, we have $2 \times 5a^2 \times - 7b^2 = - 70a^2b^2$, $2 \times 5a^2 \times 2c^2 = 20a^2c^2$, $2 \times - 7b^2 \times 2c^2 = - 28b^2c^2$; consequently, the sum of the double products is expressed by $- 70a^2b^2 + 20a^2c^2 - 28b^2c^2$, which, being added to the sum of the squares of the terms, gives $(5a^2 - 7b^2 + 2c^2)^2 = 25a^4 + 49b^4 + 4c^4 - 70a^2b^2 + 20a^2c^2 - 28b^2c^2$, as required.

It is easy to see that the common method of squaring a number which consists of more than one digit is essentially the same as to square a compound quantity whose terms are all positive, by the rule which we have given.

Thus, by squaring the number 7853, we get

$$\begin{array}{r} 7853 \\ 7853 \\ \hline 23559 \\ 39265 \\ 62824 \\ 54971 \\ \hline 61669609 \end{array}$$

$= 7853^2$; where it will be observed that when we multiply the 3 of the multiplicand by the 3 of the multiplier, we get $9 = 3^2$, and when we multiply the 5 of the multiplicand by the 5 of the multiplier, we get $25 = 5^2$, which is really 2500, on account of the local value of 5, which makes the 5 to be 50; in the same way, the 8 of the multiplicand multiplied by the 8 of the multiplier gives 800^2, and the 7 of the multiplicand multiplied by the 7 of the multiplier gives 7000^2; hence, the partial products in the above method of squaring 7853 contain 3^2, 50^2, 800^2, and 7000^2.

Again, the 8 of the multiplicand multiplied by 3 of the multiplier gives $800 \times 3 = 2400$, and the 3 of the multiplicand and multiplier being interchanged give $2 \times 800 \times 3$; and in the same way, any figure of the multiplicand and a different figure of the multiplier will give twice the product of the figures by the numbers that express their local values.

Hence, we get $7853^2 = 61669609 = 7000^2 + 800^2 + 50^2 + 3^2 + 2 \times 7000 \times 800 + 2 \times 7000 \times 50 + 2 \times 7000 \times 3 + 2 \times 800 \times 50 + 2 \times 800 \times 3 + 2 \times 50 \times 3$; agreeably to the rule.

Ex. 12.—To find the cube or third power of $x + a$, or, which is the same, to develop $(x + a)(x + a)(x + a) = (x + a)^3$.

Since $(x + a)^3 = (x + a)^2 (x + a) = (x^2 + 2xa + a^2)(x + a)$, we have to multiply $x^2 + 2xa + a^2$ by $(x + a)$. Performing the multiplication as required, we have

$$\begin{array}{llll} x^2 + 2xa & + a^2 & & \\ x + a & & & \\ \hline x^3 + 2x^2a & + xa^2 & & \\ & x^2a & + 2xa^2 & + a^3 \\ \hline x^3 + 3x^2a & + 3xa^2 & + a^3 & \end{array}$$

$= (x + a)^3$, as required.

Ex. 13.—To develop $(x-a)^3 = (x-a)^2(x-a) = (x^2 - 2xa + a^2)(x-a)$.

Here, as in Ex. 12, we have

$$\begin{array}{lllll}
x^2 & -\,2xa & +\,a^2 & & \\
x & -\,a & & & \\
\hline
x^3 & -\,2x^2a & +\,xa^2 & & \\
 & -\,x^2a & +\,2xa^2 & -\,a^3 & \\
\hline
x^3 & -\,3x^2a & +\,3xa^2 & -\,a^3 &
\end{array}$$

$= (x-a)^3$ as required.

Ex. 14.—To find the cube of any compound quantity, as $a + b + c - d + e - f +$, etc.

From what has been shown, we have $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; and since this is an identical equation, we may for b put $b + c$, and it becomes $(a+b+c)^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3 = a^3 + 3a^2(b+c) + 3a(b^2 + 2bc + c^2) + b^3 + 3b^2c + 3bc^2 + c^3 = a^3 + b^3 + 3a^2(b+c) + 3b^2(a+c) + 6abc + 3c^2(a+b) + c^3$; and if we put in this equation $c - d$ for c, we get $(a+b+c-d)^3 = a^3 + b^3 + 3a^2(b+c-d) + 3b^2(a+c-d) + 6ab(c-d) + 3(a+b)(c^2 - 2cd + d^2) + c^3 - 3c^2d + 3cd^2 - d^3 = a^3 + b^3 + c^3 + 3a^2(b+c-d) + 3b^2(a+c-d) + 3c^2(a+b-d) + 6ab(c-d) - 6cd(a+b) + 3d^2(a+b+c) - d^3$.

If, in the last equation, we put $-d + e$ for $-d$, we shall get the expansion of $(a+b+c-d+e)^3$, and so on to any extent; but it is not necessary to continue the process any further, since we easily see, from what has been done, that we can find the cube of any compound quantity by the following rule.

Observing the rule of signs; the cube of any compound quantity equals the sum of the cubes of all the terms, three times the square of each term multiplied by all the others taken with their proper signs, and six times the product of every three different terms, taken with their proper signs.

Thus, to get the cube of $2a - 3b + 4c$, we have $(2a)^3 = 8a^3$, $(-3b)^3 = -27b^3$, $(4c)^3 = 64c^3$, $3 \times (2a)^2 \times (-3b + 4c) = -36a^2b + 48a^2c$, $3(-3b)^2(2a + 4c) = 54ab^2 + 108b^2c$, $3(4c)^2(2a - 3b) = 96ac^2 - 144bc^2$, $6 \times (2a) \times (-3b) \times (4c) = -144abc$, and adding these results, we have $(2a - 3b + 4c)^3 = 8a^3 - 27b^3 + 64c^3 - 36a^2b + 48a^2c + 54ab^2 + 108b^2c + 96ac^2 - 144bc^2 - 144abc$, as required.

It is easy to see that the cube of a *number* may be found in the same way as that of a compound algebraic quantity whose terms are all positive. Thus, $327^3 = 300^3 + 20^3 + 7^3 + 3 \times 300^2 \times 27 + 3 \times 20^2 \times 307 + 3 \times 7^2 \times 320 + 6 \times 300 \times 20 \times 7 = 34965783$.

Ex. 15.—Having the coefficients in any integral power of a binomial, to find the coefficients in the next successive integral power, by detaching the coefficients.

Let $a + b$ denote the binomial, then detaching the coefficients we have $1 + 1$, and to find the coefficients in the square of $a + b$, we have

$$\begin{array}{l}1+1\\1+1\\\hline 1+1\\\phantom{1+{}}1+1\\\hline 1+2+1\end{array}$$

= the coëfficients in the square.

To get the coefficients in the cube of $a + b$, we have

$$\begin{array}{l}1+2+1\\1+1\\\hline 1+2+1\\\phantom{1+{}}1+2+1\\\hline 1+3+3+1\end{array}$$

= the coefficients in the cube.

It is hence easily seen that if we set down the sum of the coefficients in any power in order, and then set down the same sum in the same order under the coefficients already set down, removing each term one place to the right of its equal in the first sum, then if we add the coefficients thus placed, we shall have the sum of the coefficients in the next successive integral power.

Thus, $1 + 3 + 3 + 1$ being the coefficients in the cube, we get the coefficients in the fourth power by writing

$$\begin{array}{ll}1+3+3+1\phantom{{}+1} & = \text{first sum.}\\\phantom{1+{}}1+3+3+1 & = \text{second sum.}\\\hline 1+4+6+4+1 & \end{array}$$

= the sum of the coefficients in the fourth power; and in like manner, to get the sum of the coefficients in the fifth power, we write

$$\begin{array}{r}
1 + 4 + \ 6 + \ 4 + 1 \qquad\qquad \\
1 + \ 4 + \ 6 + 4 + 1 \\
\hline
1 + 5 + 10 + 10 + 5 + 1
\end{array}$$

= the sum of the coefficients in the fifth power; in the same way we can find the sum of the coefficients in the sixth, seventh, etc., powers.

For the purpose of raising a binomial to any integral power, which does not exceed the tenth, we shall give the following Table of corresponding coefficients.

TABLE OF THE COEFFICIENTS IN THE POSITIVE INTEGRAL POWERS OF A BINOMIAL, UP TO THE TENTH POWER.

$(1 + 1)^1 = 1 + 1.$

$(1 + 1)^2 = 1 + 2 + 1.$

$(1 + 1)^3 = 1 + 3 + 3 + 1.$

$(1 + 1)^4 = 1 + 4 + 6 + 4 + 1.$

$(1 + 1)^5 = 1 + 5 + 10 + 10 + 5 + 1.$

$(1 + 1)^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1.$

$(1 + 1)^7 = 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1.$

$(1 + 1)^8 = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1.$

$(1 + 1)^9 = 1 + 9 + 36 + 84 + 126 + 126 + 84 + 36 + 9 + 1.$

$(1 + 1)^{10} = 1 + 10 + 45 + 120 + 210 + 252 + 210 + 120\ 45 + 10 + 1.$

The exponents of $(1 + 1)$ in these equations indicate the powers to which the corresponding coefficients belong.

We shall give several examples to show the use of the Table.

1. To find the fourth power of $a + b$.

By the Table the coefficients in the fourth power are $1 + 4 + 6 + 4 + 1$; consequently, supplying the corresponding powers of the letters, we get $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = (a + b)^4$, as required.

2. To find the fourth power of $2x^2 + 3y^2$.

Here we have the same coefficients as before, and we must put $2x^2$ for a, and $3y^2$ for b; that is, we must use $2x^2$ and $3y^2$ in the power in the same way as we use the single letters a and b in the binomial $a + b$. Hence we get

$(2x^2 + 3y^2)^4 = (2x^2)^4 + 4(2x^2)^3 \times (3y^2) + 6(2x^2)^2 \times (3y^2)^2 + 4(2x^2) \times (3y^2)^3 + (3y^2)^4 = 16x^8 + 96x^6y^2 + 216x^4y^4 + 216x^2y^6 + 81y^4$, as required.

3. To find the seventh power of $ab - 2cd$.

By the Table, the coefficients in the seventh power are $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1$; and using ab and $-2cd$ as simple letters in a binomial, we get $(ab - 2cd)^7 = (ab)^7 + 7(ab)^6 \times (-2cd) + 21(ab)^5 \times (-2cd)^2 + 35(ab)^4 \times (-2cd)^3 + 35(ab)^3 \times (-2cd)^4 + 21(ab)^2 \times (-2cd)^5 + 7(ab) \times (-2cd)^6 + (-2cd)^7 = a^7b^7 - 14a^6b^6cd + 84a^5b^5c^2d^2 - 280a^4b^4c^3d^3 + 560a^3b^3c^4d^4 - 672a^2b^2c^5d^5 + 448abc^6d^6 - 128c^7d^7$, as required.

Ex. 16.—To find the n^{th} power of the binomial $a + b$; supposing n to denote any positive whole number.

By the notation of powers, we shall have the n^{th} power of $a + b$ expressed by $(a + b)^n = (a + b) . (a + b) . (a + b) \ldots$ to n factors (A).

If we multiply the a of the first factor of the right member of (A) by the a of the second factor, and the product by the a of the third factor, and so on to all the factors, we shall get $a . a . a \ldots$ to n factors $= a^n$ (1), by the notation of powers, and a^n will be the first term of the sought power, supposing it to be arranged according to the descending powers of a.

If we change n in (1) into $n - 1$, we get a^{n-1} for the term in the product of any $(n - 1)$ factors of (A) which is independent of b; and multiplying this term by the b of the remaining factor of (A), we get $a^{n-1}b$, which will be repeated n times, when we take every one of the $(n - 1)$ factors of (A) (which are not all the same in any two cases), and proceed as before; that is, we get $a^{n-1}b + a^{n-1}b + a^{n-1}b +$ etc., to n terms $= na^{n-1}b$ (2) for the second term of the required power.

Again, if we change n in (2) into $n - 1$, we get $a^{n-2}b + a^{n-2}b + a^{n-2}b +$ etc., to $(n - 1)$ terms $= (n - 1)a^{n-2}b$ for the term of the product of any $(n - 1)$ factors of (A) which involves the first power of b, and multiplying this term by the b of the remaining factor of (A), we get $a^{n-2}b^2 + a^{n-2}b^2 + a^{n-2}b^2 +$ etc., to $(n - 1)$ terms $= (n - 1)a^{n-2}b^2$; which will be repeated n times, when we take every one of the $(n - 1)$ factors of (A), that are not all the same in any two cases, and proceed as before; that is, we shall get $a^{n-2}b^2 + a^{n-2}b^2 +$

$a^{n-2}b^2$ + etc., to $n(n-1)$ terms $= n(n-1)a^{n-2}b^2$, which has twice as many terms as it ought to have in the sought power; for (by the process) one of the terms has arisen from multiplying a^{n-2} by the b of the first factor, and the product by the b of the second factor; and another term has resulted from the multiplication of a^{n-2} by the b of the second factor, and the product by the b of the first factor; consequently, the product of a^{n-2} by the b of the first and second factors has been taken twice, when it ought to have been taken but once; and in the same way the b in any two factors of (A) may be shown to give two terms instead of one, which we ought to have; consequently we shall have $a^{n-2}b^2 + a^{n-2}b^2 + a^{n-2}b^2$ + etc., to $\frac{n(n-1)}{1\,.\,2}$ terms $= \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2$ (3), for another term of the sought power.

If we put $n-1$ for n in (3), and multiply by b, we get $a^{n-3}b^3 + a^{n-3}b^3 +$ etc., to $\frac{(n-1)(n-2)}{1 \quad 2}$ terms $= \frac{(n-1)}{1}\frac{(n-2)}{2}a^{n-3}b^3$ for the term of the product of any $(n-1)$ factors of (A), which involves b^2, and that is multiplied by the b of the remaining factor; and proceeding in this way with every $(n-1)$ factors of (A), (the factors not being all the same in any two instances), the above equation will be repeated n times, or we shall have $a^{n-3}b^3 + a^{n-3}b^3$ + etc., to $\frac{n(n-1)(n-2)}{1\,.\,2}$ terms $= \frac{n(n-1)(n-2)}{1\,.\,2}a^{n-3}b^3$, which has three times as many terms as it ought to have in the required power; for the b in any three factors of (A) ought evidently to give but one term, whereas, by the process we have used, each b in the three factors gives a term; hence we deduce $a^{n-3}b^3 + a^{n-3}b^3 +$ etc., to $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$ terms $= \frac{n(n-1)}{1\,.\,2}\frac{(n-2)}{.\,3}a^{n-3}b^3$ (4) for another term of the sought power.

If for n we put $n-1$ in (4), and proceed in like manner, we shall find $\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}b^4$ for the fifth term of the required power, and so on; and it is evident if $m+1$ represents the number of any term, that we shall have

$\frac{n(n-1)(n-2)(n-3)\ldots.(n-m+1)}{1\,.\,2\,.\,3\,.\,4\,\ldots\ldots\,m}a^{n-m}b^m$ for the expression of the term, *which is called the general term of the expansion of* $(a+b)^n$.

Hence, collecting the terms, we get $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,.\,\frac{(n-3)}{4}a^{n-4}b^4 + \text{etc.}$ (B), for the required power, whose law of continuation is manifest; and it is easy to see that the number of terms in the right member of (B) is $n+1$, and that the last term is b^n.

Since a and b are clearly arbitrary, we may put $-b$ for b, and (B) becomes $(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \text{etc.}$ (B′), observing the rule of signs.

If in (B) we change b into $\pm b$, we shall have $(a \pm b)^n = a^n \pm na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \text{etc.}$ (C), which comprehends (B) and (B′) in a single formula, observing that for $\pm$ (*which is called the ambiguous sign*) we must use $+$ when b is to be taken as a positive quantity, and that $-$ must be used for $\pm$ when b is to be negative, or to have the sign $-$.

It is clear that (C) is an identical equation, since its right member is the expansion of its left or first member; in other terms, its right member is the same as $(a \pm b)^n$ under a different form.

Hence, although the equation has been rigorously established, only on the supposition that n is a positive whole number, yet its right member ought evidently to be considered as the development of its first member, or of $(a \pm b)^n$, whatever may be the value of *n;* that is, whether n is integral or fractional, rational or irrational, positive or negative, etc.

Regarding b as an increment of a, and $-b$ as a decrement of a, we may clearly derive (C) from the identical equation $a^n = a^n$, in the following manner.

By adding $\pm b$ to a, the first member of the equation becomes $(a \pm b)^n$; and to get the term $\pm na^{n-1}b$ from a^n, we subtract unity from its exponent, which reduces it to a^{n-1}; then we multiply together the exponent n of the power a^n, a^{n-1}, and $\pm b$, which gives $\pm na^{n-1}b$, as required.

To get the term $\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2$, we resume the term $\pm na^{n-1}b$, then, subtracting 1 from $n-1$, the exponent of a in the term, it is reduced to $\pm na^{n-2}b$; and multiplying together the index $n-1$ of a, in the term, $\pm na^{n-2}b$, and $\pm b$, and dividing by the index of b in the product, we get $\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2$, as required. In a similar way we derive the term $\pm\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3$ from the preceding term $\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2$; by diminishing the exponent of a by 1, and then multiplying together the exponent $n-2$ of a, $\frac{n(n-1)}{1\,.\,2}a^{n-3}b^2$ and $\pm b$, and dividing by the exponent of b in the product, which gives $\pm\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3$, as required. In like manner we may derive $\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}b^4$ from the last or preceding term $\pm\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3$; and so on, to any extent. Hence it is easily seen that the identical equation $a^n = a^n$, by adding $\pm b$ to a, will be changed (by the process we have used) into (C).

It may be observed that (C), when n is taken in its most general sense, is called the Binomial Formula or Theorem of Sir Isaac Newton, because it is supposed to have been discovered by him. And it may be added that the method which we have given of deriving any term of the right member of (C) from the next preceding term is substantially the same as the rule given by Newton for the development of a binomial.

We shall now give some simple examples to show the use of (C).

1. To find the third power of $7ab - 3c$.

In this question we must put $n = 3$, and use $-$ for $\pm$ in (C); which, being done, (C) becomes $(a-b)^3 = a^3 - 3a^{3-1}b + \frac{3(3-1)}{1\,.\,2}a^{3-2}b^2 - \frac{3(3-1)(3-2)}{1\,.\,2\,.\,3}a^{3-3}b^3$; or, since $3-1=2$, $\frac{3(3-1)}{1\,.\,2} = \frac{3\,.\,2}{2} = 3$, $\frac{3(3-1)(3-2)}{1\,.\,2\,.\,3} = \frac{3\,.\,2\,.\,1}{1\,.\,2\,.\,3} = 1$, and $a^{3-3} = a^0 = 1$, we get $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

To find the third power of $7ab - 3c$, we must use $7ab$ and $3c$ for a and b in the preceding equation, and we get $(7ab - 3c)^3 = (7ab)^3 - 3(7ab)^2 \times 3c + 3(7ab) \times (3c)^2 - (3c)^3 = 343a^3b^3 - 441a^2b^2c + 189abc^2 - 27c^3$, as required.

2. To find the sixth power of $1 + 3x$.

In this question we must put $n = 6$, and use $+$ for $\pm$ in (C), and it reduces to $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; hence, using 1 for a, and $3x$ for b, in this equation, we get $(1 + 3x)^6 = 1 + 6 \times 3x + 15 \times (3x)^2 + 20 \times (3x)^3 + 15 \times (3x)^4 + 6 \times (3x)^5 + (3x)^6 = 1 + 18x + 135x^2 + 540x^3 + 1215x^4 + 1458x^5 + 729x^6$, as required. If we put $x = 1$, then $1 + 3x = 4$, and we have $4^6 = 1 + 18 + 135 + 540 + 1215 + 1458 + 729 = 4096$, as it ought to do.

3. To find the fourth power of $\frac{x}{3} - \frac{y^2}{4}$.

In this question we must put $n = 4$, and use $-$ for $\pm$ in (C), and it changes to $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$; and putting $\frac{x}{3}$ for a, $\frac{y^2}{4}$ for b in this equation, we get

$$\left(\frac{x}{3} - \frac{y^2}{4}\right)^4 = \left(\frac{x}{3}\right)^4 - 4\left(\frac{x}{3}\right)^3\left(\frac{y^2}{4}\right) + 6\left(\frac{x}{3}\right)^2\left(\frac{y^2}{4}\right)^2 - 4\left(\frac{x}{3}\right)\left(\frac{y^2}{4}\right)^3 + \left(\frac{y^2}{4}\right)^4$$

$= \frac{x^4}{81} - \frac{x^3y^2}{27} + \frac{x^2y^4}{24} - \frac{xy^6}{48} + \frac{y^8}{256}$, as required. If we put $x = 1$, $y = 1$, our result reduces to $\left(\frac{1}{3} - \frac{1}{4}\right)^4 = \left(\frac{1}{12}\right)^4 = \frac{1}{81} - \frac{1}{27} + \frac{1}{24} - \frac{1}{48} + \frac{1}{256} = \frac{1}{20736}$, as it ought to do.

We shall conclude Multiplication by giving a few examples for practice.

EXAMPLES.

1. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

2. Multiply $a^4 - b^4$ by $a^2 + b^2$.

3. Multiply $x^2 - 2xa + a^2$ by $x^2 + 2xa + a^2$.

4. Multiply $9x^4 - 7x^3 + 3x - 5$ by $2x^2 - 5x + 3$, by detaching the coefficients.

5. Multiply $15a^4 - 13a^3x + 19a^2x^2 - 2x^4$ by $5a^3 + 17a^2x - 15x^3$, by detaching the coefficients.

6. Multiply $4a^4 + 3a^3x + 5ax^3 + 2x^4$ by $2a^3 + 7ax^2 + x^3$, by detaching the coefficients.

7. Multiply $a^6 + a^2$ by $a^3 - a$, by detaching the coefficients.

8. Multiply $3x - 5y^2$ by $8x^3 - 13y^4$, and the product by $15x^3 - 2y^3$.

9. Develop $(9x^2 - 2y^3)^3$, or find the third power of $9x^2 - 2y^3$.

10. Find the ninth power of $x - y$, or develop $(x - y)^9$.

11. Multiply $3ax^3 + 5bx^2 + 2cx + d$ by $4x^2 + 3x + 5$.

12. Multiply $7a^2x^4 + 9ax^5 + 11x^6$ by $7a^2x^4 - 9ax^5 + 11x^6$.

13. Multiply $a^3 - a^2x + ax^2 - x^3$ by $a + x$, and the product by $a^4 + x^4$.

14. Multiply $1 + (1 + a)x + (1 + a + b)x^2 + (1 + a + b + c)x^3$ by $1 - x$.

15. Multiply $x^4 - 37x - 22$ by $x^2 - 3$, and the product by $x^2 + 7x$, and the last product by $x - 6$.

16. Develop $(7 - 3x)^2 \,.\, (4 - 5x)^3 \,.\, (2 - 9x)^4$.

17. Multiply $4x^2 - x + 4$ by $x + 1$, and the product by $x + 2$, and the last product by $x^2 - 3$.

18. Find the product of $4x^3 + 5x^2a - 6xa^2 - 7a^3$ and $9x^2 - 11xa + 12a^2$.

19. Develop $(17a^3b^2c - 32pq)^2$ and $(2x^2 - 5yz)^7$.

20. Multiply $ax^{2n} + bx^ny^m - cy^{2m}$ by $x^n + dy^m$.

21. Multiply $2 - 3x + 3x^2 - 3x^3 + 3x^4 -$, etc., by $1 + x$.

22. Multiply $a + b + c$ by $a - b - c$, and develop $\left(1 - \frac{2}{x^2}\right)^8$.

23. Multiply $a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \ldots\ldots + b^{n-1}$ by $a - b$.

24. Multiply $\frac{3}{2}a^3 - \frac{6^5}{5} - q$ by $\frac{2}{3}a + 5b - 6c + 9$, and develop $(x^m + 2v^nwq)^4$.

SECTION V.

DIVISION.

(1.) Division may be considered as the reverse of Multiplication; that is, as having given a product and one of its factors, to find the remaining factor.

(2.) The product or quantity to be divided is called the *dividend*, and the factor by which it is to be divided is called the *divisor*. The remaining factor is called the *quotient*, because it denotes how often the dividend contains the divisor. If the divisor is contained in the dividend a certain number of times, and if there is any portion of the dividend left which does not exactly contain the divisor, the part left is called the *remainder;* if there is no remainder, the division is said to be *exact.*

(3.) It may be proper, before entering upon the rules of Division, to premise the following remarks.

By definition 8, each of the expressions, $a \div b$, $a : b$, $\frac{a}{b}$, is used to signify that a is to be divided by b; also the same thing is sometimes denoted by $b\,|\,a$. French authors frequently put the divisor to the right of the dividend, and having drawn a right line between the dividend and divisor, and another right line below the divisor, they put the quotient under the last line; thus, if a is the dividend, b the divisor, and q the quotient, then $a\Big|\frac{b}{q}$ signifies that if we divide a by b, we shall have q for the quotient.

The form $\frac{a}{b}$ is called a fractional expression, a being called the numerator and b the denominator of the fraction; so that what is called the dividend in Division is the numerator of the fraction, and the divisor is called the denominator of the fraction; also a and b are sometimes called the terms of the fraction.

It is manifest that the divisor and dividend must both be quantities of the same kind, abstract numbers; or that the divisor must be an abstract number.

When a and b are quantities of the same kind, or abstract numbers, the fraction $\frac{a}{b}$ is often called the geometrical ratio of a to b, or, more properly, it is said to *represent* the geometrical ratio of a to b.

Again, if a, b, c denote any three integral quantities, it is manifest, since ac is c times a, or c times as great as a, that $\frac{ac}{b}$ contains $\frac{a}{b}$, c times; or that $\frac{a}{b} \times c = \frac{ac}{b} = \frac{ca}{b}$: so that *the fraction* $\frac{a}{b}$ *is multiplied by the integer* c *by multiplying its numerator by* c. Also, since bc is c times as great as b, it is plain that $\frac{a}{b}$ is c times as great as $\frac{a}{bc}$, or that $\frac{a}{b} \div c = \frac{a}{bc} = \frac{a}{cb}$; hence, *the fraction* $\frac{a}{b}$ *is divided by the integer* c *by multiplying its denominator by* c.

Since $\frac{ac}{b}$ contains $\frac{a}{b}$, c times, and $\frac{ac}{b}$ contains $\frac{ac}{bc}$, c times, it follows (Ax. IV.) that $\frac{a}{b} = \frac{ac}{bc}$; so that *the value of the fraction* $\frac{a}{b}$ *is not changed by multiplying its numerator and denominator by* c.

Consequently, any integral factor, which is common to the numerator and denominator of any fraction, may be erased or stricken out of each without affecting the value of the fraction. Hence, also, any integral factor, which is common to the dividend and divisor in Division, may be stricken out of each without altering the quotient.

Let a, b, c, d denote any four integral quantities; then it is evident, since $\frac{ac}{b} = \frac{a}{b} \times c$, and since c is d times as great as $\frac{c}{d}$, that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{b} \div d = \frac{ac}{bd} = \frac{ca}{db}$. *Consequently, fractions are multiplied by each other, by taking the product of their numerators for a new numerator, and the product of their denominators for a new denominator.*

Also, since c is d times as great as $\frac{c}{d}$, it is evident that $\frac{a}{bc}$ is d times less than $\frac{a}{b} \div \frac{c}{d}$, so that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{bc} \times d = \frac{ad}{bc}$. Hence, *one fraction is divided by another by multiplying the numerator of the fraction to be divided by the denominator of the divisor, and the denominator of the fraction to be divided by the numerator of the divisor; or, which is the same thing, invert the divisor and proceed as in Multiplication.*

From what has been proved, we have $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, also $\frac{c}{d} \times \frac{a}{b} = \frac{ca}{db} = \frac{ac}{bd}$; $\therefore$ Ax. IV, $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$; and if $b = 1$, we get $a \times \frac{c}{d} = \frac{c}{d} \times a$.

Consequently, the product of fractions whose numerators and denominators are integers, is independent of the order in which the fractions are written or taken in the multiplication; and the same is true of the product of an integer and a fraction; and indeed of the product of any number of integers by any number of fractions, as is evident from what has been done in Multiplication.

(4.) Again, since in Division the dividend is supposed to equal the product of the divisor and quotient, if the dividend and divisor are both positive, the quotient must clearly be positive; but if the divisor and dividend are both negative, the quotient must be positive, since the divisor and quotient must have unlike signs in order to give the sign − to the dividend, which is their product, as appears from the rule of signs in Multiplication.

If the divisor has the sign +, and the dividend the sign −, it is manifest that the quotient must have the sign −, since the divisor and quotient must have unlike signs; also, if the divisor has the sign −, and the dividend the sign +, the quotient must have the sign −, since, by the rule of signs in Multiplication, the divisor and quotient must have like signs, in order that the dividend, which is their product, may have the sign +; consequently, since the divisor has the sign −, the quotient must also have the sign −.

Hence we deduce the following rule for the sign of the quotient in Division.

RULE.

If the divisor and dividend have like signs, the quotient must have the sign + ; *but if the divisor and dividend have unlike signs, the quotient must have the sign* —. *Or we may express the rule more concisely, by saying that like signs give* + *and unlike signs give* —.

(5.) Again, the same or different powers of any quantity are multiplied together by adding their exponents. Hence, since Division is the reverse of Multiplication, we must change the sign of the exponent of the quantity in the divisor (or conceive the sign to be changed), and then proceed as in Multiplication.

Thus, if we have to multiply a by a we get $a \times a = a^2$, and to divide a by a we have $a \times a^{-1} = a^{1-1} = a^0$, which evidently $= 1$; for $a \div a = \frac{a}{a} = 1$, since any quantity contains itself once; consequently, any quantity which has 0 for an index, is an expression for 1. The expression shows that it has been obtained by dividing the quantity by itself.

Again, to divide $6a^2b^3c$ by $3a^6b^5c^4$, we have $6a^2b^3c \times 3^{-1}a^{-6}b^{-5}c^{-4} = 2 \times 3^0a^{-4}b^{-2}c^{-3} = \frac{6a^2b^3c}{3a^6b^5c^4} = \frac{2}{a^4b^2c^3}$, so that $3^0 = 1$, $a^{-4} = \frac{1}{a^4}$, $b^{-2} = \frac{1}{b^2}$, $c^{-3} = \frac{1}{c^3}$; consequently, any quantity with a negative exponent is an expression for 1, divided by the quantity with a positive exponent; which is in conformity to the notation adopted in (11) of the definitions.

Hence, any quantity can be carried from the denominator of a fraction into the numerator, by changing the sign of its exponent, so that any fractional quantity may be changed to an integral form; thus, for $\frac{a}{b^3c^2}$ we may write $ab^{-3}c^{-2}$.

Reciprocally, any integral form may be reduced to a fractional form; thus, a^2b^3 may be expressed by $\frac{1}{a^{-2}b^{-3}}$, so that 1 is the numerator of the fraction, and the product of the quantities, with the signs of their exponents changed, is the denominator.

(6.) The rule of Division may be separated into three cases.

CASE I.

When the dividend and divisor are monomials or simple quantities.

RULE.

Divide the coefficient of the dividend by that of the divisor, and to the quotient annex the quotient obtained by dividing the literal part of the dividend by that of the divisor; and observe that like signs give + and unlike signs − for the sign of the quotient. It is often most convenient to place the divisor under the dividend in the form of a fraction, and then to erase all the factors that are common to both terms of the fraction, and the result will be the sought quotient; observing the rule of signs. But perhaps the most expeditious method, in most cases, is to change the signs of the exponents of the factors of the divisor, and then to proceed as in Multiplication.

Thus, to divide $45a^3b^5x^3$ by $9a^2b^3x$; according to the first part of the rule, 45 divided by 9 gives 5 for the quotient, and a^3 divided by a^2 gives a for the quotient, $b^5 \div b^3 = b^2$, $x^3 \div x = x^2$, $\therefore$ $5ab^2x^2$ is the sought quotient; and it is positive, because the dividend and divisor are both positive.

The same example, when done by the last part of the rule, gives $45a^3b^5x^3 \div 9a^2b^3x = 9 \times 5a^3b^5x^3 \times 9^{-1}a^{-2}b^{-3}x^{-1} = 5 \times 9^0ab^2x^2 = 5ab^2x^2$, since $9^0 = 1$.

As another example, we shall divide $29abc$ by $-42x^2yz^3$, according to the second part of the rule. Here we have $29abc \div -42x^2yz^3 = -\dfrac{29abc}{42x^2yz^3}$, the sign being −, because the dividend and divisor have unlike signs; and it is plain that the fraction can not be reduced to lower terms, since it has not explicitly any factors that are common to its numerator and denominator.

For another example, we shall divide $\frac{3}{5}a^2x^{\frac{1}{2}}$ by $\frac{4}{3}a^{\frac{2}{3}}x^{-\frac{3}{2}}$.

Here we have $\frac{3}{5}a^2x^{\frac{1}{2}} \div \frac{4}{3}a^{\frac{2}{3}}x^{-\frac{3}{2}} = \frac{3}{5}a^2x^{\frac{1}{2}} \times \frac{3}{4}a^{-\frac{2}{3}}x^{\frac{3}{2}}$, by inverting the fraction $\frac{4}{3}$ (according to what has been proved to be allowable), and for the letters proceeding according to the last part of the rule; by multiplying the coefficients, and adding the exponents of the same letter, we get the quotient equal to $\frac{9}{20}a^{2-\frac{2}{3}}x^{\frac{1}{2}+\frac{3}{2}}$, or since $2-\frac{2}{3}=\frac{6}{3}-\frac{2}{3}=\frac{4}{3}$, $\frac{1}{2}+\frac{3}{2}=\frac{4}{2}=2$, the sought quotient is plainly equal to $\frac{9}{20}a^{\frac{4}{3}}x^2$.

EXAMPLES.

1. Divide $-48a^6x^5y^7$ by $16a^8x^{10}y^3$.
2. Divide $-120b^{\frac{4}{5}}c^{\frac{8}{5}}$ by $-40b^{\frac{4}{3}}c^{-\frac{8}{5}}$.
3. Divide $-52a^3b^6c^{14}$ by $153\ ab^3c^{-3}$.
4. Divide $60xyzp^5$ by $-72x^3y^2z^3p^7$.
5. Divide $90x^{\frac{1}{2}}y^{\frac{3}{2}}z^{\frac{5}{2}}$ by $42x^{-\frac{1}{2}}y^{-\frac{3}{2}}z^{-\frac{5}{2}}$.

CASE II.

When the divisor is a monomial, and the dividend a compound or polynomial quantity

RULE.

Divide each term of the dividend by the previous rule for monomials, then the sum of the quotients, when added according to their proper signs, will be the complete quotient required.

Thus, to divide $6x^4-8x^2y^2$ by $2x^2$, we have $6x^4\div 2x^2=3x^2$, $-8x^2y^2\div 2x^2=-4y^2$, consequently there results $3x^2-4y^2$ for the sought quotient.

Also, to divide $48a^2x^3y^4-60a^4x^3y^5-144a^7x^5y^3+288a^3x^5$ by $-12a^2x^3$, we proceed as follows, viz.,

$$\frac{48a^2x^3y^4-60a^4x^3y^5-144a^7x^5y^3+288a^3x^5}{-12a^2x^3}=-\frac{48a^2x^3y^4}{12a^2x^3}+\frac{60a^4x^3y^5}{12a^2x^3}+\frac{144a^7x^5y^3}{12a^2x^3}-\frac{288a^3x^5}{12a^2x^3}=-4y^4+5a^2y^5+12a^5x^2y^3-24ax^2,$$

as required.

Again, to divide $40abc^2-8abg+12y^2$ by $8ab$, we have

$$\frac{40abc^2-8abg+12y^2}{8ab}=5c^2-g+\frac{3y^2}{2ab},$$

as required.

EXAMPLES.

1. Divide $14x^2yz - 49x^3y^2z^3$ by $-7x^2yz$, or execute what is expressed by $\frac{14x^2yz - 49x^3y^2z^3}{-7x^2yz}$.

2. Execute the expression $\frac{121a^9b^{10} - 132a^5b^7y^9 + 11a^4b^8c}{11a^5b^7}$.

3. Execute $\frac{4^8z^5abg - 4^4z^6a^2b^2g^2 - 4^6z^9a^3b^8g^3}{-4^3z^5ab}$.

4. Execute $\frac{-6m^3n^5p^7 - 24m^2g^3r^2z^2}{-3m}$.

CASE III.

When the divisor and dividend are compound quantities.

RULE I.

1. *Place the divisor at the left or right of the dividend, and draw a right or curved line between them, as in Arithmetic, being careful to arrange the divisor and dividend the same way, either according to the ascending or descending powers of a common letter.*

2. *By the rule for the division of monomials, divide the first or left-hand term of the dividend by the first or left-hand term of the divisor, and the result will be the first term of the quotient, placing the quotient to the right of the dividend, with a right or curved line between them, if the divisor is to the left of the dividend; but if the divisor is to the right of the dividend, the quotient may be put below the divisor, with a right-line between them.*

3. *Multiply the divisor by the first term of the quotient, and subtract the terms of the product from the corresponding terms of the dividend.*

Then to the remainder bring down as many of the next successive terms of the dividend as may be necessary to form a new dividend; divide the first term of the new dividend by the first term of the divisor, and the result will be the second term of the quotient; multiply the divisor by the second term of the quotient, and subtract the terms of the product from the corresponding terms of the first new dividend.

4. *Then to the remainder bring down as many of the remaining next successive terms of the given dividend as may be necessary to form a second new dividend.*

5. *From the second new dividend we proceed in the same way as before, to get the third term of the quotient, and so on, until all the terms of the given dividend have been brought down. If the last remainder is* 0, *the division is exact; but if the remainder is different from* 0, *the division is not exact, and we signify that the remainder is to be divided by the divisor, by writing the divisor under the remainder, with a right-line between them, after the manner of expressing a fraction; so that when the division is inexact, the quotient will generally be of a mixed form, being partly integral and partly fractional.*

The reason of this rule is evident from the consideration that it is the reverse of the rule of Multiplication. And it is easy to see from what has been shown in Multiplication, if the first term of the arranged dividend can not be exactly divided by the first term of the arranged divisor, or if the last term of the arranged dividend can not be exactly divided by the last term of the arranged divisor, that the dividend can not be exactly divided by the divisor; observing that the divisor and dividend are supposed to be of integral forms.

We will now illustrate our rule by a few examples.

Ex. 1.—To divide $7x - 9 + 5x^3 - 3x^2$ by $x - 1$.

Arranging the dividend and divisor according to the descending powers of x, we get (by putting the divisor to the left of the dividend) the following process.

$$
\begin{array}{l}
\underline{x-1}\,|\,5x^3-3x^2+7x-9\,|\,\underline{5x^2+2x+9} \\
\qquad\quad \underline{5x^3-5x^2\qquad\qquad} \\
\qquad\qquad\quad 2x^2+7x = \text{the 1st new dividend.} \\
\qquad\qquad\quad \underline{2x^2-2x\qquad} \\
\qquad\qquad\qquad\quad 9x-9 = \text{the 2d new dividend.} \\
\qquad\qquad\qquad\quad \underline{9x-9\qquad} \\
\qquad\qquad\qquad\quad 0+0=0=\text{the last remainder.}
\end{array}
$$

Explanation.—Having arranged the terms of the dividend and divisor, we divide the first term, $5x^3$, of the dividend by the first term, x, of the divisor, and get $5x^2$ for the result, which we put to the right of the dividend (with a curved-line between it and the dividend) for the first term of the quotient. Multiplying the divisor by $5x^2$, we get $5x^3 - 5x^2$ for the product, which we put under $5x^3 - 3x^2$, the corresponding terms of the dividend, and then subtracting $5x^3 - 5x^2$ from $5x^3 - 3x^2$, we get $2x^2$ for the remainder, to which, bringing down the next successive term, $+ 7x$, of the given dividend, we get $2x^2 + 7x$ for the first new dividend.

We now divide $2x^2$, the first term of the first new dividend, by the first term, x, of the divisor, and get $+ 2x$ for the result, which we put for the second term of the quotient. Multiplying the divisor by the second term, $2x$, of the quotient, we get $2x^2 - 2x$, which, being written under the first new dividend and subtracted from it, gives $9x$ for the remainder, to which, bringing down the remaining term, $- 9$, of the given dividend, we get $9x - 9$ for the second new dividend.

Dividing $9x$, the first term of the second new dividend, by x, the first term of the divisor, we get 9 for the result, which is the third term of the quotient. Multiplying the divisor by 9, we get $9x - 9$, which, being put under the second new dividend and subtracted, gives 0 for the remainder; consequently, since all the terms of the given dividend have been brought down, and since the last remainder equals 0, the division is exact.

We shall now put the divisor to the right of the dividend, and divide the last term of the dividend by the last term of the divisor, and so on for the successive dividends; that is, we shall proceed with the division from right to left, as in the following scheme:

$$\begin{array}{r|l}
5x^3 - 3x^2 + 7x - 9 & x - 1 = \text{the divisor.} \\ \cline{2-2}
9x - 9 & 5x^2 + 2x + 9 = \text{the quotient.} \\ \cline{1-1}
\text{1st new div.} = -3x^2 - 2x \quad\quad & \\
+2x^2 - 2x \quad\quad & \\ \cline{1-1}
\end{array}$$

$$\begin{array}{l}
\text{2d do.} = 5x^3 - 5x^2 \\
\phantom{\text{2d do.} = {}}\underline{5x^3 - 5x^2} \\
\phantom{\text{2d do.} = {}}0 + 0 = \text{naught} = \text{the last remainder.}
\end{array}$$

We see that the results are the same as by the preceding method; we also see that it would have been more convenient to have arranged the dividend and divisor according to the ascending powers of x, and to have proceeded, as in the first method, from left to right in the division.

Ex. 2.—To divide $63x^2a^3 - 61x^3a^2 + 5xa^4 - 84a^5 + x^4a + 36x^5$ by $-11xa + 9x^2 + 12a^2$.

Arranging the divisor and dividend according to the descending powers of x, or the ascending powers of a, we have the following process:

$$
\begin{array}{l|l|l}
9x^2 - 11xa + 12a^2 & 36x^5 + x^4a - 61x^3a^2 + 63x^2a^3 + 5xa^4 - 84a^5 & 4x^3 + 5x^2a - 6xa^2 - 7a^3 \\
\hline
 & \underline{36x^5 - 44x^4a + 48x^3a^2} & \\
 & \quad 45x^4a - 109x^3a^2 + 63x^2a^3 = \text{1st new dividend.} & \\
 & \quad \underline{45x^4a - 55x^3a^2 + 60x^2a^3} & \\
 & \qquad - 54x^3a^2 + 3x^2a^3 + 5xa^4 = \text{2d new dividend.} & \\
 & \qquad \underline{- 54x^3a^2 + 66x^2a^3 - 72xa^4} & \\
 & \qquad\quad - 63x^2a^3 + 77xa^4 - 84a^5 = \text{3d new dividend.} & \\
 & \qquad\quad \underline{- 63x^2a^3 + 77xa^4 - 84a^5} & \\
 & \qquad\quad 0 + 0 + 0 = \text{naught} &
\end{array}
$$

$=$ the last remainder, and the division is exact.

Ex. 3.—To divide $a^2x^3 - b^2x^3 + 2bcx^2 - 2adx^2 + aex - bex - c^2x + d^2x + ce - de$ by $ax - bx + c - d$.

It is easy to see that $a^2 - b^2$ is the coefficient of x^3, $2bc - 2ad$ that of x^2, and $ae - be - c^2 + d^2$ that of x in the dividend; so that the dividend can be put in the form $(a^2 - b^2)x^3 + (2bc - 2ad)x^2 + (ae - be - c^2 + d^2)x + ce - de$, and in a similar way we may put the divisor in the form $(a - b)x + c - d$.

In the division we shall (for simplicity), according to 15 of the Definitions, use the vertical bar for the parenthesis. Hence, arranging the divisor and dividend according to the descending powers of x, and putting the divisor to the left of the dividend, we get the following process:

Divisor. Dividend. Quotient.

$$\left.\begin{array}{r}a\\-b\end{array}\right|x+\begin{array}{r}c\\-d\end{array}\left|\begin{array}{r}a^2\\-b^2\end{array}\right|x^3\left.\begin{array}{r}-\ 2ad\\+\ 2bc\end{array}\right|x^2+\left.\begin{array}{r}ae\\-be\\-c^2\\+d^2\end{array}\right|x+\left.\begin{array}{r}ce\\-de\end{array}\right|\left.\begin{array}{r}a\\+b\end{array}\right|x^2+\left.\begin{array}{r}-c\\-d\end{array}\right|x+e$$

$$\left.\begin{array}{r}a^2\\-b^2\end{array}\right|x^3+\left.\begin{array}{r}ac\\-ad\\+bc\\-bd\end{array}\right|x^2$$

$$\left.\begin{array}{r}-ac\\-ad\\+bc\\+bd\end{array}\right|x^2+\left.\begin{array}{r}ae\\-be\\-c^2\\+d^2\end{array}\right|x = \text{the 1st new dividend.}$$

$$\left.\begin{array}{r}-ac\\-ad\\+bc\\+bd\end{array}\right|x^2+\left.\begin{array}{r}-c_2\\+d^2\end{array}\right|x$$

$$\left.\begin{array}{r}ae\\-be\end{array}\right|x+\begin{array}{r}ce\\-de\end{array} = \text{the 2d new div.}$$

$$\left.\begin{array}{r}ae\\-be\end{array}\right|x+\begin{array}{r}ce\\-de\end{array}$$

$$0 + 0 = \text{naught}$$

= the last remainder, and the division is exact.

Explanation.—Having separated the dividend from the divisor and quotient by the long vertical lines, we divide the first term $\left.\begin{array}{r}a^2\\-b^2\end{array}\right|x^3$ of the dividend, by the first term $\left.\begin{array}{r}a\\-b\end{array}\right|x$ of the divisor, and get $\left.\begin{array}{r}a\\+b\end{array}\right|x^2$ for the first term of the quotient, which, multiplied by the divisor, gives $\left.\begin{array}{r}a^2\\-b^2\end{array}\right|x^3+\left.\begin{array}{r}ac\\-ad\\+bc\\-bd\end{array}\right|x^2$ for the product, which, subtracted from $\left.\begin{array}{r}a^2\\-b^2\end{array}\right|x^3+\left.\begin{array}{r}-2ad\\+2bc\end{array}\right|x^2$, the corresponding terms of the dividend, gives $\left.\begin{array}{r}-ac\\-ad\\+bc\\+bd\end{array}\right|x^2$ for the first remainder, to which, bringing down the next term $\left.\begin{array}{r}ae\\-be\\-c^2\\+d^2\end{array}\right|x$ of the dividend, we get the first new dividend.

Dividing the first term of the first new dividend by the first term of the divisor, we get $+\left.\begin{array}{r}-c\\-d\end{array}\right|x$ for the second term of the quotient, which, multiplied by the divisor, gives the product $\left.\begin{array}{r}-ac\\-ad\\+bc\\+bd\end{array}\right|x^2+\left.\begin{array}{r}-c^2\\+d^2\end{array}\right|x$, which, subtracted from the first new dividend, gives $\left.\begin{array}{r}ae\\-be\end{array}\right|x$ for the remainder, to which, bringing down $\begin{array}{r}+ce\\-de\end{array}$ the next and last term of the given dividend, we get the second new dividend, with which we proceed in the same way as before to get the third term of the quotient, and so on, until the work is finished.

We have given this question for the purpose of showing the use of the vertical bar, and its superiority over the parenthesis, when the coefficients of the different powers of the letter (according to which the divisor and dividend are arranged) are literal and composed of many terms.

Ex. 4.—To divide $ax^{3n} + adx^{2n}y^m + bdx^ny^{2m} + by^mx^{2n} - cy^{2m}x^n - cdy^{3m}$ by $x^n + dy^m$.

Here, by using the vertical bar, and arranging the divisor and dividend according to the descending powers of x, and putting the divisor to the left of the dividend, we have the following scheme.

$$\begin{array}{rl}
x^n+dy^m & \left[ax^{3n}+\begin{array}{c|}ad\\+b\end{array}x^{2n}y^m+\begin{array}{c|}bd\\-c\end{array}x^ny^{2m}-cdy^{3m}\right] ax^{2n}+bx^ny^m-cy^{2m} \\
 & \underline{ax^{3n}+\ adx^{2n}y^m\qquad\qquad} \\
 & \qquad\qquad bx^{2n}y^m+\begin{array}{c|}bd\\-c\end{array}x^ny^{2m} = \text{the 1st new dividend.} \\
 & \qquad\qquad \underline{bx^{2n}y^m+bdx^ny^{2m}\qquad} \\
 & \qquad\qquad\qquad\quad -cx^ny^{2m}-cdy^{3m} = \text{the 2d new div'd.} \\
 & \qquad\qquad\qquad\quad \underline{-cx^ny^{2m}-cdy^{3m}} \\
 & \qquad\qquad\qquad\qquad\quad 0+0
\end{array}$$

= naught = the last remainder, and the quotient is exact.

Explanation.—Having put the dividend within braces, to separate it from the divisor and quotient, we divide the first term, ax^{3n}, of the dividend by the first term, x^n, of the divisor, which is done by subtracting the exponent n in the divisor from the exponent $3n$ in the dividend, which gives $2n$ for the exponent of x in the quotient, and we of course have ax^{2n} for the first term of the sought quotient, which, multiplied by the divisor, gives $ax^{3n} + adx^{2n}y^m$ for the product, which, subtracted from $ax^{3n} + \begin{array}{c|}ad\\+b\end{array} x^{2n}y^m$, the corresponding terms of the dividend, gives $bx^{2n}y^m$ for the remainder, to which, bringing down the next term, $\begin{array}{c|}bd\\-c\end{array} x^ny^{2m}$, of the dividend, we get the first new dividend. Then dividing the first term, $bx^{2n}y^m$, of the first new dividend by the first term, x^n, of the divisor, we get bx^ny^m for the second term of the quotient, which, multiplied by the divisor, and the product subtracted from the first new dividend, gives $-cx^ny^{2m}$ for the remainder, to which, bringing down the remaining term, $-cdy^{3m}$, of the given dividend, we get the second new dividend. Then dividing the first term of the second new dividend by the first term of the divisor, we get $-cy^{2m}$ for the third and last term of the quotient, which multiplied by the divisor, and the product subtracted from the second new dividend, nothing remains, and the work is completed.

Ex. 5.—To divide $216a^2b^2x^{2n}y^{2m} - 96a^3bx^{3n}y^m - 216ab^3x^ny^{3m} + 81b^4y^{4m} + 16a^4x^{4m}$ by $2ax^n - 3by^m$.

Arranging the dividend and divisor according to the descending powers of y, and putting the divisor to the right of the dividend, we have the following work.

$$\begin{array}{l|l}
81b^4y^{4m}-216ab^3x^ny^{3m}+216a^2b^2x^{2n}y^{2m}-96a^3bx^{3n}y^m+16a^4x^{4n} & -3by^m+2ax^n \\ \cline{2-2}
81b^4y^{4m}-\ 54ab^3x^ny^{3m} & -27b^3y^{3m}+54ab^2x^ny^{2m}-36a^2bx^{2n} \\ \cline{1-1}
\qquad -162ab^3x^ny^{3m}+216a^2b^2x^{2n}y^{2m} = \text{1st new div'nd.} & \qquad y^m+8a^3x^{3n} \\
\qquad -162ab^3x^ny^{3m}+108a^2b^2x^{2n}y^{2m} & \\ \cline{1-1}
\qquad\qquad 108a^2b^2x^{2n}y^{2m}-96a^3bx^{3n}y^m = \text{2d new dividend.} & \\
\qquad\qquad 108a^2b^2x^{2n}y^{2m}-72a^3bx^{3n}y^m & \\ \cline{1-1}
\qquad\qquad\qquad -24a^3bx^{3n}y^m+16a^4x^{4n} = \text{3d new dividend.} & \\
\qquad\qquad\qquad -24a^3bx^{3n}y^m+16a^4x^{4n} & \\ \cline{1-1}
\qquad\qquad\qquad\qquad 0 \quad + \quad 0 = \text{naught, and the division is exact.} &
\end{array}$$

Ex. 6.—To divide $a^{-n}-b^{-n}$ by $b-a$, supposing n to denote any positive whole number, and that a^{-n} is taken for the first term of the dividend, and b for that of the divisor.

In performing this division, it will be convenient to write the divisor in the form $b^{\mp1}-a$, observing that for $b^{\mp1}-a$ we take b^{-1}, and multiply the first term of any dividend by it, to get the corresponding term of the quotient, and then the term of the quotient is to be multiplied by $b^{+1}-a=b-a$, and the product subtracted from the dividend, in order to get the corresponding remainder; for from what has before been shown, to divide any quantity by b is the same thing as to multiply it by b^{-1}. Hence, we shall have the following process.

$$\begin{array}{l|l|l}
\underline{b^{\mp1}-a} & a^{-n}-b^{-n} & a^{-n}b^{-1}+a^{-n+1}b^{-2}+a^{-n+2}b^{-3}+\dots\dots \\
 & \underline{a^{-n}-a^{-n+1}b^{-1}} & \underline{\qquad\qquad +a^{-1}b^{-n},\text{ making } n \text{ terms.}}
\end{array}$$

$$a^{-n+1}b^{-1}=\text{1st new dividend.}$$

$$\underline{a^{-n+1}b^{-1}-a^{-n+2}b^{-2}}$$

$$a^{-n+2}b^{-2}=\text{2d new dividend.}$$

$$\underline{a^{-n+2}b^{-2}-a^{-n+3}b^{-3}}$$

$$a^{-n+3}b^{-3}=\text{3d new div.}$$

$$\underline{a^{-n+3}b^{-3}-a^{-n+4}b^{-4}}$$

etc., continued to

$$a^{-1}b^{-n+1}-b^{-n}$$

$$\underline{a^{-1}b^{-n+1}-b^{-n}}$$

$$0 \quad + \quad 0$$

= naught, and the division is exact.

Explanation.—The first term of the quotient, $a^{-n}b^{-1}$, is found by multiplying the first term, a^{-n}, of the dividend by b^{-1}, then the product of the first term of the quotient and the divisor, $b-a$, is $a^{-n}b^{-1+1}-a^{-n+1}b^{-1}=a^{-n}b^{0}-a^{-n+1}b^{-1}=a^{-n}-a^{-n+1}b^{-1}$ (since $b^{0}=1$); and subtracting this product from the dividend, we get $a^{-n+1}b^{-1}$ for the first new dividend: observing that we do not bring down the term $-b^{-n}$ of the given dividend until we arrive at the $(n-1)^{th}$ new dividend, since it is not needed before. Multiplying the first new dividend by b^{-1}, we get $a^{-n+1}b^{-2}$ for the second term of the quotient, which, multiplied by the divisor, $b-a$, gives $a^{-n+1}b^{-1}-a^{-n+2}b^{-2}$, which, subtracted from the first new dividend, the remainder is the second new dividend. From the second new dividend we get the third term of the quotient, as before, and so on until we arrive at the $(n-1)^{th}$ new dividend, $a^{-1}b^{-n+1} - b^{-n}$, whose first term, $a^{-1}b^{-n+1}$, multiplied by b^{-1}, gives $a^{-1}b^{-n}$ for the n^{th} term of the quotient, which, multiplied by the divisor, gives $a^{-1}b^{-n+1} - b^{-n}$, which, subtracted from $a^{-1}b^{-n+1} - b^{-n}$, nothing remains; and the division is exact.

REMARKS.

Since the dividend must equal the product of the quotient and divisor, we shall have the equation $a^{-n} - b^{-n} = (a^{-n}b^{-1} + a^{-n+1}b^{-2} + a^{-n+2}b^{-3} + \ldots\ldots + a^{-1}b^{-n})(b-a)$; and since $a^{-n} - b^{-n}$ is less than a^{-n} by the value of b^{-n}, if we add b^{-n} to the value of $a^{-n} - b^{-n}$, we shall get $a^{-n} = b^{-n} + (a^{-n}b^{-1} + a^{-n+1}b^{-2} + \text{etc.})(b-a)$; and if we put $a - b = h$, or $b - a = -h$, we have $a = b + h$, so that by putting $b + h$ for a, and $-h$ for $b - a$ in the last equation, it becomes $(b+h)^{-n} = b^{-n} - [b^{-1}(b+h)^{-n} + b^{-2}(b+h)^{-(n-1)} + b^{-3}(b+h)^{-(n-2)} + \ldots\ldots + b^{-n}(b+h)^{-1}] \times h$,(A).

It appears from (A) that $(b+h)^{-n}$ is composed of b^{-n}, which is independent of h, and of the quantity within the braces which is multiplied by h, so that we shall have $b^{-1}(b+h)^{-n} = b^{-1}b^{-n}$ — terms that depend on h, $= b^{-(n+1)}$ — terms that depend on h; and if we put $(n-1)$ for n in (A), we shall get in the same way $b^{-2}(b+h)^{-(n-1)} = b^{-(n+1)}$ — terms that depend on h; and in like manner $b^{-3}(b+h)^{-(n-2)} = b^{-(n+1)}$ — terms that depend on h, and so on.

Hence, $b^{-1}(b+h)^{-n} + b^{2}(b+h)^{-(n-1)} + \ldots\ldots + b^{-n}(b+h)^{-1}$, which is the quantity within the braces in (A), is equal to $b^{-(n+1)} + b^{-(n+1)} + b^{-(n+1)} + \text{etc.}$, to n terms, — terms that depend on h, $= nb^{-(n+1)}$ — terms that depend on h; and if we multiply by h, then the quantity within the braces in (A), when multiplied by h, will be expressed by $nb^{-(n+1)}h$ — terms that depend on h, and putting this for the part of the right-member of (A) that is multiplied by h, (A) becomes $(b+h)^{-n} = b^{-n} - nb^{-(n+1)}h$ + terms that depend on powers of h higher than the first power, (B).

Where it is to be observed that $(b+h)^{-n}$ denotes that the binomial $b + h$ is to be raised to the power whose exponent is denoted by the negative integer $-n$; and we see by (B) if the terms of the expansion of the power are arranged according to the ascending powers of h, that the two first terms of the expansion will be $b^{-n} - nb^{-(n+1)}h$.

We have shown in Multiplication that we shall have $(b + h)^n = b^n + nb^{n-1}h +$ etc., and it is easy to see, if we put $-n$ for n in this equation, that its first two terms will agree with those of (B).

QUESTIONS FOR EXERCISE.

1. Divide $a^3 + b^3$ by $a^2 - ab + b^2$. *Ans.* $a + b$.

2. Divide $59x^2 - 53x + 90x^5 - 49x^3 + 18x^4 + 10$ by $4x - 5 + 6x^2$. *Ans.* $15x^3 - 7x^2 + 9x - 2$.

3. Divide $3ab^2 - 3a^2b - b^3 + a^3$ by $-2ab + a^2 + b^2$. *Ans.* $a - b$.

4. Divide $abx^3y - abxy^3 + a^2x^4 - b^2y^3$ by $-by^3 + ax^3$. *Ans.* $ax + by$.

5. Divide $a^3 + 12ax^2 - 6a^2x - 8x^3$ by $a - 2x$. *Ans.* $a^2 - 4ax + 4x^2$.

6. Divide $27a^9 - 8b^9$ by $3a^3 - 2b^3$. *Ans.* $9a^6 + 6a^3b^3 + 4b^6$.

7. Divide $a^{6n} - 9a^{5n}b^m + 2a^{4n} - 12a^{3n}b^m - 10a^{2n}b^{2m} + 13a^{4n}b^{2m} + 15a^{3n}b^{3m}$ by $a^{2n} + 2 - 3a^nb^m$. *Ans.* $a^{4n} - 6a^{3n}b^m - 5a^{2n}b^{2m}$.

8. Divide $1 - \frac{4x}{3} - \frac{x^3}{30} + \frac{x^2}{30} - \frac{x^4}{10}$ by $1 - \frac{x}{3} + \frac{x^2}{5}$. *Ans.* $1 - x - \frac{x^2}{2}$.

9. Divide $3a^2 + a + 4a^{-1} + 2a^{-2} - 9a^{-3} - 7a^{-4} + 6a^{-6}$ by $3a - 2a^{-2} + 1$. *Ans.* $a + 2a^{-2} - 3a^{-4}$.

In this question the dividend is arranged according to the descending powers of a, because the exponents decrease by subtraction.

10. Divide $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ by $x^3 + 3x^2y + 3xy^2 + y^3$. *Ans.* $x^3 - 3x^2y + 3xy^2 - y^3$.

11. Divide $6x^6 - 12x^5 + 27x^4 + 20x - 10x^2 - 45$ by $3x^4 - 5$. *Ans.* $2x^2 - 4x + 9$.

12. Divide $81a^4 - 16b^4$ by $3a - 2b$. *Ans.* $27a^3 + 18a^2b + 12ab^2 + 8b^3$.

13. Divide $12x^4 - 33x^3 + 26x - 11 - 8x^2$ by $3x^3 + 1 - 2x$.
Ans. $4x - 11$.

14. Divide $x^2 - 5ax + 4a^2$ by $x - a$, and $a^4 - 7a^3 + 12a^2 - 7a + 1$ by $a^2 - 2a + 1$. *Ans.* $x - 4a$, and $a^2 - 5a + 1$.

15. Divide $6a^2x^3 - 13abx^3 + 5b^2x^3 + 3ahx^2 + 2acx^2 - bcx^2 - 5bhx^2 - 2adx^2 + bdx^2 - 7bx + hcx + 14ax - hdx + 7h$ by $3ax^2 - 5bx^2 - dx + cx + 7$. *Ans.* $(2a - b)x + h$.

In doing this question, it will be proper to use the vertical bar, after the manner that has been explained in the third of the preceding examples, which has been worked out.

RULE II.

1. *Arrange the divisor and dividend according to the dimensions of a common letter, as in Rule I.; then change the sign of each term of the divisor.*

2. *To get the term of the quotient that corresponds to any dividend, divide the first term of the dividend by the first term of the divisor, considering its sign as unchanged, as in Rule I.; then multiply the divisor with its sign changed by the quotient-term, and add the product to the dividend.*

3. *To the sum thus found bring down as many of the next successive terms of the given dividend as may be necessary to form a new dividend, and then proceed as before; and so on, until all the terms of the given dividend have been brought down, as in Rule I.*

The reason of this rule is evident, from the consideration that Division is the reverse of Multiplication; and it clearly amounts to the same thing as Rule I.

We will now give a few examples for the purpose of illustrating the rule.

Ex. 1.—To divide $5x^3 - 3x^2 + 7x - 9$ by $x - 1$.

Changing the signs of the terms of the divisor, it becomes $-x + 1$, and since the sign of x is to be regarded as $+$ in finding the terms of the quotient, we may, for $-x + 1$, write $\pm x + 1$; observing that in finding the terms of the quotient we must for $\pm$ use $+$, and when we multiply the divisor by any quotient-term, we must for $\pm$ use $-$.

Hence we shall divide as in the following scheme.

$$\begin{array}{l|l|l}
\underline{\pm x + 1} & 5x^3 - 3x^2 + 7x - 9 & \underline{5x^2 + 2x + 9} \\
 & \underline{-5x^3 + 5x^2} & \\
 & \qquad 2x^2 + 7x = \text{1st new dividend.} & \\
 & \qquad \underline{-2x^2 + 2x} & \\
 & \qquad\qquad 9x - 9 = \text{2d new dividend.} & \\
 & \qquad\qquad \underline{-9x + 9} & \\
 & \qquad\qquad 0 + 0 &
\end{array}$$

= naught, and the division is exact.

This example is the first that we worked by Rule I., and we see that the result is the same as there found; and the process is so evident that it needs no explanation.

Ex. 2.—To divide $2x^5 - 11x^4 + 25x^3 - 42x^2 + 41x - 15$ by $2x^3 - 3x^2 + 7x - 5$.

Changing the signs of the terms of the divisor, etc., it becomes $\pm 2x^3 + 3x^2 - 7x + 5$; then using + for ± in finding the terms of the quotient, and − for ± when the divisor is to be multiplied by any term of the quotient, we have the following solution.

$$\begin{array}{l|l|l}
\underline{\pm 2x^3 + 3x^2 - 7x + 5} & 2x^5 - 11x^4 + 25x^3 - 42x^2 + 41x - 15 & \underline{x^2 - 4x + 3} \\
 & \underline{-2x^5 + 3x^4 - 7x^3 + 5x^2} & \\
 & \qquad -8x^4 + 18x^3 - 37x^2 + 41x = \text{1st new div.} & \\
 & \qquad \underline{+8x^4 - 12x^3 + 28x^2 - 20x} & \\
 & \qquad\qquad 6x^3 - 9x^2 + 21x - 15 = \text{2d “} & \\
 & \qquad\qquad \underline{-6x^3 + 9x^2 - 21x + 15} & \\
 & \qquad\qquad 0 + 0 + 0 + 0 &
\end{array}$$

= naught, and the division is exact.

Ex. 3.—To divide $63a^5 - 33a^4b + 77a^3b^2 - 69a^2b^3 + 19ab^4 - 11b^5$ by $9a^2 + 3ab + 11b^2$.

Changing the signs of the terms of the divisor, etc., it becomes $\pm 9a^2 - 3ab - 11b^2$, which being placed to the right of the dividend, and the quotient beneath it, we shall have the following process.

$$\begin{array}{llllll|l}
63a^5 & -33a^4b & +77a^3b^2 & -69a^2b^3 & +19ab^4 & -11b^5 & \pm 9a^2 - 3ab - 11b^2 \\ \cline{7-7}
-63a^5 & -21a^4b & -77a^3b^2 & & & & 7a^3 - 6a^2b + 2ab^2 - b^3 \\ \cline{1-3}
& -54a^4b & +0 & -69a^2b^3 & & \multicolumn{2}{l}{= \text{1st new dividend.}} \\
& 54a^4b & +18a^3b^2 & +66a^2b^3 & & & \\ \cline{2-4}
& & 18a^3b^2 & -3a^2b^3 & +19ab^4 & \multicolumn{2}{l}{= \text{2d new dividend.}} \\
& & -18a^3b^2 & -6a^2b^3 & -22ab^4 & & \\ \cline{3-5}
& & & -9a^2b^3 & -3ab^4 & -11b^5 & = \text{3d new dividend.} \\
& & & 9a^2b^3 & +3ab^4 & +11b^5 & \\ \cline{4-6}
& & & 0 & +0 & +0 & = \text{the last remainder.}
\end{array}$$

$= 0$, and the division is exact.

Ex. 4.—To divide $-7x^5 + 21x^4 - 26x^3 - x^2 - 7x + 20$ by $-x^2 + 3x - 4$.

Changing the signs of the terms of the divisor, etc., it becomes $\mp x^2 - 3x + 4$, then observing that for $\mp$ we must use $-$ in finding the terms of the quotient, and $+$ when the divisor is multiplied by any quotient-term; and placing the divisor to the right of the dividend, we have the following:

$$\begin{array}{rrrrrr|l}
-7x^5 & +21x^4 & -26x^3 & -x^2 & -7x & +20 & \mp x^2 - 3x + 4 \\ \cline{7-7}
7x^5 & -21x^4 & +28x^3 & & & & 7x^3 - 2x - 5 \\ \cline{1-5}
& & 2x^3 & -x^2 & -7x & & = \text{1st new dividend.} \\
& & -2x^3 & +6x^2 & -8x & & \\ \cline{3-6}
& & & 5x^2 & -15x & +20 & = \text{2d new dividend.} \\
& & & -5x^2 & +15x & -20 & \\ \cline{4-6}
& & & 0 & +0 & +0 &
\end{array}$$

EXAMPLES FOR EXERCISE.

1. Divide $x^3 + 3x^2 + 3x + 1$ by $x + 1$.
   Ans. $x^2 + 2x + 1$.
2. Divide $x^5 - 32$ by $x - 2$.
   Ans. $x^4 + 2x^3 + 4x^2 + 8x + 16$.
3. Divide $3a^4 - 13a^2b^2 + 14b^4$ by $3a^2 - 7b^2$.
   Ans. $a^2 - 2b^2$.
4. Divide $a^3 + a^2x - ax^2 - x^3$ by $a^2 + 2ax + x^2$.
   Ans. $a - x$.
5. Divide $7x^3 + 56$ by $x + 2$. *Ans.* $7x^2 - 14x + 28$.
6. Divide $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ by $a^2 + 2ab + b^2$.
   Ans. $a^2 + 2ab + b^2$.
7. Divide $1 + x - x^5 - x^6$ by $1 - x^2$.
   Ans. $1 + x + x^2 + x^3 + x^4$.
8. Divide $x^4 - 55x^2 - 30x + 504$ by $x^2 - 10x + 21$.
   Ans. $x^2 + 10x + 24$.
9. Divide $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$.
   Ans. $a^2 + ab + b^2$.
10. Divide $12a^4 - 21a^3b - 26a^2b^2 + 30ab^3 + 25b^4$ by $4a^2 - 3ab - 5b^2$. *Ans.* $3a^2 - 3ab - 5b^2$.

(7.) It is clear from what was shown in the preliminary remarks on Division, that if we divide any divisor and its corresponding dividend by the same quantity, and then use the quotients thus obtained for a new divisor and dividend, the same quotient will result from the new divisor and dividend as from the given divisor and dividend.

Hence, having arranged the divisor and dividend as in Rule I., if we divide each term of the divisor and dividend by the first term of the divisor, we may take the results thus obtained for the new divisor and dividend; and it is easy to see that the first term of the new divisor is always 1; and the first term of the new dividend will be the first term of the quotient.

We may divide the new dividend by the new divisor, either by Rule I. or Rule II.; but it will be better to proceed as in the following rule, which is a modification of Rule II.

RULE III.

1. *Having arranged the terms of the divisor and dividend, as in Rule I., divide each term of the divisor and dividend by the first term of the divisor, and call the results the new divisor and dividend.*

2. *Change the signs of all the terms of the new divisor, except that of the first term, which is* 1, *and call the result the changed divisor.*

3. *Multiply all the terms of the changed divisor except the first term, by the first term of the new dividend; and put the products successively under the corresponding terms of the new dividend; then the sum of the results in the column that immediately follows the first term of the new dividend will be the second term of the quotient.*

4. *Multiply the same terms of the changed divisor (as before) by the second term of the quotient, and put the products successively under the corresponding terms of the new dividend and the preceding products; then the sum of the results in the column that immediately follows the second term of the quotient will be the third term of the quotient.*

5. *Then proceed in the same way as before with the third term of the quotient and the same terms of the changed divisor to get the fourth term of the quotient, and so on.*

We will now give some examples for the purpose of illustrating the rule.

Ex. 1.—To divide x^2+2x+1 by $x+1$.

Here, supposing the divisor and dividend to be arranged according to the descending powers of x, we must divide the divisor and dividend by x, or which is the same, we must multiply each of their terms by x^{-1}, and we shall have $(x+1)x^{-1}=1+x^{-1}=$ the new divisor, and $(x^2+2x+1)x^{-1}=x+2+x^{-1}=$ the new dividend; and changing the sign of the second term in the divisor, we have $1-x^{-1}$ for the changed divisor.

Hence, by the rule, we have the following process.

$$\begin{array}{l|cccc}
1 - x^{-1} & x & + 2 & + x^{-1} \\
\hline
 & & -1 & \\
 & & & -x^{-1} \\
\hline
 & x & + 1 &
\end{array}$$

= the sought quotient.

Ex. 2.—To divide $-7x^5 + 21x^4 - 26x^3 - x^2 - 7x + 20$ by $-x^2 + 3x - 4$.

Here, in order to divide the dividend and divisor by $-x^2$, the first term of the divisor, we may multiply each of their terms by $-x^{-2}$, and we get $7x^3 - 21x^2 + 26x + 1 + 7x^{-1} - 20x^{-2}$ = the new dividend, and $1 - 3x^{-1} + 4x^{-2}$ = the new divisor, and changing the signs of all the terms of the new divisor, except that of 1, the first term, we have $1 + 3x^{-1} - 4x^{-2}$ = the changed divisor.

Hence, by the rule, we must proceed as follows.

$$\begin{array}{l|cccccc}
1 + 3x^{-1} - 4x^{-2} & 7x^3 & - 21x^2 & + 26x & + 1 & + 7x^{-1} & - 20x^{-2} \\
 & & 21x^2 & - 28x & & & \\
 & & & 0 & + 0 & & \\
 & & & & -6 & + 8x^{-1} & \\
 & & & & & -15x^{-1} & + 20x^{-2} \\
\hline
 & \multicolumn{6}{l}{7x^3 + 0 - 2x - 5 = 7x^3 - 2x - 5}
\end{array}$$

= the quotient, as required; which agrees with the answer to the same question as found in Ex. 4 under Rule II.

Explanation.—If we divide the first term, $7x^3$, of the new dividend by 1, the first term of the changed divisor, we shall have, as in Rule II., $7x^3$ for the first term of the quotient, which, multiplied by all the terms of the changed divisor, except the first term 1, gives the product $(3x^{-1} - 4x^{-2}) \times 7x^3 = 21x^2 - 28x$, which is placed under the corresponding terms of the new dividend.

Then, by adding $-21x^2$ and $21x^2$, which are the terms in the column that immediately follows the first term of the quotient, we get 0 for the sum; consequently, the second term of the quotient = 0, which, multiplied by the same terms of the changed divisor, gives 0 for each product, which we put down in the scheme merely for the sake of uniformity. To get the third term of the quotient, we add the quantities which stand in the third vertical column, and get $26x - 28x = -2x$, = the third term of the quotient, since $-2x$, divided by 1, the first term of the divisor, is $-2x$; and the product of $-2x$ and the same terms of the changed divisor as before is $(3x^{-1} - 4x^{-2}) \times -2x = 6 + 8x^{-1}$, which is put under the corresponding terms of the new dividend, then the sum of the quantities in the fourth vertical column is $1 - 6 = -5$ = the fourth term of the quotient. And, as before, we get $(3x^{-1} - 4x^{-2}) \times -5 = -15x^{-1} + 20x^{-2}$, which, being put under the corresponding terms of the new dividend and the preceding products, the sums of the quantities in each of the vertical columns destroy each other, and the new dividend is exhausted; and of course the division is exact, and the sought quotient is $7x^2 - 2x - 5$.

It is easy to see that it is useless to divide by 1, the first term of the changed divisor, since any quantity is the same after it is divided by 1 as before; consequently, the quotient-terms will be found as stated in the rule in all cases. It is also easy to see that Rule III. is substantially the same as Rule II., when applied to the new divisor and dividend, when we reject the first term, 1, of the changed divisor, which is clearly allowable.

Ex. 3.—To divide $15a^4 - 6a^3b - 17a^2b^2 + 8ab^3 - 4b^4$ by $3a^2 - 4b^2$.

The terms of the dividend and divisor are divided by the first term, $3a^2$, of the divisor, by multiplying them by $3^{-1}a^{-2}$ and we shall have $5a^2 - 2ab - 17 \times 3^{-1}b^2 + 8 \times 3^{-1}a^{-1}b^3 - 4 \times 3^{-1}a^{-2}b^4$ and $1 - 4 \times 3^{-1}a^{-2}b^2$ for the new dividend and divisor; and changing the sign of the second term of the new divisor, we have $1 + 4 \times 3^{-1}a^{-2}b^2$ for the changed divisor. Hence, by the rule, we must proceed as follows.

$$\begin{array}{r|r}
1+4\times 3^{-1} & 5a^2-2ab-17\times 3^{-1}b^2+8\times 3^{-1}a^{-1}b^3-4\times 3^{-1}a^{-2}b^4 \\
a^{-2}b^2 & 20\times 3^{-1}b^2-8\times 3^{-1}a^{-1}b^3+4\times 3^{-1}a^{-2}b^4 \\
\cline{2-2}
 & \text{$5a^2-2ab+b^2$ = the exact quotient.}
\end{array}$$

For exercise, we shall refer to the examples under Rules I. and II.

(8.) DIVISION BY DETACHED COEFFICIENTS.

If we arrange the divisor and dividend in the same way, according to the successive powers of some letter (or letters) common to both of them, and supply any terms that may be wanting either in the divisor or dividend (to make the exponents of any common letter decrease or increase by the same constant difference as we pass from any term in either of them to the next successive term) by terms whose coefficients are 0, then if we omit the common letter (or letters) and use only the coefficients thus arranged, with the signs of the terms to which they belong in the division, we shall have what is called *Division by detached coefficients.*

Having obtained the coefficients in the quotients, either by Rule I. or Rule II., we can easily supply the corresponding powers of the omitted letter or letters; for if we divide the power of any omitted letter in the first term of the arranged dividend by the power of the same omitted letter in the first term of the arranged divisor, we have the power of the letter in the first term of the quotient.

The powers of any omitted letter in the following terms of the quotient may be supplied from the consideration that the powers of any omitted letter must follow the same law of increase or decrease in the terms of the quotient as in the terms of the arranged divisor and dividend.

If we prefix the coefficients in the quotient with their proper signs to the powers of the omitted letter (or letters) that correspond to them, omitting those terms whose coefficients are 0, we shall have the quotient, as required.

It is easy to see that the method of detached coefficients is nothing more than a simplification of Rules I. and II., the division being conducted in the same way, without putting down the powers of the common letter or letters according to which the divisor and dividend have been arranged, leaving the powers of the neglected letter or letters to be supplied in the quotient.

(**9.**) There is a very simple method of Division by detached coefficients, which is called Synthetic Division, that appears to have been first discovered by W. G. Horner, Esq., of Bath, England. On account of its great importance in the solution of the higher equations, we shall now proceed to give it particular notice.

HORNER'S SYNTHETIC DIVISION.

1. Having arranged the terms of the divisor and dividend as in the method of detached coefficients, we divide each coefficient in the divisor and dividend by the first coefficient in the divisor, and use the results as the detached coefficients in the divisor and dividend.

2. Then we divide the sum of the coefficients in the dividend by the sum of the coefficients in the divisor, as in Rule III., and we shall have the sum of the coefficients in the quotient; and by prefixing the coefficients in the quotient to the corresponding powers of the omitted letter or letters, we shall have the quotient of the given dividend and divisor, as required.

We will now illustrate the method by a few examples.

Ex. 1.—To divide $3x^3 - 81$ by $3x - 9$.

Here, by dividing the coefficients in the dividend and divisor by 3, the first coefficient in the divisor, they become $x^3 - 27$ and $x - 3$; then, since x^2 and x are wanting to the dividend, we supply the deficient terms by $0.x^2$, $0.x$, and the dividend becomes $x^3 + 0.x^2 + 0.x - 27$; and detaching the coefficients, we have $1 + 0 + 0 - 27$ for the detached coefficients in the dividend, and the detached coefficients in the divisor are $1 - 3$; and we have to divide $1 + 0 + 0 - 27$ by $1 - 3$, by Rule III. According to the rule, we must change the sign of the second term, $- 3$, in the divisor, and we shall have $1 + 3$ for the divisor. Hence, by Rule III., we must proceed as follows.

$$\begin{array}{r|r} 1+3 & 1+0+0-27 \\ & 3+9+27 \\ \cline{2-2} & 1+3+9 \quad\quad \end{array}$$

= the quotient, and the division is exact.

Dividing the omitted power x^3 of x in the first term of the given dividend by x in the first term of the given divisor, we have x^2 for the power of x that corresponds to the first coefficient 1, in the quotient, and x evidently corresponds to 3, the second coefficient in the quotient; consequently, $x^2 + 3x + 9$ is the quotient arising from the division of $3x^3 - 81$ by $3x - 9$.

Ex. 2.—To divide $6x^6 - 12x^5 + 27x^4 - 10x^3 + 20x - 45$ by $2x^2 - 4x + 9$.

Arranging the terms according to the descending powers of x, supplying the deficient term in the dividend, dividing the coefficients in the divisor and dividend by 2, the coefficient of the first term of the divisor, then detaching the resulting coefficients of the dividend and divisor, we shall have the sum of the coefficients $3 - 6 + \frac{27}{2} + 0 - 5 + 10 - \frac{45}{2}$ to be divided by the sum of the coefficients $1 - 2 + \frac{9}{2}$ by Rule III.; and, according to the rule, changing the signs of the last two terms of the divisor, we have $1 + 2 - \frac{9}{2}$ for the changed divisor.

Hence, according to the rule, we must proceed as follows.

$$\begin{array}{r|l} 1 + 2 - \frac{9}{2} & 3 - 6 + \frac{27}{2} + 0 - 5 + 10 - \frac{45}{2} \\ & \qquad 6 - \frac{27}{2} \qquad\qquad - 10 + \frac{45}{2} \\ \cline{2-2} & 3 + 0 + 0 + 0 - 5 + 0 + 0 \end{array}$$

= the coefficients in the quotient, and the division is exact.

Dividing x^6, the power of x in the first term of the arranged dividend, by x^2, the power of x in the first term of the arranged divisor, we have x^4 for the power of x that corresponds to the coefficient 3 in the first term of the quotient; and since the powers of x decrease by a unit as we pass from any term in the arranged divisor and dividend to the next successive term, it is clear that the sought quotient is expressed by $3x^4 + 0 . x^3 + 0 . x^2 + 0 . x - 5 = 3x^4 - 5$ by omitting the terms whose coefficients equal 0.

Ex. 3.—Divide $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ by $x^3 - 3x^2y + 3xy^2 - y^3$.

Arranging the dividend and divisor according to the descending powers of y, dividing the terms of the dividend and divisor by -1, the coefficient of the first term of the divisor, supplying the terms that are wanting in the dividend, and detaching the coefficients, we have $1 + 0 - 3 + 0 + 3 + 0 - 1$ for the coefficients in the new dividend, and

$1 - 3 + 3 - 1$ for those of the new divisor; and changing the signs of all the coefficients in the new divisor except that of the first term, we have $1 + 3 - 3 + 1$ for the coefficients in the changed divisor.

For convenience, we shall write the coefficients of the changed divisor in a vertical column, putting the first coefficient at the top, and the rest in their proper order. Then, by Rule III., we proceed as follows.

$$\begin{array}{r|rrrrrrr}
1 & 1 & +0 & -3 & +0 & +3 & +0 & -1 \\
+3 & & 3 & -3 & +1 & & & \\
-3 & & & 9 & -9 & +3 & & \\
+1 & & & & 9 & -9 & +3 & \\
 & & & & & 3 & -3 & +1 \\ \hline
 & 1 & +3 & +3 & +1 & & &
\end{array}$$

= the coefficients in the quotient; and supplying the powers of the letters that correspond to the coefficients, we evidently have $y^3 + 3y^2x + 3yx^2 + x^3$ for the quotient, as required.

We will now perform the division by placing the products of the coefficients in the divisor and any term of the quotient, in diagonal lines, under the corresponding coefficients of the new dividend, which is in accordance with the directions given in Horner's rule. And since the first of the coefficients in the changed divisor is of no use in finding the coefficients in the quotient, we shall omit it.

Hence, we shall have the following work.

$$\begin{array}{r|rrrrrrr}
 & 1 & +0 & -3 & +0 & +3 & +0 & -1 \\
+3 & & 3 & +9 & +9 & +3 & & \\
-3 & & & -3 & -9 & -9 & -3 & \\
+1 & & & & 1 & +3 & +3 & +1 \\ \hline
 & 1 & +3 & +3 & +1 & & &
\end{array}$$

= the coefficients in the divisor; the same as found before.

We think it will be evident from a comparison of this method of placing the products, with the method of placing them in horizontal lines (as we have done above), that it will generally be more simple to write them horizontally than diagonally, as directed in Horner's rule.

Ex. 4.—Divide $adx^{4n} + aex^{3n} - bdx^{2n} + \left.\begin{array}{r} cd \\ -be \end{array}\right| x^n + ce$ by $ax^{3n} - bx^n + a$

We shall now suppose the terms of the dividend and divisor to be arranged according to the descending powers of x; then it is easy to see that the difference between the exponents of x in any two successive terms of the dividend is n; but the difference of the exponents of x in the first and second terms of the divisor is $2n$; consequently there is a term wanting in the divisor between the first and second terms. It will also be noticed that the coefficients of the different powers of x in the dividend and divisor are letters; but it is easy to see that we may use them in the same way as if they were figures.

Hence, if we divide the coefficients of the dividend and divisor by a, the coefficient of the first term in the divisor, supply the deficient term in the divisor, and detach the resulting coefficients; then, by Rule III., we must proceed as follows.

$$\begin{array}{r|rrrr|r}
1 & d & +e & -\frac{bd}{a} & +\frac{cd}{a} & +\frac{ce}{a} \\
+0 & & 0 & +\frac{bd}{a} & -\frac{be}{a} & \\
+\frac{b}{a} & & & +0 & -\frac{cd}{a} & -\frac{ce}{a} \\
-\frac{c}{a} & & & & +\frac{be}{a} & \\
\hline
 & d & +e & & &
\end{array}$$

= the coefficients in the quotient, and it is evident that the quotient is $dx^n + e$.

It may be observed that the vertical bar is used in the given dividend to signify that $cd - be$ is the coefficient of x^n; also in the detached coefficients of the new dividend the vertical bar is used to signify that $\frac{cd}{a} - \frac{be}{a}$ is a coefficient in the new dividend. We have also written the detached coefficients of the changed divisor in a vertical column, for convenience and simplicity; and we have written the 0 in it on account of the deficient term in the divisor.

Then, d is the first term of the quotient, which, multiplied by 0, gives 0 to be placed under e, and, multiplied by $\frac{b}{a} - \frac{c}{a}$

gives $+\frac{bd}{a}$ to be placed under $-\frac{bd}{a}$, so that these terms destroy each other; also we have $-\frac{cd}{a}$ to be placed under $\frac{cd}{a}$, so that these terms destroy each other; then e, the second term of the quotient, multiplied by $\frac{b}{a}-\frac{c}{a}$, gives $\frac{be}{a}$ under $-\frac{be}{a}$, and $-\frac{ce}{a}$ under $\frac{ce}{a}$, and these terms destroy each other; and the quotient is as stated above.

Examples suitable for practice will be found under Rules I. and II.

If we supply any terms that may be wanting either in the new dividend or divisor, in Rule III., and then use only the coefficients (or detach them) in the division, it is clear that the rule will agree with the rule of Synthetic Division.

(10.) REMARKS ON PARTIAL OR INCOMPLETE DIVISION.

When the divisor is not an exact factor of the dividend, the division must of necessity be incomplete, since there will be always a remainder, which will not exactly contain the divisor.

If, however, we divide the dividend by the divisor in the same way as in complete division (by considering the divisor as a factor of the dividend), and carry the division sufficiently far, we can often (as in decimal fractions) diminish the remainder indefinitely, so that it may be rejected without sensibly affecting the quotient, at least so far as it may be necessary to obtain it.

EXAMPLES.

Ex. 1.—Divide 1 by $1-x$.

The division is indicated by $\frac{1}{1-x}$, which is a fraction, whose numerator is 1, and denominator $1-x$.

Since $-x+x$ equals 0, and that a quantity is not affected by adding 0 to it, we may add $-x+x$ to the numerator of the fraction $\frac{1}{1-x}$ without affecting its value, and we shall have $\frac{1}{1-x} = \frac{1-x+x}{1-x} = \frac{1-x}{1-x} + \frac{x}{1-x} = 1 + \frac{x}{1-x}$, since $\frac{1-x}{1-x} = 1$.

Also, since $-x^2 + x^2 = 0$, we shall have, in like manner as before, $\frac{x}{1-x} = \frac{x-x^2+x^2}{1-x} = \frac{x(1-x)+x^2}{1-x} = x + \frac{x^2}{1-x}$; and, in the same way, we get $\frac{x^2}{1-x} = x^2 + \frac{x^3}{1-x}$, and so on to any extent. Hence, we shall have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots + x^{n-1} + \frac{x^n}{1-x}$, (A); supposing n to stand for the number of monomial terms that precede the fraction in the right member of the equation.

The method which we have used in obtaining (A) is clearly the same as to arrange the dividend and divisor according to the ascending powers of x, or to take 1 for the first term of the divisor, and then to divide the dividend 1 by the divisor $1 - x$, by Rule I. Thus, by Rule I., we have,

$$
\begin{array}{l}
1-x \;\left|\; \begin{array}{l} 1 \\ \underline{1-x} \end{array} \right|\; 1+x+x^2+x^3+\dots x^{n-1}+\frac{x^n}{1-x} \\
\qquad\quad x \\
\qquad\quad \underline{x-x^2} \\
\qquad\qquad\quad x^2 \\
\qquad\qquad\quad \underline{x^2-x^3} \\
\qquad\qquad\qquad\quad x^3 \\
\qquad\qquad\qquad\quad \underline{x^3-x^4} \\
\qquad\qquad\qquad\qquad\quad \underline{x^4 \qquad\qquad} \\
\qquad\qquad\qquad\qquad\quad \text{etc.} \ldots\ldots \\
\qquad\qquad\quad \text{to } x^{n-1} \\
\qquad\qquad\quad \underline{x^{n-1}-x^n} \\
\qquad\qquad\qquad\quad x^n
\end{array}
$$

observing that the fraction $\frac{x^n}{1-x}$ is obtained by writing the divisor under the last remainder, as in Arithmetic.

It is easy to see that the right member of (A) is nothing more than a transformation of the first member; that is to say, the second member of the equation is the same as the first member under a different form of expression. Hence, if x is sensibly less than 1, so that when n is very great, x^n shall be diminished indefinitely, and become less than any finite quantity; then, by Ax. VII., we may reject the fraction $\frac{x^n}{1-x}$, and (A) will be reduced to $\frac{1}{1-x} = 1 + x + x^2 + x^3 +$ etc., ... to infinity, (B); and $\frac{1}{1-x}$ is said to be converted into an infinite series.

It may here be observed, that by a *series* we mean a succession of quantities of the same kind, which generally proceed according to a *certain law of formation*, which enables us to write down as many of the quantities as we choose from a knowledge of a certain number of them at the commencement of the series. Thus, we see that the monomial terms in (B) are positive, and that they constitute a series, and that if we multiply any term by x, the product will equal the next successive term; so that the series is easily continued to any extent.

A series that is continued to a finite number of quantities is finite, but one which is carried on to an infinite or indefinitely great number of quantities is an infinite series.

Any quantity when converted into a series is said to be the *generating function* of the series; thus, $\frac{1}{1-x}$ is the generating function of the series in (B).

When the quantities in a series continually diminish, so that any one of the quantities is less than the quantity that immediately precedes it, the series is said to *converge* or to be *convergent*, but if the contrary has place, the series is said to *diverge* or to be *divergent*.

Thus, if x is less than 1, the series in (B) is clearly convergent; but if x is greater than 1, the series is evidently divergent.

If the quantities in the series are equal to each other, the series is *neutral;* since there is no increase or decrease of the quantities in the series. Thus, if $x = 1$, the series in (B) is neutral, since each of its terms is 1.

When we use a series for the purpose of calculating correctly the numerical value of its generating function to a certain number of figures; in order to do it with ease, it will generally be necessary that the quantities should decrease rapidly, so that the sum of the numerical values of a few of the first quantities in the series shall give the value of the generating function to the requisite degree of exactness.

Thus, if we put $x = \frac{1}{16}$ in (B), and we wish by the series to compute the value of the generating function to five decimal places, we shall get (by putting $x = \frac{1}{16}$) $\frac{1}{1-\frac{1}{16}} = 1 + \frac{1}{16} + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{16}\right)^3 +$ etc., $= 1 + \frac{1}{16} + \frac{1}{256} + \frac{1}{4096} +$ etc. $= 1 + 0.0625 + 0.003906$, etc., $+ 0.000244$, etc., $+ 0.000015 + 0.000000 +$ etc., $= 1.066665$ etc.

Since $\frac{1}{1 - \frac{1}{16}} = \frac{16}{16-1} = \frac{16}{15} = 1.066666$ etc., $=$ the value of the generating function, we see that the series has given the correct value of it to five decimal places, with great facility in the calculation.

It is easy to see that when $x = \frac{1}{32}$, the value of $\frac{1}{1-x}$ to five decimal places can be found from the series with much greater ease than in the case of $x = \frac{1}{16}$; so that the smaller x is, the greater will be the ease of the calculation.

But if x is a very little less than 1, the series will converge with extreme slowness, so that a great many terms must be used in order to obtain the value of $\frac{1}{1-x}$ to any degree of exactness in the decimal places; for when $x = 0.99$, it will be necessary to use about eight hundred terms in order to find $\frac{1}{1-x}$ correctly to one decimal place.

If $x = 1$, or differs insensibly from it, then the series in (B) becomes $\frac{1}{1-1} = 1 + 1 + 1 + 1 + 1 +$ etc., when we reject the insensible difference between 1 and x; and since $1 - 1 =$ an indefinitely small quantity, it is evident that

$\frac{1}{1-1}$ expresses an unlimitedly great or an infinitely great quantity; since the finite quantity 1 must clearly contain the indefinitely small quantity $1-x$ ($=1-1=0$ very nearly) an infinite number of times; consequently, we may regard the series $1+1+1+$ etc., as indicating that its generating function is infinitely great.

If x is greater than 1 (or even if $1-x$ is an indefinitely small quantity), $\frac{x^n}{1-x}$ ought not to be rejected in (A); so that (B) will not serve to find the numerical value of $\frac{1}{1-x}$.

We will now show how to convert $\frac{1}{1-x}$ into a series of a different form from (B), which will enable us to find the numerical values of $\frac{1}{1-x}$ when x is sensibly greater than 1.

Thus, since $\frac{1}{1-x}=\frac{1}{-x+1}$, and that the value of a fraction is not affected by dividing its numerator and denominator by the same quantity; if we divide the numerator and denominator of $\frac{1}{-x+1}$ by $-x$, we shall have $\frac{1}{1-x}=$

$$\frac{-\frac{1}{x}}{1-\frac{1}{x}}=-\frac{\frac{1}{x}}{1-\frac{1}{x}}.$$

Again, since $-\frac{1}{x^2}+\frac{1}{x^2}=0$, $-\frac{1}{x^3}+\frac{1}{x^3}=0$, $-\frac{1}{x^4}+\frac{1}{x^4}=0$, and so on to $-\frac{1}{x^{n+1}}+\frac{1}{x^{n+1}}=0$ (n being a positive integer); and because a quantity is not affected by adding 0 to it; we get $\frac{1}{x}=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^2}=\frac{1}{x}\left(1-\frac{1}{x}\right)+\frac{1}{x^2}=\frac{1}{x}\left(1-\frac{1}{x}\right)+\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^3}=\frac{1}{x}\left(1-\frac{1}{x}\right)+\frac{1}{x^2}\left(1-\frac{1}{x}\right)+\frac{1}{x^3}$, and so on to $\frac{1}{x}=\frac{1}{x}\left(1-\frac{1}{x}\right)+\frac{1}{x^2}\left(1-\frac{1}{x}\right)+\frac{1}{x^3}\left(1-\frac{1}{x}\right)+\ldots+\frac{1}{x^n}\left(1-\frac{1}{x}\right)+\frac{1}{x^{n+1}}$; consequently, if we divide both sides of this equation by $1-\frac{1}{x}$, we shall have $\frac{\frac{1}{x}}{1-\frac{1}{x}} = \frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots.+\frac{1}{x^n}+\frac{1}{x^{n+1}\left(1-\frac{1}{x}\right)}$, or, since $x^{n+1}=x^n\times x$, we get $\frac{1}{x^{n+1}\left(1-\frac{1}{x}\right)}$, $=\frac{1}{x^n(x-1}$, and of course $\left[\text{since } \frac{1}{1-x} = -\frac{\frac{1}{x}}{1-\frac{1}{x}}\right]$ we have $\frac{1}{1-x} = -\left[\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots.+\frac{1}{x^n}+\frac{1}{x^n(x-1}\right]$, (C).

The process by which we have obtained (C) is clearly the same thing as to divide 1 by $-x+1$, using $-x$ as the first term of the divisor. Thus, if we divide 1 by $-x+1$, by Rule I., we have

$$\begin{array}{lll}
\underline{-x+1} & 1 & \underline{-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}} \\
& \underline{1-\frac{1}{x}} & \\
& \frac{1}{x} & \\
& \underline{\frac{1}{x}-\frac{1}{x^2}} & \\
& \frac{1}{x^2} & \\
& \underline{\frac{1}{x^2}-\frac{1}{x^3}} & \\
& \frac{1}{x^3} & \\
& \ldots. & \\
& \text{etc., to } \underline{\frac{1}{x^n}-\frac{1}{x^{n+1}}} & \\
& \frac{1}{x^{n+1}} &
\end{array}$$

consequently, we easily obtain (C), as given above.

If x is sensibly greater than 1, so that $x-1$ is a finite quantity, it is clear that the integer n may be taken so great that $\frac{1}{x^n(x-1)}$ shall become an insensible quantity, which may be rejected; which being done, (C) is reduced to $\frac{1}{1-x} = -\left[\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\text{etc.},\right]$, (D), which is an infinite series, whose law of formation is evident.

It is clear that the series in (D) will enable us to calculate the generating function $\frac{1}{1-x}$ to any required degree of exactness when $x-1$ is not a very small quantity.

Thus, if we put $x = 32$, (D) becomes $\frac{1}{1-32} = -\left[\frac{1}{32}+\left(\frac{1}{32}\right)^2+\left(\frac{1}{32}\right)^3+\text{etc.},\right] = -[0.03125 + 0.000976 \text{ etc.}, + 0.000030 \text{ etc.} +] = -0.032256$, etc.; but $\frac{1}{1-32} = -\frac{1}{31} = -0.032258$, etc., so that three terms of the series give the correct value of the generating function to five decimal places.

If, however, $x-1$ is an indefinitely small quantity, the series will not serve to calculate the numerical value of the generating function, since it is indefinitely great.

It follows, from what has been done, that if in any calculation we meet with the series $1+x+x^2+x^3+$ etc., to infinity, such that x is sensibly less than 1, then we may put $\frac{1}{1-x}$ for it. But if $1-x$ is an indefinitely small positive quantity, then the series will indicate that the generating function is infinitely great.

If x is sensibly greater than 1, we must put the series $-\left[\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\text{etc., to infinity}\right]$, for the given series; because the generating function has been expanded according to the ascending powers of x, when it ought to have been expanded according to the descending powers of x, or in the form $-[x^{-1}+x^{-2}+x^{-3}+\text{etc.}]$, (which are the descending powers, since they decrease by subtraction), and $-[x^{-1}+x^{-2}+x^{-3}+\text{etc.}] = -\left[\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\text{etc.}\right]$.

Now, since $-\left[\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\text{etc., to infinity}\right]$ is equal to $\frac{1}{1-x}$, we may put $\frac{1}{1-x}$ for $1+x+x^2+x^3+$ etc., to infinity, however great the value of x may be. Hence, we shall have in all cases $\frac{1}{1-x}=1+x+x^2+x^3+$ etc., to infinity (E), where it is to be observed if x is sensibly greater than 1, that ($=$) the sign of equality must not be understood to mean that the numerical value of the series is the same as that of $\frac{1}{1-x}$, but only that it is the expansion of $\frac{1}{1-x}$ according to the ascending powers of x, when it ought to have been the expansion according to the descending powers of x. What is here said does not prevent our using $1+x+x^2+x^3+$ etc., in calculation (when x is greater than 1), since we can put the series in (D) for it; or, which is the same, we may always put the generating function $\frac{1}{1-x}$ for it.

It may be shown, in a similar way, if we meet with the series $-\left[\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\text{etc., to infinity}\right]$ in any calculation, that we can always put $\frac{1}{1-x}$ for it, without having any regard to the value of x.

It is evident that similar observations must always have place, when we convert any fractional expression into an infinite series. It may be added that the great use of a series is to compute the value of the quantity of which it is the expansion, or which is the same, the generating function of the series; consequently, the series must be such that its numerical value shall equal that of the generating function when we reject indefinitely small quantities, so that we see the reasonableness of the preceding observations.

Ex. 2.—Divide 1 by $1+x$.

If we take 1 for the first term of the divisor, then we proceed as follows.

$$\frac{1}{1+x}=\frac{1+x-x}{1+x}=1-\frac{x}{1+x}=1-x+\frac{x^2}{1+x}=1-$$

$x + x^2 - \frac{x^3}{1+x} = 1 - x + x^2 - x^3 + \text{etc.}, \ldots \pm \frac{x^n}{1+x}$,(A); observing that n is a positive integer, and that for $\pm$ we must use $+$ if n is an even number, and that $-$ must be used for $\pm$ when n is an odd number.

If x is sensibly less than 1, then n may be taken so great that $\pm \frac{x^n}{1+x}$ shall be less than any finite quantity; consequently $\pm \frac{x^n}{1+x}$ may be rejected, and we shall have $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \text{etc.}$, (B), the law of continuation of the series being such that if we multiply any term by $-x$, the product will be the next successive term.

If x equals 1, or differs insensibly from 1, then $\pm \frac{x^n}{1+x}$ becomes sensibly $\pm \frac{1}{2}$, and of course can not be rejected however great n may be, so long as it is finite; and if we regard x as absolutely equal to 1, we can not reject $\pm \frac{x^n}{1+x}$ if we even suppose n to be infinite; because $\pm \frac{x^n}{1+x}$ will become absolutely $\pm \frac{1}{2}$.

Because x in (B) is less than 1, we shall have x greater than $xx = x^2$, and x^2 greater than $x^2x = x^3$, and so on; and since the terms are alternately $+$ and $-$, $1 - x$ is less than the value of the whole series, since we must add the positive quantity $x^2 - x^3 + x^4 - x^5 + \text{etc.}$, to it to make up the series; also $1 - x + x^2 - x^3$ is less than the value of the whole series, since we must add the positive quantity $x^4 - x^5 + x^6 - x^7 + \text{etc.}$ to it, to complete the series; and in this way we see that the sum of any even number of terms is less than the true value of the series. Also 1 is greater than the value of the series, since we must subtract the positive quantity $x - x^2 + x^3 - x^4 + \text{etc.}$ from 1, to get the series; and in the same way $1 - x + x^2$ is greater than the value of the whole series, since we must subtract the positive quantity $x^3 - x^4 + x^5 - x^6 + \text{etc.}$, from $1 - x + x^2$ to get the series; and in this manner it appears that the sum of any odd number of the successive terms of (B) is greater than the value of the whole series.

From what has been done, it follows that $\frac{1}{1+x}$ may be considered as having the sums of an even and odd number of the successive terms of the series (B) for certain *limiting values or limits*, since $\frac{1}{1+x}$ is greater than the sum of the even number, and less than the sum of the odd number of terms.

Hence, if we add any number of the terms of the series, and use the integer m to stand for the number of terms added, it is clear that the difference between $\frac{1}{1+x}$ and the sum of m terms of the series will be numerically less than the $(m+1)^{th}$ term of the series; since the sums of m and $(m+1)$ terms of the series are *limits* to the value of $\frac{1}{1+x}$.

It is clear that what has been said is applicable to any converging series whose terms are alternately + and —.

We will now convert $\frac{1}{1+x}$ into a series by taking x for the first term of the divisor;

since $\frac{1}{1+x} = \frac{1}{x+1} = \frac{\frac{1}{x}}{1+\frac{1}{x}} = \frac{1}{x} \times \left[\frac{1}{1+\frac{1}{x}}\right]$, and that (as in (A)) we have $\frac{1}{1+\frac{1}{x}} = 1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \text{etc.}, \ldots . \pm \frac{1}{x^n\left(1+\frac{1}{x}\right)}$; we shall get $\frac{1}{1+x} = \frac{1}{x}\left[1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \text{etc.}, \ldots . \pm \frac{1}{x^n\left(1+\frac{1}{x}\right)}\right] = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \ldots . \pm \frac{1}{x^n(1+x)}$

(C); observing that n is a positive integer, and that for ± we must use — when n is an even number, but if n is an odd number, then for ± we must use +.

If x is sensibly greater than 1, we may take n so great that $\pm \frac{1}{x^n(1+x)}$ may be rejected, and (C) will be reduced to

$\frac{1}{1+x} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \frac{1}{x^5} -$ etc., (D); which is the series to be used to calculate $\frac{1}{1+x}$, when x is sensibly greater than 1.

When x equals 1, or differs insensibly from it, we can not reject $\pm \frac{1}{x^n(1+x)}$, and of course we can not find $\frac{1}{1+x}$ from (D), but must use (C), or take the generating function for the series.

It follows, from what has been done, that when x is sensibly less than 1, we must use the series (B) to find $\frac{1}{1+x}$; but if x is sensibly greater than 1, we must use (D); and if x differs insensibly from 1 or equals 1, we must take the generating function itself, or use (A) or (C) to calculate it.

Ex. 3.—Divide a by $b - c$, by Rule III., or convert $\frac{a}{b-c}$ into a series.

Here, dividing the dividend and divisor by b, the first term of the divisor, we have $\frac{a}{b}$ for the new dividend, and $1 - \frac{c}{b}$ for the new divisor, and changing the sign of the second term of the new divisor, we have $1 + \frac{c}{b}$ for the changed divisor. Hence, by Rule III., we have

$$1 + \frac{c}{b} \,\Big|\, \frac{a}{b}$$

$$\frac{a}{b} + \frac{ac}{b^2} + \frac{ac^2}{b^3} + \frac{ac^3}{b^4} + \dots + \frac{\frac{a}{b} \times \left(\frac{c}{b}\right)^n}{1 - \frac{c}{b}}$$

= the quotient; observing that n is an integer which equals the number of terms that precede the remainder, which is expressed by $\frac{\frac{a}{b} \times \left(\frac{c}{b}\right)^n}{1 - \frac{c}{b}} = \frac{a \times \left(\frac{c}{b}\right)^n}{b - c}$.

Hence we have $\frac{a}{b-c} = \frac{a}{b} + \frac{ac}{b^2} + \frac{ac^2}{b^3} + \frac{ac^3}{b^4} + \dots\dots + \frac{a \times \left(\frac{c}{b}\right)^n}{b-c}$, (A), for the quotient or series, as required.

If $\frac{c}{b}$ is essentially less than 1, we may evidently take n so great that $\frac{a \times \left(\frac{c}{b}\right)^n}{b-c}$ may be rejected, which will reduce (A) to $\frac{a}{b-c} = \frac{a}{b} + \frac{ac}{b^2} + \frac{ac^2}{b^3} + \frac{ac^3}{b^4} +$ etc., to infinity, (B), which may be used to calculate the numerical value of $\frac{a}{b-c}$ when $\frac{c}{b}$ is sensibly less than 1; but if $\frac{c}{b}$ is sensibly greater than 1, we can not find the numerical value of $\frac{a}{b-c}$ by (B), and we must either use (A), or return to the generating function $\frac{a}{b-c}$.

If c is greater than b, we must take $-c$ for the first term of the divisor, and we have to divide a by $-c+b$, or dividing the divisor and dividend by $-c$, the first term of the divisor, we have $-\frac{a}{c}$ to be taken for the new dividend, and $1-\frac{b}{c}$ for the new divisor in Rule III., and changing the sign of the second term in the new divisor, we have $1+\frac{b}{c}$ for the changed divisor.

Hence, by Rule III., we shall get

$$\begin{array}{l|l} 1+\frac{b}{c} & -\frac{a}{c} \\ \hline \end{array}$$

$$-\left[\frac{a}{c} + \frac{ab}{c^2} + \frac{ab^2}{c^3} + \dots + \frac{\frac{a}{c} \times \left(\frac{b}{c}\right)^n}{1-\frac{b}{c}}\right] =$$

the quotient.

Hence, since $\frac{\frac{a}{c} \times \left(\frac{b}{c}\right)^n}{1-\frac{b}{c}} = \frac{a \times \left(\frac{b}{c}\right)^n}{c-b}$, we shall have $\frac{a}{b-c}$

$$= -\left[\frac{a}{c}+\frac{ab}{c^2}+\frac{ab^2}{c^3}+\ldots\ldots+\frac{a\times\left(\frac{b}{c}\right)^n}{c-b}\right],$$ (C), as required.

If $\frac{b}{c}$ is sensibly less than 1, we may take n so great that $\frac{a\times\left(\frac{b}{c}\right)^n}{c-b}$ may be rejected, which will reduce (C) to $\frac{a}{b-c}$

$$= -\left[\frac{a}{c}+\frac{ab}{c^2}+\frac{ab^2}{c^3}+\frac{ab^3}{c^4}+\text{etc., to infinity}\right],$$ (D).

Hence, if $\frac{b}{c}$ is sensibly less than 1, we may find the numerical value of $\frac{a}{b-c}$ from (D); but if $\frac{b}{c}$ is sensibly greater than 1, we may find the numerical value of $\frac{a}{b-c}$ from (B). If $\frac{b}{c}$ or $\frac{c}{b}$ differ insensibly from 1, $\frac{a}{b-c}$ will be indefinitely great, provided a is a finite quantity, which we suppose it to be.

If, in any calculation, we meet with the series (B), we may always put $\frac{a}{b-c}$ for it, and in like manner we may put $\frac{a}{b-c}$ for the series (D).

If, in (A), (B), (C), and (D), we change the sign of c, or put $-c$ for c, they will give $\frac{a}{b+c}$ when converted into a series, adapted to the cases of $\frac{c}{b}$ sensibly less than 1 in (A) and (B); and $\frac{b}{c}$ sensibly less than 1 in (C) and (D).

Where it is to be observed that similar remarks to those made in Ex. 2 will have place, they being so evident, we do not think it necessary to consider them here.

It is evident that any fraction whose denominator is a binomial, and whose numerator does not exactly contain its denominator, may be reduced to a series in a manner similar to that used in this example.

Ex. 4.—To convert $\frac{A}{a+bx}$ into a series, arranged according to the ascending powers of x, by Rule III.

Having divided the dividend and divisor, according to the rule, by a, the first term of the divisor, we have $\frac{A}{a}$ and $1+\frac{b}{a}x$ for the new dividend and divisor; and changing the sign of the second term of the new divisor, we have $1-\frac{b}{a}x$ for the changed divisor. Hence, by Rule III., we shall get

$$\begin{array}{l|l} 1-\frac{b}{a}x & \frac{A}{a} \\ \cline{2-2} & \frac{A}{a}-\frac{Ab}{a^2}x+\frac{Ab^2}{a^3}x^2-\frac{Ab^3}{a^4}x^3+\text{etc.}, = \text{the quotient;} \end{array}$$

consequently $\frac{A}{a+bx}=\frac{A}{a}-\frac{Ab}{a^2}x+\frac{Ab^2}{a^3}x^2-\frac{Ab^3}{a^4}x^3+$ etc., (A), as required.

It is easy to see that the law of formation of the series is such, that if we multiply any term by $-\frac{b}{a}x$, the product will equal the next successive term.

It may be observed that (A) is often called a *recurring series of the first order*, because in order to find any term we recur to the preceding term; also, $-\frac{b}{a}x$ is called *the scale of relation of any two terms*, because the term that precedes any term being multiplied by $-\frac{b}{a}x$, gives the term; and $-\frac{b}{a}$ is called *the scale of relation of the coefficients of any two successive powers of x.*

Ex. 5.—To expand $\frac{A+Bx}{a+bx+cx^2}$ into a series of ascending powers of x, by Rule III.

Here we find $\frac{A}{a}+\frac{B}{a}x$ for the new dividend, and $1-\frac{b}{a}x-\frac{c}{a}x^2$ for the changed divisor. Hence, by the rule, we get,

$$
\begin{array}{l|llll}
1-\frac{b}{a}x-\frac{c}{a}x^2 & \frac{A}{a} & +\frac{B}{a}x & & \\
 & & -\frac{Ab}{a^2}x & +\frac{A}{a}\times\left(-\frac{c}{a}x^2\right) & +\left.\begin{matrix}\frac{B}{a}\\-\frac{Ab}{a^2}\end{matrix}\right|x\times\left(-\frac{c}{a}x^2\right) \\
 & & & \left.\begin{matrix}\frac{B}{a}\\-\frac{Ab}{a^2}\end{matrix}\right|x\times\left(-\frac{b}{a}x\right) & +\left.\begin{matrix}\frac{A}{a}\times\left(-\frac{c}{a}x^2\right)\\ \left.\begin{matrix}\frac{B}{a}\\-\frac{Ab}{a^2}\end{matrix}\right|x\times\left(-\frac{b}{a}x\right)\end{matrix}\right|\times\left(-\frac{b}{a}x\right)+\text{etc.}
\end{array}
$$

---

and $\frac{A}{a}+\frac{aB-Ab}{a^2}.x+\frac{A(b^2-ac)-Bab}{a^3}.x^2+\frac{A(2abc-b^3)+B(ab^2-a^2c)}{a^4}.x^3$
$+$ etc. $=$ the quotient, as required.

For $\frac{A}{a}$ is clearly the first term of the quotient, which, multiplied by the second term $-\frac{b}{a}x$ of the changed divisor, gives $-\frac{Ab}{a^2}x$ to be added to the term $\frac{B}{a}x$ of the dividend for the second term of the quotient; also, the first term of the quotient, multiplied by the third term $-\frac{c}{a}x^2$ of the changed divisor, gives $\frac{A}{a}\times\left(-\frac{c}{a}x^2\right)$, which involves x^2, and is a part of the third term of the quotient; the other part of the third term of the quotient is got by multiplying the second term of the quotient, which is expressed by $\left.\begin{array}{r}\frac{B}{a}\\ -\frac{Ab}{a^2}\end{array}\right| x$ (using the vertical bar), into the second term of the changed divisor, and there results $\left.\begin{array}{r}\frac{B}{a}\\ -\frac{Ab}{a^2}\end{array}\right| x \times \left(-\frac{b}{a}x\right)$, which, added to $\frac{A}{a} \times \left(-\frac{c}{a}x^2\right)$, gives the third term of the quotient.

Then the second term of the quotient $\left.\begin{array}{r}\frac{B}{a}\\ -\frac{Ab}{a^2}\end{array}\right| x$, multiplied by the third term $-\frac{c}{a}x^2$ of the changed divisor, gives $\left.\begin{array}{r}\frac{B}{a}\\ -\frac{Ab}{a^2}\end{array}\right| x \times \left(-\frac{c}{a}x^2\right)$ for a part of the fourth term of the quotient which involves x^3; the other part of the fourth term of the quotient is expressed by

$$\left.\begin{array}{l}\frac{A}{a} \times \left(-\frac{c}{a}x^2\right)\\ \left.\begin{array}{r}\frac{B}{a}\\ -\frac{Ab}{a^2}\end{array}\right| x \times \left(-\frac{b}{a}x\right)\end{array}\right| \times \left(-\frac{b}{a}x\right),$$

and is obtained by multiplying the third term of the quotient (expressed by using the vertical bar) into the second term $-\frac{b}{a}x$ of the changed divisor.

In like manner, if we multiply the fourth term of the quotient by the second term of the changed divisor, and the third term of the quotient by the third term of the changed divisor, then by adding the products according to their signs, we shall have the fifth term; and in the same way we get the sixth term from the fifth and fourth terms, and so on.

From what has been done, we have $\frac{A + Bx}{a + bx + cx^2} = \frac{A}{a} + \frac{aB - bA}{a^2}.x + \frac{A(b^2 - ac) - Bab}{a^3}.x^2 + \frac{A(2abc - b^3) + B(ab^2 - a^2c)}{a^4}.x^3$ + etc., (A), for the series, as required.

Since any term of (A) is found by multiplying the next preceding term by $-\frac{b}{a}x$, and the next preceding term but one by $-\frac{c}{a}x^2$, and then adding the products, (A) is called a recurring series of the second order, whose scale of relation is said to be the multipliers $\left(-\frac{b}{a}x, -\frac{c}{a}x^2,\right)$, and $\left(-\frac{b}{a}, -, \frac{c}{a}\right)$ is called the scale of the coefficients, observing that the series is said to be of the second order, because any term after the second term is deduced from the two preceding terms by means of the two multipliers in the scale, as has been shown.

Universally, if we have any rational algebraic fraction whose numerator and denominator are arranged according to the ascending powers of a common letter, as x; then if the greatest exponent of x in the numerator is less than the greatest exponent of x in the denominator, we can expand the fraction into a recurring series arranged according to the ascending powers of x, in the same way as in Ex. 5, and the series will be of the same order as the number of multipliers in the scale of relation of its terms; and it is easy to see that the number of multipliers in the scale is always one less than the number of terms in the denominator of the developed fraction.

Hence, if we denote the number of terms in the denominator of the fraction by the integer $m + 1$, then m will denote the number of multipliers in the scale, and the series will be a recurring series of the m^{th} order; it is also clear that the m first terms of the series will not follow any particular law; so that the series properly begins to be recurring at the $(m + 1)^{th}$ term.

Thus, if we have the fraction $\frac{A + Bx + Cx^2}{a + bx + cx^2 + dx^3}$, then if we divide its numerator and denominator by a, the first term of the divisor, we shall have $\frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3}=$

$$\frac{\frac{A}{a}+\frac{B}{a}x+\frac{C}{a}x^2}{1+\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3}.$$

Hence, if we take $\frac{A}{a}+\frac{B}{a}x+\frac{C}{a}x^2$ for the new dividend, and $1+\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3$ for the new divisor in Rule III., we shall have $1-\frac{b}{a}x-\frac{c}{a}x^2-\frac{d}{a}x^3$ for the changed divisor. Then, if we divide as in Rule III., and denote the first three terms of the quotient by $A^1+B^1x+C^1x^2$, we shall get

$$1-\frac{b}{a}x-\frac{c}{a}x^2-\frac{d}{a}x^3\left|\begin{array}{l}\frac{A}{a}+\frac{B}{a}x+\frac{C}{a}x^2 \\ \hline A^1+B^1x+C^1x^2+\left(-\frac{d}{a}A^1-\frac{c}{a}B^1-\frac{b}{a}C^1\right)x^3\end{array}\right.$$

$+$ etc., $=$ the quotient, as is evident from Rule III.

It is easy to see that the scale of relation of the terms of the series is $\left(-\frac{b}{a}x,\ -\frac{c}{a}x^2,\ -\frac{d}{a}x^3,\right)$, so that the number of multipliers in the scale is one less than the number of terms in the denominator of the given fraction $\frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3}$, and of course we have $\frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3}=A^1+B^1x+C^1x^2+\left(-\frac{d}{a}A^1-\frac{c}{a}B^1-\frac{b}{a}C^1\right)x^3+\left[-\frac{d}{a}B^1-\frac{c}{a}C^1-\frac{b}{a}\left(-\frac{d}{a}A^1-\frac{c}{a}B^1-\frac{b}{a}C^1\right)\right]x^4+$ etc., which is a recurring series of the third order, in conformity to what has been said.

Resuming the fraction

$\frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3}=\frac{Cx^2+Bx+A}{dx^3+cx^2+bx+a}$; if we divide its numerator and denominator by dx^3, we get

$\frac{A+Bx+Cx^2}{a+bx+cx^2+dx^3}=\frac{\frac{C}{dx}+\frac{B}{dx^2}+\frac{A}{dx^3}}{1+\frac{c}{d}\cdot\frac{1}{x}+\frac{b}{d}\cdot\frac{1}{x^2}+\frac{a}{d}\cdot\frac{1}{x^3}}$, or if (for simplicity) we denote $\frac{1}{x}$ by y we have $\frac{A + Bx + Cx^2}{a + bx + cx^2 + dx^3} =$

$$y \times \frac{\frac{C}{d} + \frac{B}{d}y + \frac{A}{d}y^2}{1 + \frac{c}{d}y + \frac{b}{d}y^2 + \frac{a}{d}y^3}.$$

Hence, if we divide $\frac{C}{d} + \frac{B}{d}y + \frac{A}{d}y^2$ by $1 + \frac{c}{d}y + \frac{b}{d}y^2 + \frac{a}{d}y^3$, by Rule III., we shall have a recurring series, whose scale of relation is $\left(-\frac{c}{d}y, -\frac{b}{d}y^2, -\frac{a}{d}y^3,\right)$, and if we multiply each term of the series by y, and then put $\frac{1}{x}$ for y in the series, we shall have the fraction $\frac{A + Bx + Cx^2}{a + bx + cx^2 + dx^3}$ developed into a recurring series arranged according to the descending powers of x.

In like manner we may develop any fraction (of the kind we have considered) either according to the ascending or descending powers of x; and each series thus found will be a recurring series.

If we wish to obtain the numerical value of the fraction from the series, we must use that one of the two whose terms converge, as is evident from what was formerly shown.

We shall now show how to treat a fraction that has a term in its numerator independent of x, and the greatest exponent of x in the numerator not less than the greatest exponent of x in the denominator.

Thus, let $\frac{A + Bx + Cx^2 + Dx^3}{a + bx + cx^2}$ stand for the fraction, then if we divide the numerator by the denominator (after the denominator and numerator have been arranged according to the descending powers of x) we shall get

$$\frac{A + Bx + Cx^2 + Dx^3}{a + bx + cx^2} = \frac{Dx}{c} + \frac{Cc - Db}{c^2} + \frac{A^1 + B^1x}{a + bx + cx^2},$$

where we have (for simplicity) represented $\frac{Ac^2 + Dab - Cac}{c^2}$ by A^1, and $\frac{Bc^2 + D(b^2 - ac) - Cbc}{c^2}$ by B^1; so that the given fraction is reduced to terms that are freed from the given denominator, and to the fraction $\frac{A^1 + B^1x}{a + bx + cx^2}$, whose denominator is the denominator of the given fraction; provided always, that the numerator of the given fraction is not exactly divisible by the denominator.

We can now convert $\frac{A^1 + B^1x}{a + bx + cx^2}$ into a recurring series, which is arranged either according to the ascending or descending powers of x, by the methods that have been previously given.

It is clear that any rational algebraic fraction of the same kind (as that above) may be treated in a similar way.

EXAMPLES FOR EXERCISE.

1. Convert $\frac{1 - x}{1 + x}$ into a series of ascending powers of x.

Ans. $\frac{1 - x}{1 + x} = 1 - 2x + 2x^2 - 2x^3 + \text{etc.}$

2. Expand $\frac{1}{3x - 1}$ into a series of descending powers of x.

Ans. $\frac{1}{3x - 1} = \frac{1}{3x} + \left(\frac{1}{3x}\right)^2 + \left(\frac{1}{3x}\right)^3 + \text{etc.}$

3. Expand $\frac{1 - 3x}{1 + 2x^2}$ according to the ascending powers of x.

Ans. $\frac{1 - 3x}{1 + 2x^2} = 1 - 3x - 2x^2 + 6x^3 + 4x^4 - 12x^5 - 8x^6 + 24x^7 + 16x^8 - \text{etc.}$

4. Develop $\frac{3 + 2x}{1 - x + x^2}$ into a series of ascending powers of x.

Ans. $\frac{3 + 2x}{1 - x + x^2} = 3 + 5x + 2x^2 - 3x^3 - 5x^4 - 2x^5 + 3x^6 + 5x^7 + 2x^8 - \text{etc.}$

5. Develop the same fraction, $\frac{3 + 2x}{1 - x + x^2}$, into a series of descending powers of x.

Ans. $\frac{3 + 2x}{1 - x + x^2} = \frac{2}{x} + \frac{5}{x^2} + \frac{3}{x^3} - \frac{2}{x^4} - \frac{5}{x^5} - \frac{3}{x^6} + \frac{2}{x^7} + \frac{5}{x^8} + \frac{3}{x^9} - \text{etc.}$

6. Convert $\frac{1}{1 - 2x + x^2}$ into a series of ascending powers of x.

Ans. $\frac{1}{1 - 2x + x^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 +$ etc.

7. Expand $\frac{1}{1 + 3x + 3x^2 + x^3}$ into a series of ascending powers of x.

Ans. $\frac{1}{1 + 3x + 3x^2 + x^3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 +$ etc.

We will now proceed to treat of some cases of incomplete division, which will enable us to change certain algebraic expressions into new forms, which will be useful to us hereafter.

EXAMPLES.

Ex. 1.—To divide $ax + b$ by $x - m$.

Here we have

$$\begin{array}{r|l|l} x - m & ax + b & a \\ \cline{1-1} \cline{3-3} & ax - am & \\ \cline{2-2} & \qquad am + b & \end{array}$$

= the remainder, which is the same as the given dividend if we put m for x; and since the dividend must (from the nature of division) equal the product of the divisor and quotient together with the remainder, we shall have $ax + b = a(x - m) + am + b$; so that $ax + b$ is changed to $a(x - m) + am + b$.

Ex. 2.—To divide $ax^2 + bx + c$ by $x - m$

Here we get

$$\begin{array}{r|l|l} x - m & ax^2 + bx + c & ax + am + b \\ \cline{1-1} \cline{3-3} & ax^2 - amx & \\ \cline{2-2} & \qquad (am + b)x + c & \\ & \qquad (am + b)x - am^2 - mb & \\ \cline{2-3} & \qquad\qquad am^2 + mb + c & \end{array}$$

= the remainder, which can be immediately obtained from the given dividend by changing x into m.

By the first example, we can change the quotient $ax + am + b$ into $a(x - m) + 2am + b$; and we shall of course have $ax^2 + bx + c = a(x-m)^2 + (2am+b)(x-m) + am^2 + bm + c$ for the transformation of $ax^2 + bx + c$.

Ex. 3.—To divide $ax^3 + bx^2 + cx + d$ by $x - m$.

Here we have

$$
\begin{array}{l|l|l}
x-m & ax^3 + bx^2 + cx + d & ax^2 + (am+b)x + am^2 + bm + c \\
\cline{1-1}\cline{3-3}
 & ax^3 - amx^2 & \\
\cline{2-2}
 & \quad (am+b)x^2 + cx & \\
 & \quad (am+b)x^2 - (am^2+bm)x & \\
\cline{2-2}
 & \qquad (am^2+bm+c)x+d & \\
 & \qquad (am^2+bm+c)x-am^3-bm^2-cm & \\
\cline{2-2}
 & \qquad\qquad am^3+bm^2+cm+d &
\end{array}
$$

= the remainder, which is deduced at once from the given dividend by changing x into m.

If in the last example we change b into $am + b$, and c into $am^2 + bm + c$, $ax^2 + bx + c$ will become $ax^2 + (am+b)x + am^2 + bm + c = a(x-m)^2 + (3am+b)(x-m) + 3am^2 + 2bm + c$ = the transformation of our quotient.

Hence, we deduce $ax^3 + bx^2 + cx + d = a(x-m)^3 + (3am + b)(x-m)^2 + (3am^2 + 2bm + c)(x-m) + am^3 + bm^2 + cm + d$, which is the transformation of $ax^3 + bx^2 + cx + d$.

Ex. 4.—To transform $Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots . + Sx + T$, (A), so that x shall be changed into $x - m$, supposing n to stand for a positive whole number, and the coefficients A, B, C, ... T to be independent of x.

The transformation may clearly be effected as in the preceding examples; but we shall give another method, which is in some respects more simple.

Thus, since $x = x - m + m$, if we represent $x - m$ by y, we have $x = y + m$, so that we may evidently put $y + m$ for x in (A), which will change it to $A(y+m)^n + B(y+m)^{n-1} + C(y+m)^{n-2} + \ldots . + S(y+m) + T$, (B).

Because $(y+m)^n = (y+m) \times (y+m)^{n-1} = y(y+m)^{n-1} + m(y+m)^{n-1}$, we get $A(y+m)^n = y[A(y+m)^{n-1}] + Am(y+m)^{n-1}$, (1); and adding $B(y+m)^{n-1}$ to both sides of (1), we get $A(y+m)^n + B(y+m)^{n-1} = y[A(y+m)^{n-1}] + (Am + B)(y+m)^{n-1}$, (2); $\therefore$ if we change A in (1) into $Am + B$, and n into $n-1$, we get $(Am+B)(y+m)^{n-1} = y[(Am+B)(y+m)^{n-2}] + (Am^2+Bm)(y+m)^{n-2}$; consequently, if we add $C(y+m)^{n-2}$ to both sides of (2), we get $A(y+m)^n + B(y+m)^{n-1} + C(y+m)^{n-2} = y[A(y+m)^{n-1} + (Am+B)(y+m)^{n-2}] + (Am^2+Bm+C)(y+m)^{n-2}$, (3).

If we put $Am^2 + Bm + C$ for A in (1), and change n into $n-2$, it becomes $(Am^2+Bm+C)(y+m)^{n-2} = y[(Am^2+Bm+C)(y+m)^{n-3}] + (Am^3+Bm^2+Cm)(y+m)^{n-3}$; consequently, if we add $D(y+m)^{n-3}$ to both sides of (3), we shall get $A(y+m)^n + B(y+m)^{n-1} + C(y+m)^{n-2} + D(y+m)^{n-3} = y[A(y+m)^{n-1} + (Am+B)(y+m)^{n-2} + (Am^2+Bm+C)(y+m)^{n-3}] + (Am^3+Bm^2+Cm+D)(y+m)^{n-3}$, (4).

It is easy to see, from the law of the preceding transformations, that if we continue the above process until it has been repeated n times, we shall have $A(y+m)^n + B(y+m)^{n-1} + C(y+m)^{n-2} + \ldots\ldots + S(y+m) + T = y[A(y+m)^{n-1} + (Am+B)(y+m)^{n-2} + (Am^2+Bm+C)(y+m)^{n-3} + (Am^3 + Bm^2 + Cm + D)(y+m)^{n-4} + \ldots\ldots + (Am^{n-2} + Bm^{n-3} + Cm^{n-4} + \ldots\ldots + R)(y+m) + Am^{n-1} + Bm^{n-2} + Cm^{n-3} + \ldots\ldots + S] + Am^n + Bm^{n-1} + Cm^{n-2} + \ldots\ldots + Sm + T$, (C).

The quantity $Am^n + Bm^{n-1} + Cm^{n-2} + \ldots\ldots + Sm + T$, which is independent of y in (C), is evidently the remainder which the division of (A) by $x - m$ would give, and it is deduced immediately from (A) by changing x into m.

Hence, in order that (A) may be exactly divisible by $x - m$, it will be necessary that m should be such as to reduce the aforesaid remainder to 0; or, which is the same, m must be a root of the equation $Am^n + Bm^{n-1} + Cm^{n-2} + \ldots\ldots + Sm + T = 0$; which is in conformity to the remarks made under Ex. 4 of the third case of Multiplication.

The quantity $A(y+m)^{n-1} + (Am+B)(y+m)^{n-2} + (Am^2 + Bm + C)(y+m)^{n-3} + \text{etc.}\ldots\ldots + (Am^{n-2} + Bm^{n-3} + \ldots\ldots + R)(y+m) + Am^{n-1} + Bm^{n-2} + \ldots\ldots + S$, which is within the braces in the right member of (C), and which is multiplied by y, is evidently of the same form as (B), and may be transformed by (C) merely by putting $n-1$ for n, $Am + B$ for B, $Am^2 + Bm + C$ for C, and so on; then the part of the transformation which is independent of y will clearly be the coefficient of y in the transformation of (B), as required; and it is easy to see that $nAm^{n-1} + (n-1)Bm^{n-2} + (n-2)Cm^{n-3} + \dots + 2Rm + S$ is the coefficient of y.

The part of the last transformation that has y for a factor (as in (C)) being transformed as before by (C), the part of it which is independent of y will clearly be the coefficient of y^2, and so on, until the coefficients of all the powers of y in the transformation of (B) are obtained. It will not, however, be necessary formally to obtain them; for we see from what has been done, that they can easily be found by the following very simple rule.

RULE.

Write the coefficients of (A) *or* (B) *in a horizontal line, supplying any term that may be wanting in the given quantity* (A) (*to make the exponents of x decrease by a unit as we pass from any term to the next successive term*) *by* 0, *thus we have* $A + B + C + \dots + S + T$, (D).

Then multiply A *by* m, *according to their signs, and add* B *to the product, and denote the result by* B^1, *and we have* $Am + B = B^1$; *multiply* B^1 *by* m, *according to their signs, and add* C *to the product, and denote the result by* C^1, *and we have* $Am^2 + Bm + C = B^1m + C = C^1$; *proceed in the same way as before with* C^1 *and* D, *and so on; and the result, whose number is denoted by the number of units in* n, *will be the quantity in the transformed expression of* (B), *which is independent of* y.

The coefficient of y is found in the same way, by using B^1, C^1, etc., instead of B, C, etc.; observing that the coefficient of y is the result whose number is $n - 1$.

If we put $Am + B^{I} = B^{II}$, $B^{II}m + C^{I} = C^{II}$, $C^{II}m + D^{I} = D^{II}$ and so on; then using A, B^{II}, C^{II}, etc., as before, and taking the result whose number is $n - 2$, we have the coefficient of y^2. Proceeding in like manner with the preceding results, we get the coefficient of y^3, and so on, for the coefficients of y^4, y^5, etc., until all the coefficients are found.

By writing the coefficients, (D), of the given equation, in the form of a dividend, and m for the corresponding quotient, we may express the rule, and the manner of applying it to particular examples, in the following

SCHEME.

$$\begin{array}{l|l}
A+B+C\quad+D\quad\ldots\ldots+R\quad+S\quad+T & m \\
\underline{\quad Am+B^{\text{I}}m+C^{\text{I}}m+\ldots\ldots+Q^{\text{I}}m+R^{\text{I}}m+S^{\text{I}}m} & \\
\quad B^{\text{I}}+C^{\text{I}}\quad+D^{\text{I}}\quad+\ldots\ldots+R^{\text{I}}\quad+S^{\text{I}}\quad+T^{\text{I}} & \\
\multicolumn{2}{l}{\underline{\quad Am+B^{\text{II}}m+C^{\text{II}}m+\ldots\ldots+Q^{\text{II}}m+R^{\text{II}}m}} \\
\multicolumn{2}{l}{\quad B^{\text{II}}+C^{\text{II}}\quad+D^{\text{II}}\quad+\ldots\ldots+R^{\text{II}}\quad+S^{\text{II}}} \\
\multicolumn{2}{l}{\underline{\quad Am+B^{\text{III}}m+C^{\text{III}}m+\ldots\ldots+Q^{\text{III}}m}} \\
\multicolumn{2}{l}{\quad B^{\text{III}}+C^{\text{III}}\quad+D^{\text{III}}\quad+\ldots\ldots+R^{\text{III}}} \\
\multicolumn{2}{l}{\underline{\quad Am+B^{\text{IV}}m+\text{etc.}\quad+\text{etc.}}} \\
\multicolumn{2}{l}{\quad B^{\text{IV}}+C^{\text{IV}}\quad+\text{etc.}\quad+\text{etc.}}
\end{array} \qquad (E).$$

and so on, until we arrive at the coefficient of y^n in the transformed equation, which (coefficient) is clearly A, the coefficient of x^n in the expression to be transformed. Where it is to be observed that T^{I} is used to express the term of the transformed equation which is independent of y, and S^{II} expresses the coefficient of y, and R^{III} that of y^2, and so on.

Remark.—Our rule is substantially the same as that given by W. G. Horner, Esq. (of Bath, England), for the purpose of solving equations; but it has been obtained by a much more simple and obvious method.

Horner's method consists in dividing (A) the equation to be transformed by $x - m$, then the remainder is the term of the transformed equation which is independent of y. Then he divides the quotient by $x - m$, and the remainder is the coefficient of y. And dividing the last quotient by $x - m$, the remainder is the coefficient of y^2. Proceeding in the same way with the last quotient, we get the coefficient of y^3, and so on for the coefficients of y^4, y^5, etc.

Horner abridges the required divisions very much by detaching the coefficients as in (D), and then dividing by $1 - m$, by Synthetic Division, or Rule III.

We will now illustrate the rule by some particular examples, which we shall perform according to the scheme, (E).

EXAMPLES.

Ex. 1.—To transform $3x^2 + 5x + 7$, supposing that $x = y + 2$.

Here we have $n = 2$, $m = 2$, $A = 3$, $B = 5$, $C = 7$; then by (E) we have the following process.

$$\begin{array}{rrr|l}
3 & +5 & +7 & 2 \\
 & 6 & +22 & \\ \cline{1-3}
 & 11 & +29 & \\
 & 6 & & \\ \cline{1-3}
 & 17 & &
\end{array}$$

Then, since 29 corresponds to T^{I} in (E), and 17 to S^{II}, and that 3 is the coefficient of y^2, we have $3y^2 + 17y + 29$ for the transformation, as required.

Ex. 2.—To transform $x^3 - 3x^2 + 2x - 11$, when we put $x = y + 2$.

Here $n = 3$, $m = 2$, $A = 1$, $B = -3$, $C = 2$, $D = -11$, and we have, by proceeding as in the scheme, (E), the following work.

$$\begin{array}{rrrr|l}
1 & -3 & +2 & -11 & 2 \\
 & 2 & -2 & +0 & \\ \cline{1-4}
 & -1 & +0 & -11 & \\
 & 2 & +2 & & \\ \cline{1-3}
 & 1 & +2 & & \\
 & 2 & & & \\ \cline{1-2}
 & 3 & & &
\end{array}$$

Hence, since -11 corresponds to T^{I}, and 2 to S^{II}, 3 to R^{III} in the scheme, and that the coefficient of x^3 in the given expression is 1, we shall have $y^3 + 3y^2 + 2y - 11$ for the sought transformation.

Ex. 3.—To transform $x^5 - 10x^3 + 6x + 1$, by putting $x = y - 3$.

Here we have $n = 5$, $m = -3$, $A = 1$, $B = 0$, $C = -10$, $D = 0$, $E = 6$, $F = 1$; hence, proceeding as in (E), we get

$$\begin{array}{rrrrrr|l}
1 & +0 & -10 & +0 & +6 & +1 & -3 \\
 & -3 & +9 & +3 & -9 & +9 & \\ \cline{1-6}
 & -3 & -1 & +3 & -3 & +10 & \\
 & -3 & +18 & -51 & +144 & & \\ \cline{1-5}
 & -6 & +17 & -48 & +141 & & \\
 & -3 & +27 & -132 & & & \\ \cline{1-4}
 & -9 & +44 & -180 & & & \\
 & -3 & +36 & & & & \\ \cline{1-3}
 & -12 & +80 & & & & \\
 & -3 & & & & & \\ \cline{1-2}
 & -15 & & & & &
\end{array}$$

Since 10, 141, -180, 80, -15, severally correspond to T^{I}, S^{II}, R^{III}, Q^{IV}, P^{V} in the scheme, and that $A = 1$, we shall have $y^5 - 15y^4 + 80y^3 - 180y^2 + 141y + 10$ for the required transformation.

Ex. 4.—To transform x^2, when $x = y + m$.

In this case we have $n = 2$, $A = 1$, $B = 0$, $C = 0$; consequently we get

$$\begin{array}{ccccc|l} 1 & + & 0 & + & 0 & m \\ & & m & + & m^2 & \\ \hline & & m & & & \\ & & m & & & \\ \cline{3-3} & & 2m & & & \end{array}$$

hence $y^2 + 2my + m^2$ is the required transformation; and we have $x^2 = (y + m)^2 = y^2 + 2my + m^2$, a well-known result.

Ex. 5.—To transform x^3 when $x = y - m$.

In this instance $n = 3$, and for m we must use $-m$; also $A = 1$, $B = 0$, $C = 0$, $D = 0$; consequently, we shall get

$$\begin{array}{ccccccc|l} 1 & + & 0 & + & 0 & + & 0 & -m \\ & - & m & + & m^2 & - & m^3 & \\ & - & m & + & 2m^2 & & & \\ \hline & - & 2m & + & 3m^2 & & & \\ & - & m & & & & & \\ \cline{1-3} & - & 3m & & & & & \end{array}$$

Hence, since $-m^3$, $3m^2$, $-3m$ severally correspond to T^{I}, S^{II}, R^{III} in the scheme, and that $A = 1$, we get $x^3 = (y - m)^3 = y^3 - 3my^2 + 3m^2y - m^3$, a well-known result.

Ex. 6.—Transform $x^2 - 36$, when $x = y + 6$.

Here we have $n = 2$, $m = 6$, $A = 1$, $B = 0$, $C = -36$; and by (E) we get

$$\begin{array}{ccccc|l} 1 & + & 0 & - & 36 & 6 \\ & & 6 & + & 36 & \\ \hline & & 6 & + & 0 & \\ & & 6 & & & \\ \cline{1-3} & & 12 & & & \end{array}$$

Hence, since 0, 12 correspond to T^{I}, S^{II} in (E), we get

$y^2 + 12y$ for the sought transformation; and we shall have $y^2 + 12y = x^2 - 36$.

If $x^2 - 36 = 0$, or $x^2 = 36$, then we must have $y^2 + 12y = (y + 12)y = 0$, which is satisfied either by putting $y = 0$, or $y + 12 = 0$. When $y = 0$, we get $x = y + 6 = 0 + 6 = 6$, and since $x^2 = 36$, it follows that $x = 6$ is the square root of 36. Also, $y + 12 = 0$ gives $y = -12$, and since $x = y + 6$, we must have $x = -12 + 6 = -6$; consequently, -6 must also be a square root of 36. Hence, we see that the square root of 36 may be either $+6$ or -6.

Ex. 7.—Transform $x^3 - 27$ when $x = y + 3$.

Here $m = 3$, $n = 3$, $A = 1$, $B = 0$, $C = 0$, $D = -27$, and by (E) we get

$$\begin{array}{rrrrl}
1 & +0 & +\ 0 & -27\,\vert & 3\\
 & 3 & +\ 9 & +27\,\vert & \\
\hline
 & 3 & +\ 9 & +\ 0 & \\
 & 3 & +18 & & \\
\cline{1-3}
 & 6 & +27 & & \\
 & 3 & & & \\
\cline{1-2}
 & 9 & & &
\end{array}$$

Hence, since 0, 27, 9 correspond to T^{I}, S^{II}, R^{III}, in (E), we get $y^3 + 9y^2 + 27y$ for the sought transformation. Hence we shall have $y^3 + 9y^2 + 27y = x^3 - 27$, and if $x^3 - 27 = 0$, or $x^3 = 27$, we must have $y^3 + 9y^2 + 27y = (y^2 + 9y + 27)y = 0$, which is satisfied by putting $y = 0$, or $y^2 + 9y + 27 = 0$; the expression $y = 0$ gives $x = y + 3 = 0 + 3 = 3$. Consequently, since $x^3 = 27$, we must have $x = 3 =$ the third or cube root of 27.

Remark.—We see from this and the preceding example that our rule may be applied to the extraction of the roots of numbers.

Ex. 8.—Transform $x^2 - 7x + 12$, when $x = y + 4$.

Here $n = 2$, $m = 4$, $A = 1$, $B = -7$, $C = 12$, and by (E) we shall get

$$\begin{array}{rrrl}
1 & -7 & +12\,\vert & 4\\
 & 4 & -12\,\vert & \\
\hline
 & -3 & +\ 0 & \\
 & 4 & & \\
\cline{1-2}
 & 1 & &
\end{array}$$

Hence we have $y^2 + y$ for the sought transformation; and of course we must have $y^2 + y = x^2 - 7x + 12$. Consequently, if $x^2 - 7x + 12 = 0$, we must have $y^2 + y = (y+1)y = 0$, which is satisfied either by $y = 0$, or $y + 1 = 0$. The value $y = 0$ gives $x = y + 4 = 4$, and the value $y + 1 = 0$ gives $y = -1$, and $x = y + 4 = 3$, and we have $x - 4 = 0$, or $x - 3 = 0$. Hence $x - 4$ and $x - 3$ must be the factors of $x^2 - 7x + 12$; indeed, we have $(x - 4)(x - 3) = x^2 - 7x + 12$.

Ex. 9.—Transform $x^3 - 3x^2 + 2x - 6$, when $x = y + 3$.

Here we have $n = 3$, $m = 3$, $A = 1$, $B = -3$, $C = 2$, $D = -6$; hence, by (E), we get

$$\begin{array}{rrrr|l}
1 - & 3 + & 2 - & 6 & 3 \\
 & 3 + & 0 + & 6 & \\ \hline
 & 0 + & 2 + & 0 & \\
 & 3 + & 9 & & \\ \cline{1-3}
 & 3 + & 11 & & \\
 & 3 & & & \\ \cline{1-2}
 & 6 & & &
\end{array}$$

Consequently, since 0, 11, 6, correspond to T^{r}, S^{n}, R^{m}, and that the coefficient of x^3 is 1, we have $y^3 + 6y^2 + 11y$ for the required transformation.

If $x^3 - 3x^2 + 2x - 6 = 0$, then $y^3 + 6y^2 + 11y = (y^2 + 6y + 11)y = 0$; which is satisfied by putting $y = 0$, or $y^2 + 6y + 11 = 0$; the value $y = 0$ reduces $x = y + 3$ to $x = 3$ or $x - 3 = 0$; and consequently $x - 3$ is a factor of $x^3 - 3x^2 + 2x - 6$; indeed we have $x^3 - 3x^2 + 2x - 6 = (x - 3)(x^2+2)$.

Remark.—It is clear from this, and the preceding examples, that our rule can be applied to the resolution of equations; or, which comes to the same thing, to the resolution of quantities of the form (A) into their factors. And the principle is, so to assume m in $x = y + m$, that the term which does not involve y in the transformed equation shall equal 0, or differ from 0 by a quantity that may be rejected as of no consequence in the required degree of exactness of x.

EXAMPLES FOR EXERCISE.

1. Transform $5x^2 - 7x - 50$, when $x = y + 4$.

Ans. $5y^2 + 33y + 2$ is the transformation.

2. Transform $x^3 - 9$, when $x = y + 2$.
Ans. $y^3 + 6y^2 + 12y - 1$.

3. Transform $x^3 - 7x - 7$, when $x = y + 3$.
Ans. $y^3 + 9y^2 + 20y - 1$.

4. Transform $x^3 + 11x^2 - 102x + 181$, when $x = y - 17$.
Ans. $y^3 - 40y^2 + 391y + 181$.

5. Transform $x^4 + 2x^3 - 7x^2 - 8x + 12$, when $x = y + 1$.
Ans. $y^4 + 6y^3 + 5y^2 - 12y$.

6. Transform $x^4 - 19x^3 + 132x^2 - 302x + 200$, when $x = y + 2$.
Ans. $y^4 - 11y^3 + 42y^2 + 30y - 12$.

7. Transform $x^2 + 2ax - b$, when $x = y - a$.
Ans. $y^2 - (a^2 + b)$.

8. Transform $x^3 - 3ax^2 + bx - c$, when $x = y + a$.
Ans. $y^3 - (3a^2 - b)y - (2a^3 - ab + c)$.

(11.) APPENDIX TO MULTIPLICATION AND DIVISION.

We have, in Multiplication and Division, supposed the multiplier and divisor (each) in every operation to retain the same form. But it is easy to see that we may, if we please, continually change the form of the multiplier or divisor (at will) in any case or during the same operation; provided always, the *value* of the multiplier or divisor is not changed.

We will now give some examples for the purpose of showing that it will often be useful to change the forms of the multiplier or divisor (continually), in order to transform expressions that may occur in calculation.

EXAMPLES.

Ex. 1.—Let m stand for any positive integer, and suppose it is required to find the sum of the series $(m - 1) + (m - 2) + (m - 3) + (m - 4) + \ldots + 1$, (1), which is called a decreasing arithmetical progression; 1, the difference between any two successive terms, being called the ratio of decrease; also $(m - 1)$ and 1 are called its first and last terms, or the extremes of the series, and since $(m - 1)$ contains as many units as there are terms in the series, $(m - 1)$ is called the number of terms in the progression.

To find the sum of the series (1), we shall multiply each of its terms by 2, when put under the form $2 = m - (m - 2)$ for the multiplier of the first term $(m - 1)$ of the series, and under the form $2 = (m - 1) - (m - 3)$ for the multiplier of the second term $(m - 2)$, and $2 = (m - 2) - (m - 4)$ is the multiplier of the third term $(m - 3)$, and so on; so that we continually change the form of the multiplier without changing its value.

The product of the first term of the series and its multiplier is $m(m - 1) - (m - 1)(m - 2)$; and if in this we change m into $m - 1$, we shall have $(m - 1)(m - 2) - (m - 2)(m - 3)$ for the product of the second term of the series and its multiplier; and if in the last product we change m into $m - 1$, we get $(m - 2)(m - 3) - (m - 3)(m - 4)$, and so on to the last product, which is expressed by $[m - (m - 2)] . [m - (m - 1)] - [m - (m - 1)] . [m - m]$; whose last term is 0, since its factor $m - m$ equals 0.

Hence, adding the preceding products, by putting corresponding terms in vertical columns under each other, we get the following results.

$$\begin{array}{llllll}
m(m-1) & -(m-1)(m-2) & & & & \\
 & (m-1)(m-2) & -(m-2)(m-3) & & & \\
 & & (m-2)(m-3) & -(m-3)(m-4) & & \\
 & & & (m-3)(m-4) & - & \\
 & & & & & (m-4)(m-5) \\
 & & & & & \cdots\cdots\cdots
\end{array}$$

and so on to the last product inclusive, whose sum is $m(m - 1)$; since the terms in vertical columns destroy each other, and that the last term of the last product is 0.

Hence, since twice the series is transformed to $m(m - 1)$, we shall have $(m - 1) + (m - 2) + (m - 3) + (m - 4) + \ldots . + 1 = \frac{m(m - 1)}{2}$, (2), as required.

Because $m = (m - 1) + 1 =$ the sum of the extreme terms of the series, and that $m - 1 =$ the number of terms, we deduce from (2) the following rule for finding the sum of the terms of the series.

RULE.

The sum of the terms of the series equals half the product of the sum of the extreme terms by the number of terms.

Ex. 2.—To transform the series $(m-1)(m-2)+(m-2)(m-3)+(m-3)(m-4)+(m-4)(m-5)+\dots\dots+[m-(m-2)]\,.\,[m-(m-1)]$, (1), in such a way that its sum can be found.

Here we multiply each term of the series by 3, which is to be put under the form $m-(m-3)$ for the multiplier of the first term $(m-1)(m-2)$ of the series; and $(m-1)-(m-4)$ is the multiplier of the second term $(m-2)(m-3)$; and $(m-2)-(m-5)$ is the multiplier of the third term $(m-3)(m-4)$, and so on to $[m-(m-3)]-(m-m)$, which is the multiplier of the last term.

The product of the first term of the series and its multiplier is $m(m-1)(m-2)-(m-1)(m-2)(m-3)$, and putting in this $m-1$ for m, we have $(m-1)(m-2)(m-3)-(m-2)(m-3)(m-4)$ for the product of the second term of the series and its multiplier; and changing in this last product m into $m-1$, we have $(m-2)(m-3)(m-4)-(m-3)(m-4)(m-5)$ for the product of the third term of the series and its multiplier, and so on, to the product of the last term of the series and its multiplier, which is $[m-(m-3)]\,.\,[m-(m-2)]\,.\,[m-(m-1)]$, by omitting $m-m=0$ in the multiplier.

Hence, if we add all the preceding products, as in the preceding example, their sum will be reduced to $m(m-1)(m-2)$, which is of course equal to 3 times the sum of all the terms of the given series; consequently we have $(m-1)(m-2)+(m-2)(m-3)+(m-3)(m-4)+(m-4)(m-5)+\dots\dots+[m-(m-2)]\,.\,[m-(m-1)]=\dfrac{m(m-1)(m-2)}{3}$, (2), as required.

Ex. 3.—To transform the series $(m-1)(m-2)(m-3)+(m-2)(m-3)(m-4)+(m-3)(m-4)(m-5)+\dots\dots+[m-(m-3)]\,.\,[m-(m-2)]\,.\,[m-(m-1)]$ (1).

Here we must multiply each term by 4, which is greater by a unit than the number of factors in each of the terms of the given series.

For the multiplier of the first term of the series we must put $4=m-(m-4)$, and for the multiplier of the second term of the series we must put $4=(m-1)-(m-5)$, and $4=(m-2)-(m-6)$ is the multiplier of the third term, and so on. Hence, (as in the preceding examples), we easily get $(m-1)(m-2)(m-3) + (m-2)(m-3)(m-4) + (m-3)(m-4)(m-5) + \dots + [m-(m-3)] \times [m-(m-2] . [m-(m-1)] = \frac{m(m-1)(m-2)(m-3)}{4}$, (2), as required.

Universally, if n is any positive integer, and we have the series of successive products, each containing n factors, as $(m-1)(m-2)(m-3) \times \dots \times (m-n) + (m-2)(m-3)(m-4) \times \dots \times [m-(n+1)] + (m-3)(m-4)(m-5) \times \dots \times [m-(n+2)] + (m-4)(m-5)(m-6) \times \dots \times [m-(n+3)] + \dots + [m-(m-n)] . [m-(m-n+1)] . [m-(m-n+2)] \times \dots \times [m-(m-1)]$, such that any term of the series is deduced from the preceding term by changing m into $m-1$; then the multiplier of each term must be $n+1$, which is greater than the number of factors in each term of the series by a unit.

For the multiplier of the first term of the series, we must put $n+1 = m-(m-n-1)$; and changing m into $m-1$, we have $n+1 = (m-1)-(m-n-2)$ for the multiplier of the second term of the series; and changing in this multiplier m into $m-1$, we have $(m-2)-(m-n-3)$ for the multiplier of the third term of the series, and so on.

Consequently, if we denote the sum of all the terms of the series (for brevity) by S, we shall have (as before) $S = \frac{m(m-1)(m-2)(\times \dots \times (m-n)}{n+1}$, (A), for the sum of all the terms of the series, as required.

Ex. 4.—Using m to represent any positive integer, as heretofore, it is required to find the sum of the series $m^2 + (m-1)^2 + (m-2)^2 + (m-3)^2 + \dots + 1^2$; (1).

Since $m^2 = m + m^2 - m$, we get $m^2 = m + m(m-1)$; and changing m into $m-1$, we have $(m-1)^2 = (m-1) + (m-1)(m-2)$; and changing m into $m-1$, we get $(m-2)^2 = (m-2) + (m-2)(m-3)$, and so on. Consequently, by adding the preceding results, we get $m^2 + (m-1)^2 + (m-2)^2 + (m-3)^2 + \dots + 1^2 = m + (m-1) + (m-2) + (m-3) + \dots + 1 + m(m-1) + (m-1)(m-2) + (m-2)(m-3) + \dots + 2 \times 1$. If for m in (2) of Ex. 1 we put $m+1$, it will give $m + (m-1)$

$+ (m-2) + (m-3) + \ldots + 1 = \frac{m(m+1)}{2}$; and if in (2) of Ex. 2, we put $m+1$ for m, we shall get $m(m-1) + (m-1)(m-2) + (m-2)(m-3) + \ldots + 2 \times 1 = \frac{(m-1)m(m+1)}{3}$.

Hence, we deduce $m^2 + (m-1)^2 + (m-2)^2 + (m-3)^2 + \ldots + 1^2 = \frac{m(m+1)}{2} + \frac{(m-1)m(m+1)}{3} = \frac{m(m+1)(2m+1)}{6}$ (2), as required.

Ex. 5.—To find the sum of the series $m^3 + (m-1)^3 + (m-2)^3 + (m-3)^3 + \ldots + 1^3$ (1).

Since $m^3 = m + m^3 - m = m + m(m^2 - 1)$, and that $m^2 - 1 = (m+1)(m-1)$, we of course get $m^3 = m + (m+1)m(m-1)$; and changing m into $m-1$, we have $(m-1)^3 = (m-1) + m(m-1)(m-2)$; and changing in this, m into $m-1$, we get $(m-2)^3 = (m-2) + (m-1)(m-2)(m-3)$, and so on. Adding the preceding results, we have $m^3 + (m-1)^3 + (m-2)^3 + (m-3)^3 + \ldots + 1^3 = m + (m-1) + (m-2) + (m-3) + \ldots + 1 + (m+1)m(m-1) + m(m-1)(m-2) + (m-1)(m-2)(m-3) + \ldots + 3 \times 2 \times 1$.

As in the last example, we have $m + (m-1) + (m-2) + (m-3) + \ldots + 1 = \frac{m(m+1)}{2}$; and by putting $m+2$ for m in (2) of Ex. 3, we get $(m+1) . m(m-1) + m(m-1)(m-2) + (m-1)(m-2)(m-3) + \ldots + 3 \times 2 \times 1 = \frac{(m+2)(m+1)m(m-1)}{4}$. Hence, we deduce $m^3 + (m-1)^3 + (m-2)^3 + \ldots + 1^3 = \frac{m(m+1)}{2} + \frac{(m-1)m(m+1)(m+2)}{4} = \frac{m^2(m+1)^2}{4}$ (2), as required.

Ex. 6.—To find the sum of the series $\frac{1}{1 . 2} + \frac{1}{2 . 3} + \frac{1}{3 . 4} + \ldots + \frac{1}{n(n+1)}$ (1), in which the notation $1 . 2$ is used to signify the product of 1 and 2; also $2 . 3$ is used to express the product of 2 and 3, and so on; also n is used for a whole number, which contains as many units as there are units in the number of any term, when the terms are counted from the left to the right.

To find the sum of the series, we multiply each term by 1; when put under the forms $1 = 2 - 1$ for the multiplier of the first term, $1 = 3 - 2$ for the multiplier of the second term, $1 = 4 - 3$ for the multiplier of the third term, and so on, to $(n + 1) - n$, which is the multiplier of the term whose number is n.

The product of $\frac{1}{1\,.\,2}$ and $2 - 1$ is $\frac{2}{1\,.\,2} - \frac{1}{1\,.\,2} = 1 - \frac{1}{2}$; and the product of the second term and its multiplier is $\frac{1}{2} - \frac{1}{3}$; the product of the third term and its multiplier is $\frac{1}{3} - \frac{1}{4}$; and so on, to the term whose number is n, whose product by its multiplier is $\frac{1}{n} - \frac{1}{n+1}$. Hence, by adding the preceding products, and erasing the terms that are numerically equal and affected with opposite signs, which destroy each other, we shall have $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$ (2), which equals the sum of the series, as required.

Ex. 7.—To find the sum of the series $\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \frac{1}{3\,.\,4\,.\,5} + \dots + \frac{1}{n(n+1)(n+2)}$ (1).

Here we must multiply each term by 2, which is less by a unit than the number of factors in the denominator of each term. We must put $2 = 3 - 1$ for the multiplier of the first term; and $2 = 4 - 2$ for the multiplier of the second term; and $2 = 5 - 3$ for the multiplier of the third term, and so on, to $(n + 2) - n$, which is the multiplier of the term whose number is n.

The product of the first term and its multiplier is $\frac{1}{1\,.\,2} - \frac{1}{2\,.\,3}$; and $\frac{1}{2\,.\,3} - \frac{1}{3\,.\,4}$ is the product of the second term and its multiplier; and $\frac{1}{3\,.\,4} - \frac{1}{4\,.\,5}$ is the product of the third term and its multiplier; and so on, to the term whose number is n, whose product by its multiplier is $\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$. Hence, by adding the preceding products, and erasing the terms that destroy each other, we shall have twice the sum of the series expressed by $\frac{1}{1\,.\,2} - \frac{1}{(n+1)(n+2)} = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$; and dividing by 2, we get the series $\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \frac{1}{3\,.\,4\,.\,5} + \ldots\ldots + \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ for the sum of n terms of the series, as required.

Ex. 8.—To find the sum of the series $\frac{1}{1\,.\,2\,.\,3 \times \ldots\ldots (p+1)} + \frac{1}{2\,.\,3\,.\,4 \times \ldots\ldots \times (p+2)} + \frac{1}{3\,.\,4\,.\,5 \times \ldots\ldots \times (p+3)} + \ldots\ldots + \frac{1}{n(n+1)(n+2) \times \ldots\ldots \times (p+n)}$ (1), supposing p to stand for a positive integer, such that $p+1$ equals the number of factors in the denominator of each of the fractional terms of the series.

Here we must multiply each term of the series by p; and we must put $p = (p+1) - 1$ for the multiplier of the first term of the series; and put $p = (p+2) - 2$ for the multiplier of the second term; also, $p = (p+3) - 3$ is the multiplier of the third term, and so on, to $p = (p+n) - n$, which is the multiplier of the term whose number is n.

If we denote the sum of the n terms of the series by S, then (as before) we shall have $S = \frac{1}{p \times 1\,.\,2\,.\,3 \times \ldots\ldots p} - \frac{1}{p(n+1)(n+2) \times \ldots\ldots \times (p+n)}$, (A), for the sum of the series, as required.

Ex. 9.—To convert $\frac{m(m-1)}{2}$ into a series.

By putting 2 under the form $2 = m - (m - 2)$, and dividing $m(m - 1)$ by $m - (m - 2)$, using m for the first term of the divisor, we get

$$\begin{array}{r|l} m-(m-2) & m(m-1 \qquad \underline{|(m-1)} \\ & m(m-1-(m-1)(m-2) \\ \hline & \qquad\qquad (m-1)(m-2) \end{array}$$

= the remainder; so that $\dfrac{m(m-1)}{2} = (m-1) + \dfrac{(m-1)(m-2)}{2}$. If we now put $2 = (m - 1) - (m - 3)$, and divide $(m - 1)(m - 2)$ by $(m - 1) - (m - 3)$, using $(m - 1)$ for the first term of the divisor, we get, as before, $\dfrac{(m-1)(m-2)}{2} = (m-2) + \dfrac{(m-2)(m-3)}{2}$; and putting $2 = (m - 2) - (m - 4)$, we also have $\dfrac{(m-2)(m-3)}{2} = (m-3) + \dfrac{(m-3)(m-4)}{2}$, and so on.

Hence, we get $\dfrac{m(m-1)}{2} = (m-1) + (m-2) + (m-3) + \text{etc.}$ (1), which will clearly never terminate when m is not a positive integer.

If in any calculation we meet with the infinite series $(m - 1) + (m - 2) + (m - 3) + \text{etc.}$, we may evidently put $\dfrac{m(m-1)}{2}$ for it; because the series is the expansion of $\dfrac{m(m-1)}{2}$, as has been shown.

Ex. 10.—To expand $\dfrac{m(m-1)(m-2)}{3}$ into a series.

Here we put $3 = m - (m - 3)$, and divide $m(m-1)(m-2)$ by $m - (m - 3)$, using m for the first term of the divisor, and we have $\dfrac{m(m-1)(m-2)}{3} = (m-1)(m-2) + \dfrac{(m-1)(m-2)(m-3)}{m-(m-3)}$; and in the divisor $m - (m - 3)$ changing m into $m - 1$, we have $\dfrac{(m-1)(m-2)(m-3)}{(m-1)-(m-4)} =$

$(m-2)(m-3)+\frac{(m-2)(m-3)(m-4}{(m-1)-(m-4)}$; and in the divisor $(m-1)-(m-4)$, if we change m into $m-1$, we have $(m-2)-(m-5)=(m-1)-(m-4)$, and then $\frac{(m-2)(m-3)(m-4)}{(m-2)-(m-5)}=(m-3)(m-4)+\frac{(m-3)(m-4)(m-5)}{(m-3)-(m-6)}$, and so on.

Hence, $\frac{m(m-1)(m-2)}{3}=(m-1)(m-2)+(m-2)(m-3)+(m-3)(m-4)+$ etc., (1), which will never terminate if m is not a positive integer. If we meet with the infinite series $(m-1)(m-2)+(m-2)(m-3)+$ etc., in any calculation, we may evidently put $\frac{m(m-1)(m-2)}{3}$ for it.

Universally, if n is any positive integer, and we meet with the infinite series $(m-1)(m-2)(m-3)\times\ldots\ldots\times(m-n)+(m-2)(m-3)(m-4)\times\ldots\ldots\times(m-n-1)+(m-3)(m-4)(m-5)\times\ldots\ldots\times(m-n-2)+$ etc., in any calculation, we may clearly put $\frac{m(m-1)(m-2)\times\ldots\ldots\times(m-n)}{n+1}$, (1), for the series, because the series is the expansion of $\frac{m(m-1)(m-2)\times\ldots\ldots\times(m-n)}{n+1}$, as is evident from what has been done.

SECTION VI.

OF INTEGERS; THEIR DIVISORS, FORMS, ETC.

(**1.**) When an integral number is such that it can not be produced by the multiplication of any whole numbers except unity and itself, it is called a *prime number*. Thus, 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, etc., are prime numbers.

(**2.**) Any whole number which can be produced by the multiplication of whole numbers, which are different from unity and the number itself, is called a *composite number;*

and the number is said to be a *multiple* of any one of its factors. Thus, 4, 6, 8, 9, 10, 12, 14, 15, etc., are composite numbers; and $4 = 2 \times 2$ is a multiple of 2, and $6 = 2 \times 3$ is a multiple of 2 or 3, and so on.

(3.) Any divisor of an integral number, including unity and the number itself among the divisors, is called a *measure* of the number. Thus 1 is a measure of 1, 1 and 2 are measures of 2, 1 and 3 are measures of 3; 1, 2, 4 are measures of 4, and so on.

(4.) If we include unity among the factors of any integral number, and exclude the number itself, or do not consider the number as a factor of itself, then any integral divisor of the number is often called an *aliquot part* of the number. Thus 1 is the half of 2, because 2 contains 1 twice; also 2 is the third part of 6, because 6 contains 2, 3 times, and so on.

(5.) Any two or more integers or integral quantities that have one or more common factors are said to have any one of the common factors for a common measure or divisor, and the product of all the common factors is called their *greatest common measure or divisor.*

(6.) If unity or one is the greatest common divisor, the numbers or quantities are said to be *prime* to each other.

Thus, 2 and 3 are measures and divisors of 6, 18, and 30, and 6 is the greatest common measure or divisor of the same numbers; also 8 and 15 are prime to each other, since 1 is their greatest common divisor.

(7.) Any integer or integral quantity that can be divided by two or more integers or integral quantities, is called their *common multiple;* and the least integer or integral quantity that can be divided by two or more integers or integral quantities, is called their *least common multiple.*

Thus, since 45 can be divided by 3 and 5, 45 is a common multiple of 3 and 5; but the least integer that can be divided by 3 and 5 is 15, so that 15 is the least common multiple of 3 and 5.

(8.) Any quantity (whether it is integral or not) when multiplied by an integer, gives a product that is often called a multiple of the quantity. Thus, $7a$, $3b$, $2c$ are called multiples of a, b, c, without reference to the natures of a, b, c.

(**9.**) We will now proceed to establish the following important proposition. *If a and b denote any two whole numbers, each greater than unity, and p any prime number which does not divide either a or b, then the product ab of a and b is not divisible by p.*

For if we take either of the numbers as a, then if a is greater than p, we may put $a = mp + a^1$, supposing m to be the quotient of the division of a by p, and a^1 the remainder, which by the nature of division is less than the divisor p.

Hence we have $ab = mpb + a^1b$, and if ab is divisible by p, its equal $mpb + a^1b$ must be divisible by p; and since mpb is a multiple of p, it is of course divisible by p, consequently a^1b must be divisible by p. Hence, in ab we may suppose a to be less than p; and since p is a prime number, it will not be divisible by a without a remainder; hence, if we put m^1 for the quotient of the division of p by a, and a^1 for the remainder, we shall have $p = m^1a + a^1$, and of course $pb = m^1ab + a^1b$; then, since pb is divisible by p, its equal $m^1ab + a^1b$ must be divisible by p; consequently, if ab is divisible by p, a^1b must be divisible by p.

In like manner, if we divide p by a^1, and use a^{II} for the remainder, we may show that $a^{\text{II}}b$ must be divisible by p when a^1b is divisible by p, and so on.

Hence, if ab is divisible by p, then must each of the products a^1b, $a^{\text{II}}b$, $a^{\text{III}}b$, $a^{\text{IV}}b$, and so on, be divisible by p. But since a^1 is less than p, and a^{II}, the remainder of the division of p by a^1, is less than a^1; and in the same way a^{III} less than a^{II}, a^{IV} less than a^{III}, and so on, it is evident that we shall soon have one of these integers equal to unity; for there can not be an infinite number of different positive integers, as a, a^1, a^{II}, a^{III}, etc., each less than p. Consequently, if in $a^{\text{V}}b$ we have $a^{\text{V}} = 1$, we must have $a^{\text{V}}b = b$, divisible by p, which is against the hypothesis, since b is not divisible by p; hence ab is not divisible by p, as was to be shown.

(**10.**) *It follows from what has been done, that if* p *divides the product* ab, *and does not divide* a, *that it must of neces-*

sity divide b, *and the proof is the same as in the proposition.*

(**11.**) Also, if P is the product of any number of whole numbers which is not divisible by p, and c another whole number which is not divisible by p, then, by the proposition, Pc is not divisible by p.

Hence, if the integers a and b are not divisible by p, since their product ab is not divisible by p, we may put ab for P, and we shall have abc not divisible by p.

In the same way, if the integer d is not divisible by p, the product $abcd$ is not divisible by p, and so on for any number of integers.

(**12.**) Reversely, if p divides the product $abcd$, etc., of any number of integers, it must divide one or more of the integers; for if it does not, it can not divide their product, which is against the hypothesis.

(**13.**) If each of the numbers a, b, c, d, etc., is a prime number, and A another prime number different from unity, which does not enter into the product $abcd$, etc., then A can not divide the product; for it can not divide either of the prime numbers a, b, c, d, etc., and of course can not divide their product.

(**14.**) Still supposing a, b, c, d, etc., to be prime numbers, and n, n', n'', n''', etc., any positive integers, one or more of which may equal unity, should the case require it; then a^n, $b^{n'}$, $c^{n''}$, $d^{n'''}$, etc., which is the product of the n^{th} power of a by the n'^{th} power of b, by the n''^{th} power of c, and so on, can not be divided by the prime number A. For since $a^n =$ the product $aaaa$, etc., to n factors, and that A does not divide either of these factors, it does not divide their product a^n; and in the same way A does not divide $b^{n'}$, $c^{n''}$, $d^{n'''}$, etc., consequently it does not divide their product.

(**15.**) Again, the product $a^n b^{n'} c^{n''}$, etc., is of course divisible by a^n, but it is not divisible by a^m if m is greater than n; for let $m = n + 1$, then $a^m = a^{n+1} = a^n a$; and to divide by $a^n a$ is the same as to divide by a^n, and then to divide the quotient by a; dividing the given product by a^n reduces it to $b^{n'} c^{n''}$, etc., which can not be divided by a, since a does not enter it as a factor, and because b, c, etc., being prime numbers, $b^{n'}c^{n''}$, etc., can not be divided by the prime number a, which does not enter their product; consequently $a^nb^{n'}c^{n''}$, etc., can not be divided by a^m if m is greater than n, but if m equals n, or is less than n, then the product $a^nb^{n'}b^{n''}$, etc., is clearly divisible by a^m. And similar remarks are applicable to the division of $a^nb^{n'}c^{n''}$, etc., by any integral power of any one of the prime numbers b, c, d, etc.

(**16.**) If we put $a^nb^{n'}c^{n''}$, etc., $= \text{P}^1$, and suppose that the product of the integers A, B, C, D, etc., is equal to P^1, we shall get ABCD, etc., $= \text{P}^1$. Then it is evident, from what has been shown, that A, B, C, D, etc., can not contain any prime numbers that are different from the prime numbers a, b, c, d, etc.; also the sum of the exponents of a in A, B, C, D, etc., must equal n, the sum of the exponents of b must equal n', and so on for the exponents of c, d, etc.

(**17.**) It follows, from what has been done, that any integral number, P^1, can be put under the form $a^nb^{n'}c^{n''}$, etc. For if we divide P^1 by a, and the quotient by a, and so on, as often as it can be done, then the number of divisions will give the value of n, the exponent of a; and in the same way by dividing by b we get the exponent of b, and so on (for the exponents of c, d, etc.; the simplest way will be to divide the last quotient of the division by a by b, and the quotient by b, and so on), as often as it can be done; and then to divide the last quotient of these divisions by c, in the same way as before, and so on for all the prime divisors. Thus, to put 324 under the above form, we divide by 2 and get 162 for the quotient, which, divided by 2 again, gives 81 for the quotient, then we have $324 = 2^2 \times 81$.

And dividing 81 by the prime 3, we get 27 for the quotient, which divided by 3 gives 9 for the quotient, and 9 divided by 3 gives 3 for the quotient, and 3 divided by 3 gives 1 for the quotient; and since there are four divisions, we have $81 = 3^4$, consequently we have $324 = 2^2 \times 3^4$; and a is represented by 2, and its exponent n is also represented by 2; also b is expressed by 3, and its exponent is expressed by 4, and the other primes, c, d, etc., are each equal to unity, so that we have $324 = 2^2 \times 3^4$ expressed in the form $a^nb^{n'}$, etc.

In the same way, we can express 1890 by the form $1890 = 2 \times 3^3 \times 5 \times 7$; and 2592 is expressed by $2592 = 2^5 \times 3^4$; we also have $234 = 2 \times 3^2 \times 13$.

(**18.**) Now, since 2 and $3^2 = 9$ are factors of all the numbers, it follows that 2 and $3^2 = 9$ are common measures or divisors of the numbers; and $2 \times 3^2 = 18$ is the greatest common measure or divisor of the same numbers.

(**19.**) Also, $2^5 \times 3^4 \times 5 \times 7 \times 13 = 1179360$ is the least common multiple of the same numbers; which is found by taking 2 with the greatest exponent that it has in any one of the numbers, and 3 with the greatest exponent that it has in any one of the numbers, and so on; then the product of all the powers of 2, 3, etc., thus found, is the least common multiple of all the numbers, as is evident from the nature of the least common multiple, as has heretofore been defined.

(**20.**) *In like manner, if we resolve any whole numbers into products of powers of prime numbers, and select the greatest power of each prime that is common to all the products, then the product of the powers of the primes thus obtained will be the greatest common measure or divisor of the numbers.*

(**21.**) *If* 1 *is the only prime that is common to the products, the numbers will of course have no other common divisor than* 1; *and if no two products have any other divisor than* 1, *the numbers are said to be prime to each other.*

(**22.**) *Again, supposing the numbers resolved into products of powers of primes as before, if we select the greatest power of each prime in the products, then the product of all the powers thus obtained will be the least common multiple of the numbers; observing that the same power of any prime is to be taken but once in the product. It hence follows, if the numbers are prime to each other, that their product will be their least common multiple.*

(**23.**) *It is clear that we may proceed in a similar way to find the greatest algebraic divisor and least common multiple of algebraic expressions of integral forms, by resolving them into products of powers of algebraic primes, and then proceeding as before.*

EXAMPLES.

Ex. 1.—To find the greatest common divisor and least common multiple of 27 and 72.

Here we have $27 = 3^3$, and $72 = 3^2 \times 2^3$, and 3^2 is the greatest power and the only power of any prime (except 1) which is common to the resolved numbers; consequently $3^2 = 9$ is the greatest common divisor of the numbers, as required. Also 3^3 is the greatest power of the prime 3 that enters into the resolved numbers; and 2^3 is the only power of the prime 2 that enters into the resolved numbers; consequently $2^3 \times 3^3 = 216 =$ the least common multiple of the numbers, as required.

Ex. 2.—To find the greatest common divisor and least common multiple of 91, 819, 2145, and 2431.

Here we have $91 = 13 \times 7$, $819 = 3^2 \times 7 \times 13$, $2145 = 3 \times 5 \times 11 \times 13$, $2431 = 11 \times 13 \times 17$.

The only power of any prime that is common to all the resolved numbers is $13^1 = 13$; consequently 13 is the greatest common divisor of the numbers, as required.

Also, since 3^2, 5^1, 7^1, 11^1, 13^1, 17^1 are the greatest powers of the primes that enter into the resolved numbers, we have $3^2 \times 5 \times 7 \times 11 \times 13 \times 17 = 765765$, which equals the least common multiple of the numbers, as required.

Ex. 3.—To find the greatest common divisor and least common multiple of 7, 19, 36, 37.

Since the numbers have no common divisor except 1, it is of course their greatest common divisor.

And to find the least common multiple of the numbers, we have $7 \times 19 \times 36 \times 37 = 177156$ for the least common multiple, as required.

Ex. 4.—To find the greatest common divisor and least common multiple of a^5b^2c, $3a^2bd$, $4a^2b^2d^2e$.

Since a^2 and b are the only factors common to all the quantities, we have a^2b for their greatest common measure.

And selecting the highest powers of the prime factors of the quantities, we have $1 \times 3 \times 4 \times a^5 \times b^2 \times c \times d^2 \times e = 12a^5b^2cd^2e$, which is the least common multiple of the quantities.

Ex. 5.—To find the greatest common divisor and least common multiple of $42a^2b^3c^4$, $54a^5b^5c^3$, and $6a^3b^3d^5$.

Since $42 = 6 \times 7$, $54 = 6 \times 9$, the greatest common divisor of the coefficients is 6, and a^2b^3 being the only powers of prime literal factors that are common to all the quantities, we of course have $6a^2b^3$ for the greatest common measure. And since $2^1 \times 3^3 \times 7^1 = 378 =$ the least common multiple of the coefficients of the quantities, and that $a^5b^5c^4d^5 =$ the least common multiple of the literal parts of the quantities, we evidently have $378a^5b^5c^4d^5$ for the required least common multiple of the quantities.

Ex. 6.—To find the greatest common measure and least common multiple of $a^2 - b^2$, $a^3 + b^3$, $(a + b)^2$.

From what was formerly shown, we know that $a + b$ is the greatest common divisor of all the quantities, for $a^2 - b^2 = (a + b)(a - b)$, and $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, $(a + b)^2$, $= (a + b)(a + b)$.

Also, $(a + b)^2 \times (a - b)(a^2 - ab + b^2) = (a^2 - b^2)(a^3 + b^3)$ is evidently the least common multiple, as required.

EXAMPLES FOR EXERCISE.

1. Find the greatest common divisor of 119, 323, and 1088. *Ans.* 17.

2. Find the least common multiple of 13, 17, and 19. *Ans.* 4199.

3. Find the least common multiple of $13x^3y^5z^2$, $39x^6y^7$, and $26x^7a^2b$. *Ans.* $78a^2bx^7y^7z^2$.

4. Find the greatest common divisor of $114a^4b^3c^2$, $437a^6b^6c^3$, $133a^{10}b^{14}c^{15}d$. *Ans.* $19a^4b^3c^2$.

5. Find the greatest common divisor of $3(a^2 - b^2)$, $6(a^3 - b^3)$, $9ab^2 - 9b^3$. *Ans.* $3(a - b)$.

(**24.**) Since it is necessary to have prime numbers, in order to find the greatest common measure and least common multiple of integers by the methods that have been given, we shall here insert the common way of determining them.

Write down the odd numbers as follows, viz., 1, 3, 5, 7, $\dot{9}$, 11, 13, $\dot{15}$, 17, 19, $\dot{21}$, 23, $\dot{25}$, $\dot{27}$, 29, 31, $\dot{33}$, $\dot{35}$, 37, $\dot{39}$, 41, 43, $\dot{45}$, 47, $\dot{49}$, $\dot{51}$, 53, $\dot{55}$, $\dot{57}$, 59, 61, $\dot{63}$, $\dot{65}$, 67, $\dot{69}$, 71, 73, $\dot{75}$, $\dot{77}$, 79, $\dot{81}$, 83, $\dot{85}$, $\dot{87}$, 89, $\dot{91}$, $\dot{93}$, $\dot{95}$, 97, $\dot{99}$. 101; then mark all the multiples of 3, 5, 7, etc., with a dot above them, and all the numbers not marked will plainly be prime numbers.

Thus we have found 1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101 all to be prime numbers; and by carrying the series of odd numbers sufficiently far, we may find as many prime numbers as we please.

(**25.**) There is a method of determining whether a given number is prime or not, which it may be well to notice. If a number is not prime, it will have two factors at least, which will equal each other when the number is a square; but when the number is not a square, one of the factors will clearly be less than the square root of the number, and the other will be greater.

Hence, if we find that the number is not divisible by any prime number that is less than its square root, we may infer with certainty that the number is prime. Thus, to determine whether 79 is a prime number or not, we find that it is not divisible by either of the numbers 2, 3, 5, 7, $\therefore$ it is a prime number; also 131 is not divisible by either of the numbers 2, 3, 5, 7, 11, $\therefore$ it is a prime number; but 91 is divisible by one of the numbers 2, 3, 5, 7, viz., by 7, $\therefore$ 91 is a composite number.

We shall soon give another method of finding the greatest common measure, or least common multiple, of two or more numbers, which does not require the use of prime numbers.

(**26.**) Again, in order that an integral quantity may be resolved into any number, n, of equal factors, it is evidently necessary (from what has been done) that it should have been formed by taking the product of n equal factors; if it has not been formed thus, or if it does not consist of the product of n equal factors, it of course can not be resolved into n equal factors, except approximately; and the quantity is called a surd or an irrational quantity.

Thus, if we have to take the cube root of the integral quantity a^2, supposing a not to consist of three equal factors, it evidently can not be done except approximately, for the cube root requires a^2 to be resolved into three equal factors, and one of the factors to be taken; and it is evident that if a^2 consists of three equal factors, each of them must equal the square of a factor of which a consists of three that are equal to each other.

Consequently, if a is not a cube, or if it does not consist of three equal factors, we say that a^2 is an irrational quantity with respect to the cube root; that is, that $\sqrt[3]{a^2} = (a^2)^{\frac{1}{3}}$, which denotes the cube root is an irrational quantity. Now the square root of a^2 is evidently a, or $-a$; so that a quantity which is irrational with respect to a root of one kind, may sometimes be rational with respect to a root of another kind.

(**27.**) To make what has been said more evident, we observe if a denotes any positive integral quantity which does not consist of n equal integral factors, that it does not consist of the product of n equal fractions, whose numerators and denominators are composed of a finite number of figures.

For, if possible, let $a^{\frac{1}{n}} = \frac{m}{p}$, where m and p are finite and integral; by taking the n^{th} power, we get $(a^{\frac{1}{n}})^n = a = \frac{m^n}{p^n}$, an impossible result, for the fraction $\frac{m}{p}$ may be supposed to be in its lowest terms, or such that its numerator and denominator have no common factor except unity; consequently, from what has been proved, m^n and p^n have no common factor except unity; so that we have the integral quantity a equal to the irreducible fraction $\frac{m^n}{p^n}$, which is impossible.

Hence, by (4) *of the definitions,* $\sqrt[n]{a} = a^{\frac{1}{n}}$, *which denotes the* nth *root of* a, *is an irrational quantity.*

(**28.**) Now, although $a^{\frac{1}{n}}$ can not be expressed by a fraction whose numerator and denominator are finite integers, yet two fractions, whose denominators are equal, and whose numerators differ only by a unit, may be found, so that $a^{\frac{1}{n}}$ shall be greater than the smaller fraction, and smaller than the greater fraction; *and the fractions thus found are said to be limits of* $a^{\frac{1}{n}}$.

For, if we suppose r to be the greatest integral n^{th} root contained in ap^n, when p is supposed to be a positive integer > 1, then, since ap^n is not an exact n^{th} power, we shall have $ap^n > r^n$ and $ap^n < (r+1)^n$, or, which is the same thing, we shall have $a^{\frac{1}{n}} > \frac{r}{p}$ and $a^{\frac{1}{n}} < \frac{r+1}{p}$; and $\frac{r}{p}$, $\frac{r+1}{p}$ are said to be limits of $a^{\frac{1}{n}}$. If r and p are indefinitely great numbers, it is evident that the fractions $\frac{r}{p}$, $\frac{r+1}{p}$ will differ infinitely little from each other; for their difference, $\frac{1}{p}$, will become an infinitely small quantity, since p, the divisor of 1 the numerator, is indefinitely great.

It is customary in practice to find r and p to a great number of figures, and to take $\frac{r}{p}$ for the value of $a^{\frac{1}{n}}$, and to say that the value of $a^{\frac{1}{n}}$ is correctly found to within the fraction $\frac{1}{p}$; but it is evident that a more correct value of $a^{\frac{1}{n}}$ will often be obtained by taking $\frac{r+\frac{1}{2}}{p} = \frac{2r+1}{2p}$ for the value of $a^{\frac{1}{n}}$.

From what has been done, it follows that $a^{\frac{1}{n}}$ can not be expressed by a fraction whose numerator and denominator are finite integers.

To illustrate what has been said, we shall find some of the limits of the square root of 2; or, which is the same, shall find limits for $\sqrt{2}$.

If we assume $p = 10$, then $2p^2 = 200$, and the greatest integral square contained in 200 is $196 = 14^2 = r^2$, $\therefore \sqrt{2} > \frac{14}{10}$, and $\sqrt{2}$ is $< \frac{15}{10}$, so that $\sqrt{2}$ has $\frac{14}{10}$ and $\frac{15}{10}$ for two of its limits; and by assuming $p = 100$, we get $\frac{141}{100}$, $\frac{142}{100}$ for closer limits of $\sqrt{2}$, and $\frac{1414}{1000}$, $\frac{1415}{1000}$ are nearer limits, and $\frac{14142}{10000}$, $\frac{14143}{10000}$ are yet nearer limits of $\sqrt{2}$; and in this manner we may proceed indefinitely to find nearer limits of $\sqrt{2}$

(29.) Again, if we represent any irrational quantity by a, and let m denote any very small rational quantity of the same kind, as a, then by taking m, $2m$, $3m$, $4m$, etc., we may evidently find some whole number, as p, such that the inequalities, $pm < a$, $pm + m > a$ shall have place; then a will have pm and $pm + m$ for its limits. Now it is manifest that by supposing m to be very small, and consequently p very great, we may bring these limits as near to equality as we please; for we may suppose m to be diminished *ad infinitum*, and p increased accordingly.

In like manner, if b represents any irrational quantity, and n a very small rational quantity of the same kind as b, we may find the integer q such that the inequalities $qn < b$, $qn + n > b$ shall have place; in which we may suppose q to be taken so great that n, the difference of the limits qn, $qn + n$, of b, shall be less than any given quantity.

(**30.**) We are now prepared to show that the product ab is equal to the product ba; which may be done as follows, viz., since pm is less than a, and qn less than b, we have $pm \times qn < ab$ and $qn \times pm < ba$, and since pm, qn are rational quantities, we have, by what has been proved in Multiplication and Division, $pm \times qn = qn \times pm$, $\therefore pm \times qn < ab$ and $pm \times qn < ba$ have place at the same time.

We may show, in a similar way, that the inequalities $(pm + m) . (qn + n) > ab$, $(pm + m) . (qn + n) > ba$ have place at the same time. Hence, the products ab, ba have the same limits, which are $pm \times qn$ and $(pm + m) . (qn + n) = (pm + m) . qn + (pm + m) . n = pm . qn + pm . n + qn . m + m . n$; in which pm, qn represent rational finite quantities, and m, n indefinitely small quantities. Now $pm . n + qn . m + m . n$ is the difference of the limits, which will be diminished indefinitely by diminishing m and n indefinitely; for since the finite quantity pm is multiplied by the indefinitely small quantity n, the product $pm . n$ will be indefinitely small; in the same manner the product $qn . m$ is indefinitely small, and the product $m . n$ is also indefinitely smaller than the other products, since each of its factors is indefinitely small. Hence, evidently, by supposing m and n to be less than any given quantities, the difference of the limits will be less than any given quantity; consequently, since ab and ba differ less from each other than their limits, it follows that they differ less from each other than any given difference; consequently (Ax. VII.) we must have $ab = ba$, as required.

A similar demonstration is applicable when one of the quantities, a, b, is rational, and the other irrational. And in like manner it may be shown that the product of any number of irrational quantities, or of quantities some of which are rational and others irrational, is independent of the order of the factors.

(**31.**) *Since all real quantities are either rational or irrational, it follows, from what has been shown, together with what was formerly proved, that the product of any number of real quantities is independent of the order of the factors.*

Hence, according to custom, the product of literal quantities may be indicated by writing the letters alphabetically. Thus, the product of a, v, s, t, x is expressed by $astvx$, and $bpqr$ expresses the product of r, b, q, p.

(**32.**) OF THE GREATEST COMMON MEASURE OR DIVISOR.

The product of all the particular divisors that are common to two or more numbers or quantities of the same kind, is called their greatest common measure or divisor.

Thus, if we have two numbers or quantities, am and bm, such that m is their greatest common divisor, then a and b can have no other divisor than 1; for if they can, m will not be the greatest common divisor of am and bm, which is against the hypothesis. If we suppose am to be greater than bm, and divide am by bm, using p to represent the integral part of the quotient, then we shall have the remainder of the division expressed by $am - pbm = (a - pb)m$, which, by the nature of division, is less than the divisor bm.

Since a and b can have no other common divisor than 1, it is clear that b and $a - pb$ can have no other divisor than 1; for if they can, since it divides b and $a - pb$, and as it divides b in $-pb$, it must divide a, for otherwise it can not divide $a - pb$; hence, if b and $a - pb$ can have any other common divisor than 1, a and b must also have a common divisor other than 1, which is against the hypothesis.

It hence appears that m is the greatest common divisor of bm and the remainder of the division of am by bm, which is expressed by $(a-pb)m$, so that m is the greatest common divisor of am, bm, and $(a-pb)m$.

In the same way, if we divide bm by $(a-pb)m$, we may show that m is the greatest common divisor of the remainder of this division, bm and $(a-pb)m$ and am; also, if we divide $(a-pb)m$ by the last remainder, m will still be the greatest common divisor of the remainder and the corresponding divisor and dividend, and so on, so long as the division can be continued. Hence, in order to find the greatest common measure or divisor of two numbers or quantities of the same kind, we have the following

RULE.

Divide the greater number or quantity by the less, and the last divisor by the last remainder, and so on, till nothing remains, then the last divisor will be the greatest common measure or divisor required.

If the greatest common divisor equals 1, *the numbers or quantities are prime to each other.*

Also, if the successive divisors and remainders decrease indefinitely, so as ultimately to become less than any given quantities, without any remainder having actually become equal to 0. *the quantities will have no common measure, and are said to be* incommensurable.

(**33.**) If the greatest common divisor of three or more numbers or quantities is to be found, then we find (by the rule) the greatest common divisor of any two of them; and then we find (as before) the greatest common divisor of the common divisor just found, and of any one of the remaining numbers or quantities, and so on, for any common divisor found and any one of the remaining numbers or quantities, until all the numbers or quantities have been used; then will the last divisor found evidently be the greatest common divisor.

(**34.**) *Remark.*—In finding the greatest common divisor of two numbers or quantities, we may, if we please, omit any divisor of one of them, provided it is not a divisor of the other; for the divisor thus rejected evidently can not affect the common divisor; we may also multiply one of the numbers or quantities by any factor which is not a divisor of the remaining number or quantity; for it is evident that the factor thus introduced can not affect the common divisor.

It may also be observed that the common divisor of any two numbers or quantities of the same kind is clearly a common divisor of their sum and difference. And analogous observations will have place when we have to find the greatest common divisor of more than two numbers or quantities.

EXAMPLES.

Ex. 1.—To find the greatest common divisor of 27 and 72.

Dividing 72 by 27 we have the quotient 2, and remainder 18; then dividing the last divisor, 27, by the remainder, 18, we have the quotient 1 and remainder 9; and dividing the last divisor, 18, by the remainder, 9, we have the quotient 2 and remainder 0; consequently 9 is the greatest common divisor required.

Ex. 2.—To find the greatest common divisor of 11648 and and 12051.

Proceeding as in Ex. 1, we have the following process.

$$\begin{array}{rrrrrrr}
\underline{11648}\,| & 12051\,| & \underline{1} & & & & \\
 & \underline{11648} & & & & & \\
 & \underline{403}\,| & 11648\,| & \underline{28} & & & \\
 & & \underline{806} & & & & \\
 & & 3588 & & & & \\
 & & \underline{3224} & & & & \\
 & & \underline{364}\,| & 403\,| & \underline{1} & & \\
 & & & \underline{364} & & & \\
 & & & \underline{39}\,| & 364\,| & \underline{9} & \\
 & & & & \underline{351} & & \\
 & & & & \underline{13}\,| & 39\,| & \underline{3} \\
 & & & & & \underline{39} & \\
 & & & & & 0 &
\end{array}$$

and 13 is the greatest common divisor required.

Ex. 3.—To find the greatest common divisor of 826 and 897.

Dividing 897 by 826, we get the quotient 1 and remainder 71. Since 71 is a prime number, which does not exactly divide 826, it is unnecessary to continue the division any further; for the greatest divisor sought must be a divisor of 71 and 826, and since 71, and 1, are the only divisors of 71, and as 71 does not divide 826, it follows that 1 is the greatest common divisor of 71 and 826. Hence, since the greatest divisor of 71 and 826 must be the greatest divisor of 826 and 897, 1 is their greatest common divisor, and they are of course prime to each other.

Ex. 4.—To find the greatest common divisor of 301, 364, and 819.

Here, by proceeding as in the preceding examples, we find 91 to be the greatest common divisor of 364 and 819. And in the same way we get 7 for the greatest common divisor of 91 and 301, and of course 7 is the greatest common divisor of the three given numbers. If we divide the given numbers by their greatest common divisor, 7, the quotients will be 43, 52, and 117, which clearly have no other common divisor than 1, and the numbers are said to be reduced to *their lowest terms*, with reference to their greatest common divisor.

Ex. 5.—To find the greatest common divisor of 1 and $\sqrt{2}$, if possible.

We have heretofore shown that $\sqrt{2}$ has $\frac{14142135}{10000000}$ and $\frac{14142136}{10000000}$ for its less and greater limits.

Hence, dividing $\sqrt{2}$ by 1, and the divisor 1 by the remainder, and so on, according to the rule, we shall have the following process.

$$\begin{array}{lllllll}
1\,) & \sqrt{2} & (1 & & & & \\
 & 1 & & & & & \\
\cline{2-2}
 & \sqrt{2}-1\,) & 1 & (2 & & & \\
 & & 2\sqrt{2}-2 & & & & \\
\cline{3-3}
 & & 3-2\sqrt{2}\,) & \sqrt{2}-1 & (2 & & \\
 & & & 6-4\sqrt{2} & & & \\
\cline{4-4}
 & & & 5\sqrt{2}-7\,) & 3-2\sqrt{2} & (2 & \\
 & & & & 10\sqrt{2}-14 & & \\
\cline{5-5}
 & & & & 17-12\sqrt{2}\,) & 5\sqrt{2}-7 & (2 \\
 & & & & & 34-24\sqrt{2} &
\end{array}$$

and so on, indefinitely; for it is manifest that we shall never arrive at a remainder which is 0.

It is easy to see that the remainders successively diminish, and that they will ultimately become indefinitely small. For the first remainder, $\sqrt{2} - 1 = 0.4142135 +$, the second remainder $3 - 2\sqrt{2} = 0.17157 +$, the third remainder $5\sqrt{2} - 7 = 0.07106 +$, and so on, the remainders continually diminishing without limit. Hence we conclude that 1 and $\sqrt{2}$ are incommensurable, or that they have no common divisor.

The same conclusion may be obtained in the following manner, viz., it is easy to see that the greatest common divisor of 1 and $\sqrt{2}$ must be the same as that of 1 and $\sqrt{2} - 1$ and of 1 and $\sqrt{2} + 1$. Hence, instead of dividing 1 by $\sqrt{2} - 1$, we may divide $\sqrt{2} + 1$ by 1, and we shall have

$$\begin{array}{r|c|l} \underline{1} & \sqrt{2}+1 & \underline{2} \\ & 2 & \\ \hline & \sqrt{2}-1 & 1 \ \underline{.} \end{array} \quad :$$

and instead of dividing 1 by $\sqrt{2} - 1$, we may again divide $\sqrt{2} + 1$ by 1, and so on, so that we have continually to divide $\sqrt{2} + 1$ by 1. Hence, since the division never terminates, it follows that 1 and $\sqrt{2}$ are incommensurable.

This method coincides with the process given by Legendre in his Geometry, for the purpose of showing that the diagonal and side of a square have no common measure.

Remark.—From what has been done in the solution of this example, we may be enabled to conceive of the division of quantity into parts that are less than any given quantities.

Ex. 6.—To find the greatest common divisor of $15a^4 + 15a^3b - 21a^2b^2 - 21ab^3$, and $5a^5 - 7a^3b^2 + 5a^2b^3 - 7b^5$.

Since $3a$ is a common factor of the terms of the first quantity, and not of the terms of the second quantity, it may be rejected because it does not enter into the common divisor. Hence, dividing each term of the first quantity by $3a$, and using the result as the divisor of the other quantity, and arranging the terms of the divisor and dividend according to the descending powers of a, we have, by Rule I. of Division, the following process.

$$\begin{array}{r|l|l}
\begin{array}{r}5a^3+5a^2b-\\ 7ab^2-7b^3\end{array} & \begin{array}{r}
5a^5+0\quad-7a^3b^2+\ 5a^2b^3+0-7b^5\\
5a^5+5a^4b-7a^3b^2-\ 7a^2b^3\qquad\qquad\\
\hline
-5a^4b+0\quad+12a^2b^3+0\qquad\\
-5a^4b-5a^3b^2+\ 7a^2b^3+7ab^4\qquad\\
\hline
5a^3b^2+\ 5a^2b^3-7ab^4-7b^5\\
5a^3b^2+\ 5a^2b^3-7ab^4-7b^5\\
\hline
0+\quad 0\quad+\ 0\ +0
\end{array} & a^2-ab+b^2
\end{array}$$

Hence, since the division is exact, $5a^3+5a^2b-7ab^2-7b^3$ is the greatest common divisor.

The process is so evident that it needs no other explanation than to observe that we have supplied the terms that were wanting in the dividend by naughts.

Ex. 7.—To find the greatest common divisor of $3a^3+5a^2b+6ab^2+10b^3$, and $7a^3-9a^2b+14ab^2-18b^3$.

Since 3 does not divide the first quantity, we shall multiply the second quantity by 3, and it becomes $21a^3-27a^2b+42ab^2-54b^3$; whose first term $21a^3$ can now be exactly divided by the first term $3a^3$ of the other quantity. Hence, taking the first quantity for a divisor, and three times the second for a dividend, and arranging according to the descending powers of a, we have (by Rule I. of Division) 7 for the quotient, and $-62a^2b-124b^3$ for the remainder. Since $-62b$ does not divide the first divisor, it must be rejected in the remainder, which will be reduced to a^2+2b^2, which is the greatest required divisor, since it divides the first divisor.

Ex. 8.—To find the greatest common divisor of $9a^6b^2-24a^5b^3+54a^4b^4-60a^3b^5+57a^2b^6-21ab^7$, and $36a^6b^2-60a^5b^3+138a^4b^4-90a^3b^5+126a^2b^6$.

Since $3ab^2$ is a factor of each term of the given quantities, it is consequently a factor of the sought divisor. And to find if the quantities have any compound common divisor, we divide 2 times the first by $3ab^2$, and the second by $6a^2b^2$; then dividing the first result by the second, we get

$$
\begin{array}{l|r|l}
6a^4-10a^3b+23a^2 & & \\
\underline{b^2-15ab^3+21b^4} & 6a^5-16a^4b+36a^3b^2-40a^2b^3+38ab^4-14b^5 & \underline{a-b} \\
 & \underline{6a^5-10a^4b+23a^3b^2-15a^2b^3+21ab^4\qquad\qquad} & \\
 & -\ 6a^4b+13a^3b^2-25a^2b^3+17ab^4-14b^5 & \\
 & \underline{-\ 6a^4b+10a^3b^2-23a^2b^3+15ab^4-21b^5} & \\
 & 3a^3b^2-\ 2a^2b^3+\ 2ab^4+\ 7b^5 &
\end{array}
$$

= the remainder. And taking the last divisor for a dividend, and the remainder, after the useless factor b^2 has been rejected, for a divisor, we get

$$
\begin{array}{l|r|l}
\underline{3a^3-2a^2b+2ab^2+7b^3} & 6a^4-10a^3b+23a^2b^2-15ab^3+21b^4 & \underline{2a-2b} \\
 & \underline{6a^4-\ 4a^3b+\ 4a^2b^2+14ab^3\qquad\quad} & \\
 & -\ 6a^3b+19a^2b^2-29ab^3+21b^4 & \\
 & \underline{-\ 6a^3b+\ 4a^2b^2-\ 4ab^3-14b^4} & \\
 & 15a^2b^2-25ab^3+35b^4 &
\end{array}
$$

= the remainder. Then, taking the last divisor for a dividend, and the last remainder for a divisor, after the factor $5b^2$ has been rejected, we have

$$
\begin{array}{l|l|l}
\underline{3a^2-5ab+7b^2} & 3a^3-2a^2b+2ab^2+7b^3 & \underline{a+b\qquad\qquad} \\
 & \underline{3a^3-5a^2b+7ab^2\qquad\qquad} & \\
 & \qquad 3a^2b-5ab^2+7b^3 & \\
 & \qquad \underline{3a^2b-5ab^2+7b^3} & \\
 & \qquad 0+0+0 &
\end{array}
$$

= the remainder. Hence, the last divisor $3a^2-5ab+7b^2$ is the required compound divisor, because the last remainder is 0. If we now multiply the compound by the simple divisor, we have $(3a^2-5ab+7b^2)\times 3ab^2 = 9a^3b^2-15a^2b^3+21ab^4$ for the required greatest common divisor of the given quantities.

Ex. 9.—To find the greatest common divisor of $6(a+b)x^3 - (2a+2b-9c+9d)x^2 - 3(c-d-e)x - e$, and $2(a+b)x^4 + (2a+2b+3c-3d)x^3 + (2a+2b+3c-3d+e)x^2 + (3c-3d+e)x + e$.

By taking 9 times the second quantity for a dividend, and the first quantity for the corresponding divisor, and using the vertical bar for the parenthesis, we get

$$
\begin{array}{l|l|l}
\left.\begin{array}{r}6a\\6b\end{array}\right| x^3+\left.\begin{array}{r}-2a\\-2b\\9c\\-9d\end{array}\right| x^2+\left.\begin{array}{r}-3c\\3d\\3e\end{array}\right| x-e
&
\left.\begin{array}{r}18a\\18b\end{array}\right| x^4+\left.\begin{array}{r}18a\\18b\\27c\\-27d\end{array}\right| x^3+\left.\begin{array}{r}18a\\18b\\27c\\-27d\\9e\end{array}\right| x^2+\left.\begin{array}{r}27c\\-27d\\9e\end{array}\right| x+9e
& 3x+4
\\
&
\left.\begin{array}{r}18a\\18b\end{array}\right| x^4+\left.\begin{array}{r}-6a\\-6b\\27c\\-27d\end{array}\right| x^3+\left.\begin{array}{r}-9c\\9d\\9e\end{array}\right| x^2-\quad 3ex
&
\\ \cline{2-2}
&
\qquad\quad\left.\begin{array}{r}24a\\24b\end{array}\right| x^3+\left.\begin{array}{r}18a\\18b\\36c\\-36d\end{array}\right| x^2+\left.\begin{array}{r}27c\\-27d\\12e\end{array}\right| x+9e
&
\\
&
\qquad\quad\left.\begin{array}{r}24a\\24b\end{array}\right| x^3+\left.\begin{array}{r}-8a\\-8b\\36c\\-36d\end{array}\right| x^2+\left.\begin{array}{r}-12c\\12d\\12e\end{array}\right| x-4e
&
\\ \cline{2-2}
&
\qquad\qquad\qquad\left.\begin{array}{r}26a\\26b\end{array}\right| x^2+\left.\begin{array}{r}39c\\-39d\end{array}\right| x+13e
&
\end{array}
$$

= the remainder. Rejecting the useless factor 13 in the remainder, it becomes $\left.\begin{array}{r}2a\\2b\end{array}\right| x^2+\left.\begin{array}{r}3c\\-3d\end{array}\right| x+e=2(a+b)x^2+3(c-d)x+e$, which will be found, on trial, to divide the last divisor, and of course it is the required divisor of the given quantities.

Ex. 10.—To find the greatest common divisor of $a^5-a^3b^2+a^2b^3-b^5$, $a^5+a^3b^2+a^2b^3+b^5$, and a^5+b^5.

The greatest common divisor of the first two quantities is

$a^3 + b^3$, and the greatest common divisor of $a^3 + b^3$ and $a^5 + b^5$ is $a + b$; consequently, $a + b$ is the required divisor.

(**35.**) It may be proper here to notice that there are some particular methods by which the greatest common divisor can often be found with more facility than by the general rule.

1. One of these methods consists in resolving the quantities into factors, and then taking the product of the factors that are common to them for their greatest common divisor. Thus, if we take the quantities $3a^2 - 2a - 1$ and $4a^3 - 2a^2 - 3a + 1$, given in Bonnycastle's Algebra, Case I. of Fractions, since the quantities are both reduced to 0 by putting $a = 1$, we infer that they involve $a - 1$ as a common factor; and dividing each of them by $a - 1$, we easily find that we shall have $3a^2 - 2a - 1 = (3a + 1)(a - 1)$, $4a^3 - 2a^2 - 3a + 1 = (4a^2 + 2a - 1)(a - 1)$, which show that $a - 1$ is the greatest common divisor.

Again, in the same case, Bonnycastle requires the greatest common divisor of $7a^2 - 23ab + 6b^2$ and $5a^3 - 18a^2b + 11ab^2 - 6b^3$.

Since the quantities are both reduced to 0 by putting $a = 3b$, it follows that they have $a - 3b$ for a common factor. And it is easy to put the expressions in the forms, $7a^2 - 23ab + 6b^2 = (7a - 2b)(a - 3b)$, $5a^3 - 18a^2b + 11ab^2 - 6b^3 = (5a^2 - 3ab + 2b^2) \times (a - 3b)$, which show $a - 3b$ to be the greatest common divisor.

As this method has been heretofore used, we shall not prosecute it any further.

2. Another method is deduced from the circumstance, that any factor which is common to any two quantities is common to any number of times one of them increased or diminished by any number of times the other.

Thus, in the same case from Bonnycastle, it is required to find the greatest common divisor of $a^2 - ab - 2b^2$ and $a^2 - 3ab + 2b^2$.

It is easy to see that, if we take the sum of the quantities, we shall have $2a(a - 2b)$, and if we take their difference we have $2b(a - 2b)$; it hence follows that $a - 2b$ is the sought divisor.

Again, in Case II. of Bonnycastle, the greatest common divisor of the quantities $6a^2 + 7ax - 3x^2$ and $6a^2 + 11ax + 3x^2$ is required. Here, proceeding in the same way as before, we get $2a + 3x$ for the sought divisor.

EXAMPLES FOR EXERCISE.

1. Find the greatest common divisor of 1143 and 1791. *Ans.* 9.

2. Find the greatest common divisor of 637, 889, and 4949. *Ans.* 7.

3. Find the greatest common divisor of $x^2 - 2xy + y^2$ and $x^2 - y^2$. *Ans.* $x - y$.

4. Find the greatest common divisor of $3x^3 - 2x^2y - 3xy^2 + 2y^3$ and $3x^3 - 2x^2y + 3xy^2 - 2y^3$. *Ans.* $3x - 2y$.

5. Find the greatest common measure of $a^8 - b^8$ and $a^6 - a^5b + ab^5 - b^6$. *Ans.* $a^2 - b^2$.

6. Find the greatest common measure of $21x^4 - 69x^3 + 51x^2 - 99x$ and $70x^5 - 174x^4 + 238x^3 - 322x^2 + 148x - 88$. *Ans.* $7x^2 - 2x + 11$.

7. Give the greatest common measure of $2x^4 + 4x^3 + 3x^2 + 2x + 1$ and $8x^3 + 12x^2 + 6x + 2$. *Ans.* $x + 1$.

8. Give the greatest common divisor of $18z^3 + 45z^2y - 8zy^2 - 20y^3$ and $54z^4 + 135z^3y - 16zy^3 - 40y^4$. *Ans.* $6z^2 + 11zy - 10y^2$.

9. Determine the greatest common divisor of $a^4 + a^3b + ab^3 + b^4$, $a^5 + a^4b - ab^4 - b^5$, and $a^7 + a^6b - ab^6 - b^7$. *Ans.* $(a + b)^2$.

10. Determine the greatest common divisor of $x^3 + 2x^2 - 4x + 1$, $x^5 + 5x^4 - 7x^3 - 11x^2 + 2x + 10$, and $x^6 - 3x^3 - 5x + 7$. *Ans.* $x - 1$.

11. Find the greatest common divisor of $(y^2 + 1)x^3 - mx^2 - (y^2 + 1)x + m$, and $(p-1)(y^2+1)x^2 + [n(y^2-1) - m(p-1)]x - mn$, by using the vertical bar for the parenthesis. *Ans.* $(y^2 + 1)x - m$.

(36.) OF THE LEAST COMMON MULTIPLE.

The least common multiple of two or more numbers, or quantities of the same kind, is the least number or quantity which can be divided by each of them without a remainder.

If we use am and bm to stand for two numbers or quantities of the same kind, having m for their greatest common divisor, it is easy to see that their least common multiple must be the product of a, b, and m, or abm. For the least common multiple must involve a and m as factors, in order to be divisible by am, and it must involve b and m as factors in order that it may be divisible by bm.

Since $abm = \frac{am \times bm}{m} = \frac{am}{m} \times bm = \frac{bm}{m} \times am$, we easily find the least common multiple of two or more numbers, or quantities of the same kind, by the following

RULE.

Divide the product of any two of the numbers or quantities by their greatest common divisor; or divide one of them by their greatest common divisor, and multiply the quotient by the other, and the result is the least common multiple of the two numbers or quantities. Then the least common multiple of one of the remaining numbers or quantities and the least common multiple just found, is clearly the least common multiple of the three numbers or quantities used. If there are more than three numbers or quantities, we proceed in the same way to find the least common multiple of one of the remaining numbers or quantities, and of the least common multiple of the three just found; and so on, for any number of quantities.

EXAMPLES.

Ex. 1.—To find the least common multiple of 36 and 48.

Since 12 is the greatest common divisor of 36 and 48, we have $36 \div 12 = 3$, and $3 \times 48 = 144 =$ the required least common multiple.

Ex. 2.—To find the least common multiple of 36, 48, and 33.

By the first example, 144 is the least common multiple of 36 and 48; and since 3 is the greatest common divisor of 144 and 33, we have $33 \div 3 = 11$, and by the rule $144 \times 11 = 1584$, is the least common multiple required.

Ex. 3.—To find the least common multiple of $a^2 - b^2$ and $a^3 + b^3$.

It is easy to see that $a + b$ is the greatest common divisor of the quantities; and since $a^2 - b^2 = (a + b)(a - b)$, we have $(a^2 - b^2) \div (a + b) = a - b$. Hence we shall have $(a^3 + b^3)(a - b) = a^4 - a^3b + ab^3 - b^4$ for the least common multiple.

Ex. 4.—To find the least common multiple of $a^2 - b^2$, $a^3 + b^3$, and $a^6 - b^6$.

By Ex. 3, $(a^3 + b^3)(a - b)$ is the least common multiple of $a^2 - b^2$, $a^3 + b^3$; and since $a^6 - b^6$ is divisible by $(a^3 + b^3)(a - b)$, we shall have $a^6 - b^6$ for the sought least common multiple.

Ex. 5.—To find the least common multiple of $3a^2 - 2a - 1$ and $4a^3 - 2a^2 - 3a + 1$.

Since $a - 1$ is the greatest common divisor of the quantities, and that $(3a^2 - 2a - 1) \div (a - 1) = 3a + 1$, we get $(4a^3 - 2a^2 - 3a + 1)(3a + 1) = 12a^4 - 2a^3 - 11a^2 + 1$ for the sought least common multiple.

EXAMPLES FOR EXERCISE.

1. Find the least common multiple of 13 and 19.

Ans. $19 \times 13 = 247$.

2. Find the least common multiple of 6, 7, and 21.

Ans. 42.

3. Find the least common multiple of $a + b$ and $x - y$.

Ans. $(a + b)(x - y)$.

4. Find the least common multiple of $x + y$ and $x - y$.

Ans. $x^2 - y^2$.

5. Find the least common multiple of $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, and $a^2 - b^2$. *Ans.* $(a^2 - b^2)^2$.

6. Find the least common multiple of $a^4 + b^4$ and $a^3 + b^3$.

Ans. $(a^4 + b^4)(a^3 + b^3)$.

7. Find the least common multiple of $a^3 + b^3$, $a^5 + b^5$, and $a^7 + b^7$.

Ans. $(a^7 + b^7)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)(a^2 - ab + b^2)$.

8. Find the least common multiple of $a^2 - ab - 2b^2$ and $a^2 - 3ab + 2b^2$. *Ans.* $(a^2 - ab - 2b^2)(a - b)$.

9. Find the least common multiple of $6a^2 + 7ax - 3x^2$ and $6a^2 + 11ax + 3x^2$. *Ans.* $(6a^2 + 7ax - 3x^2)(3a + x)$.

(27.) *Remark.*—Many arithmetical and algebraic questions are easily solved by the use of the least common multiple. For examples, take the following questions.

1. A and B denote two individuals, such that A alone can do a piece of work in 12 days, and B alone can do the work in 14 days; then, in what time can A and B together do the work?

The least common multiple of 12 and 14 is 84. And since $84 \div 12 = 7$, and $84 \div 14 = 6$, it follows that A alone in 84 days can do 7 times the work, and B alone in 84 days can do 6 times the work; consequently, both together in 84 days can do 13 times the work. Hence, if we divide 84 by 13, we shall have $\frac{84}{13} = 6\frac{6}{13}$ = the number of days in which A and B together can do the work, as required.

2. Two pipes, A and B, are such that A will fill a certain cistern in an hour and a half, and B will fill it in two hours and a quarter. In what time will A and B together fill the cistern?

Take a quarter of an hour for the unit of time, then 6 and 9 will respectively represent the times in which A and B can fill the cistern.

Now 18 is the least common multiple of 6 and 9; and since $18 \div 6 = 3$ and $18 \div 9 = 2$, it follows that A alone will fill the cistern 3 times and B alone will fill the cistern 2 times in 18 units of time; consequently A and B together will fill the cistern 5 times in 18 units of time. Hence, if we divide 18 by 5, we have $\frac{18}{5} = 3\frac{3}{5}$ units of time in which A and B together will fill the cistern; or, since our unit of time equals 15 minutes, we have $3\frac{3}{5} \times 15 = 54$ = the number of minutes in which A and B together will fill the cistern.

3. A man and his wife usually drank out a cask of beer in 12 days, but when the man was from home it lasted the woman 30 days. How many days would the man alone be in drinking it?

The least common multiple of 12 and 30 is 60; and since $60 \div 12 = 5$ and $60 \div 30 = 2$, it follows that the man and woman would drink out the cask of beer 5 times in 60 days, and that the woman alone would drink out the cask of beer

2 times in 60 days. Hence, by subtraction, 5 times — 2 times $= 3$ times, is the number of times that the man alone would drink out the cask of beer in 60 days. Consequently, if we divide 60 by 3, we have $60 \div 3 = 20 =$ the number of days in which the man alone would drink out the cask of beer.

4. Let A, B, and C denote three individuals, such that A can do a certain piece of work in the time a, and B can do the same piece of work in the time b, and C can do it in the time c. Required, the time in which A, B, and C can jointly perform the work?

Since a, b, and c are under the form of primes, or are not explicitly composed of factors, their product abc must be taken for their least common multiple. Hence, $abc \div a = bc$, $abc \div b = ac$, $abc \div c = ab$, and $bc + ac + ab =$ the number of times that A, B, and C together can do the work in abc days. Hence, we shall have $\frac{abc}{ab + ac + bc} =$ the time in which A, B, C, jointly, can perform the work.

If $a = 6$ days, $b = 8$ days, $c = 24$ days, then putting 6 for a, 8 for b, and 24 for c, we shall have $\frac{abc}{ab + ac + bc} = \frac{6 \times 8 \times 24}{48 + 144 + 192} = \frac{1152}{384} = 3 =$ the number of days in which A, B, and C can jointly perform the work.

5. If A and B can perform a piece of work in a days, A and C in b days, B and C in c days, then in what time can A, B, and C jointly perform the work?

Here, abc expresses the least common multiple of a, b, and c. And $abc \div a = bc =$ the number of times that A and B can do the work in abc days; also, $abc \div b = ac =$ the number of times that A can do the work in abc days; and $abc \div c = ab =$ the number of times that B and C can do the work in abc days.

Hence, A and B, A and C, B and C, or $2A + 2B + 2C$, can perform $(ab + ac + bc)$ times the work in abc days, or $A + B + C$ can perform $\frac{ab + ac + bc}{2}$ times the work in abc days. Hence we shall have $abc \div \frac{ab+ac+bc}{2} = \frac{2abc}{ab+ac+bc}$ = the time in which A, B, and C together can perform the work. If $a=8$, $b=9$, $c=10$, then the time in which A, B, and C can jointly perform the work will be found to be $5\frac{115}{121}$ days.

Again, since A + B + C can do $\frac{ab+ac+bc}{2}$ times the work in abc days, and B + C can do the work ab times in abc days, we shall get, by subtraction, $\frac{ab+ac+bc}{2}-ab=\frac{ac+bc-ab}{2}=$ the number of times in which A alone can do the work in abc days. Hence, $\frac{2abc}{ac+bc-ab}=$ the time in which A alone can perform the work; and in like manner $\frac{2abc}{ab+bc-ac}$ is the time in which B alone can do the work, and $\frac{2abc}{ab+ac-bc}$ is the time in which C alone can do the work.

If $a=8$, $b=9$, $c=10$, it will be found that A can do the work in $14\frac{34}{49}$ days, and that B can do the work in $17\frac{23}{41}$ days, and C can do the work in $23\frac{7}{31}$ days.

SECTION VII.

OF FRACTIONS.

(1.) If we represent any quantity by 1, and wish to show that the quantity is to be divided into a equal parts, and that b of the parts are to be used, we proceed as follows, viz.:

Conceive the quantity to consist of a equal parts; then, since 1 stands for the quantity, it is clear that $\frac{1}{a}$ will stand for one of the equal parts into which the quantity is to be divided, since the quotient $\frac{1}{a}$ expresses one part out of the a parts which are supposed to compose the quantity.

Since b of the equal parts are to be used, we must multiply $\frac{1}{a}$ by b, and we shall have $\frac{1}{a} \times b$ for the proper expression of the portion of the quantity that is to be used. Because $\frac{1}{a} \times b$ is an expression for b equal parts out of the a equal parts into which the quantity (that is represented by 1) is supposed to be broken or divided, it is called a *fraction;* and since a denominates or shows into how many equal parts the quantity is to be divided, it is called the *denominator of the fraction;* and because b enumerates or shows how many of the equal parts into which the quantity is divided are to be taken, it is called the *numerator of the fraction;* and since the value of the fraction depends on its numerator and denominator, *we call the numerator and denominator the terms of the fraction.*

Because $\frac{1}{a} \times b$ is the same as $\frac{1}{a} + \frac{1}{a} + \frac{1}{a} +$ etc., to b terms, and since $\frac{1}{a} + \frac{1}{a} + \frac{1}{a} +$ etc., to b terms (by addition) equals $\frac{b}{a}$, we shall have $\frac{1}{a} \times b = \frac{b}{a}$, which shows that the fraction can be expressed by writing the denominator below the numerator, with a right line between them; and it is clear that the fraction $\frac{b}{a}$ expresses the quotient of the division of the numerator by the denominator. Hence, generally, any quotient which is indicated by writing the divisor below the dividend, with a right line between them, is called a *fraction*, particularly if the quotient expresses a portion of some quantity that is represented by 1; *the dividend being called the numerator and the divisor the denominator of the fraction; also the numerator and denominator are called the terms of the fraction.*

(**2.**) Any fraction whose numerator is less than its denominator is called *a proper fraction.* Thus, $\frac{3}{4}, \frac{4}{5}, \frac{7}{13}$ are proper fractions.

(**3.**) Any fraction whose numerator is not less than its denominator is an *improper fraction*. Thus, $\frac{2}{2}, \frac{3}{2}, \frac{8}{5}$ are improper fractions.

(**4.**) Any quantity which is not of a fractional form is said to be of an *integral form*. Thus, 2, 3, a, b are integral forms.

(**5.**) Any quantity whose expression is partly integral and partly fractional is called a *mixed number or form*. Thus, $2\frac{1}{3}, 3\frac{4}{5}, a + \frac{b}{c}, p + \frac{q-s}{r+v}$ are mixed numbers or forms.

(**6.**) Any single expression which denotes any number of parts of the quantity which is expressed by 1, is called a *single or simple fraction*. Thus, $\frac{3}{4}, \frac{5}{9}, \frac{a}{b}, \frac{c-d}{a+b}$ are single fractions.

(**7.**) A fraction of a fraction is called a *compound fraction*. Thus, $\frac{3}{4}$ of $\frac{7}{8}$ and $\frac{a}{b} \times \frac{c}{d}$ are compound fractions.

(**8.**) Because the value of a quantity is not affected by dividing it by 1, we may express any integer or integral quantity in the form of a fraction, by writing 1 under it. Thus, for 3 and a, we may write $\frac{3}{1}, \frac{a}{1}$.

(**9.**) It is evident, from the rule of signs in Division, if the entire numerator and denominator of a fraction have like signs, that the fraction is positive; but if they have unlike signs, the fraction is negative.

And it is clear that we may change the signs of the entire numerator and denominator of a fraction without affecting its sign.

(**10.**) If any term, either of the numerator or denominator of a fraction has no sign expressed, the sign + is to be understood. Also, if no sign is prefixed to the line that separates the numerator from the denominator of a fraction, the sign + must be understood.

(**11.**) If the sign — is prefixed to the line that separates the numerator from the denominator of a fraction, the meaning is that the fraction is to be subtracted, or that the signs of its entire numerator or denominator must be changed. Thus, $\frac{a}{b} - \frac{c-d}{e-f}$ requires the sign of $c-d$ or of $e-f$ to be changed, and is equivalent to $\frac{a}{b} + \frac{-c+d}{e-f}$ or to $\frac{a}{b} + \frac{c-d}{-e+f}$.

(**12.**) Fractions are said to be reduced, when they are changed from one form of expression to any other equivalent form.

(13.) REDUCTION OF FRACTIONS.

CASE I.

To reduce fractions to their lowest terms.

RULE.

Erase all the factors that are common to the numerator and denominator; or, which comes to the same thing, divide both the numerator and denominator by their greatest common divisor, and the fraction will be reduced as required. According to what was shown in Division, the reduced fraction will be equivalent to the given fraction.

Thus, $\frac{a^2b}{a^2c} = \frac{b}{c}$ by erasing the factor a^2 from the numerator and denominator.

Also, $\frac{a^3+b^3}{a^2-b^2} = \frac{a^2-ab+b^2}{a-b}$ by dividing the numerator and denominator by their greatest common divisor $a+b$.

EXAMPLES.

1. Reduce $\frac{5863}{6437}$ to its lowest terms. *Ans.* $\frac{143}{157}$.

2. Reduce $\frac{a^4-b^4}{a^3+b^3}$ to its lowest terms.

Ans. $\frac{a^3-a^2b+ab^2-b^3}{a^2-ab+b^2}$.

3. Reduce $\frac{a^2x^2+2ax^3+x^4}{a^3+3a^2x+3ax^2+x^3}$ to its lowest terms.

Ans. $\frac{x^2}{a+x}$.

4. Reduce $\frac{a^3-b^3}{a^6-b^6}$ to its lowest terms.

Ans. $\frac{a^2+ab+b^2}{a^4+a^3b+a^2b^2+ab^3+b^4}$.

5. Reduce $\frac{75a^3b^3(x^3-y^3)^{\frac{5}{2}}}{45a^4b^4(x^3-y^3)^{\frac{2}{3}}}$ to its lowest terms.

Ans. $\frac{5(x^3-y^3)^{\frac{11}{6}}}{3ab}$.

6. Reduce $\frac{x^3-3x^2+5x-3}{x^4+7x^2-3x-5}$ to its lowest terms.

Ans. $\frac{x^2-2x+3}{x^3+x^2+8x+5}$.

7. Reduce $\frac{a^5-a^3b^2+a^2b^3-b^5}{a^5+a^3b^2+a^2b^3+b^5}$ to its lowest terms.

Ans. $\frac{a^2-b^2}{a^2+b^2}$.

8. Reduce $\frac{x^3+x^2+x+1}{x^5+x^4+x^3+x^2+x+1}$ to its lowest terms.

Ans. $\frac{x^2+1}{x^4+x^2+1}$.

9. Reduce $\frac{6a^4-10a^3b+23a^2b^2-15ab^3+21b^4}{6a^5-16a^4b+36a^3b^2-40a^2b^3+38ab^4-14b^5}$ to its lowest terms.

Ans. $\frac{2a^2+3b^2}{2a^3-2a^2b+4ab^2-2b^3}$.

CASE II.

To reduce a mixed quantity to a fraction, usually called an *improper fraction.*

RULE.

Multiply the integral part by the denominator of the fraction; then add the product and the numerator of the fraction, according to their signs; and the result being placed over the denominator, with a line drawn between them, will be the fraction required.

Thus, $a \pm \frac{b}{c} = \frac{ac \pm b}{c}$; for since $a = \frac{ac}{c}$, we get $a \pm \frac{b}{c} = \frac{ac}{c} \pm \frac{b}{c} = \frac{ac \pm b}{c}$; for clearly the sum of the quotients of ac divided by c, and $\pm b$ divided by c, must equal the quotient of $ac \pm b$ divided by c.

EXAMPLES.

1. Reduce $1 + \frac{b}{a}$ to a fraction.

Here $1+\frac{b}{a}=\frac{a+b}{a}$; for $1=\frac{a}{a}$ and $1+\frac{b}{a}=\frac{a}{a}+\frac{b}{a}=\frac{a+b}{a}$.

2. Reduce $3a-\frac{2b}{5cd}$ to a fraction.

Here $3a=\frac{15acd}{5cd}$, and $3a-\frac{2b}{5cd}=\frac{15acd}{5cd}-\frac{2b}{5cd}=\frac{15acd-2b}{5cd}$.

3. Reduce $6\sqrt{x^2-y^2}+\frac{5ab}{4cd}$ to a fraction.

Since $6\sqrt{x^2-y^2}=\frac{24cd\sqrt{x^2-y^2}}{4cd}$, we shall have $6\sqrt{x^2-y^2}+\frac{5ab}{4cd}=\frac{24cd\sqrt{x^2-y^2}+5ab}{4cd}$.

4. Reduce $27-\frac{47}{11}$ to a fraction. *Ans.* $\frac{250}{11}$.

5. Reduce $2\sqrt{x^2+y^2}-\frac{x^2+y^2}{\sqrt{x^2+y^2}}$ to a fraction or integer.

Ans. $\frac{x^2+y^2}{\sqrt{x^2+y^2}}=\sqrt{x^2+y^2}$.

6. Reduce $15-\frac{7cd}{a^2b^3}$ to a fraction. *Ans.* $\frac{15a^2b^3-7cd}{a^2b^3}$.

7. Reduce $19abc^{\frac{1}{2}}+\frac{17mp^2}{13abc^{\frac{3}{2}}}$ to a fraction.

Ans. $\frac{247a^2b^2c^2+17mp^2}{13abc^{\frac{3}{2}}}$

8. Reduce $12\sqrt{ac}-\frac{5bc}{13mp}$ to a fraction.

Ans. $\frac{156mp\sqrt{ac}-5bc}{13mp}$.

9. Reduce $a+b-\frac{a^2+b^2}{a+b}$ to a fraction. *Ans.* $\frac{2ab}{a+b}$.

10. Reduce $2a-\frac{3a^2-2b^2}{2a}$ to a fraction. *Ans.* $\frac{a^2+2b^2}{2a}$.

11. Reduce $3a^2 - 5b^2 + \dfrac{7a^2 - 11b^2}{14a - 19b}$ to a fraction.

Ans. $\dfrac{42a^3 - (57b - 7)a^2 - 70b^2a - 11b^2 + 95b^3}{14a - 19b}$.

CASE III.

To reduce an improper fraction to an integral or mixed quantity.

RULE.

Divide the numerator by the denominator for the integral part, and if there is a remainder write it over the denominator, with its proper sign, for the fractional part.

This case is the reverse of the last, and the reason of the rule requires no explanation.

EXAMPLES.

1. Reduce $\dfrac{18}{6}$, $\dfrac{15}{4}$, $\dfrac{a^2 + b}{a}$, to integral or mixed quantities.

Here we get $\dfrac{18}{6} = 3$, $\dfrac{15}{4} = 3\dfrac{3}{4}$, $\dfrac{a^2 + b}{a} = a + \dfrac{b}{a}$.

2. Reduce $\dfrac{a^2 - b^2}{a + b}$ and $\dfrac{a^3 + b^3}{a + b}$ to integral quantities.

Ans. $a - b$ and $a^2 - ab + b^2$.

3. Reduce $\dfrac{65x^3 - 7x^2 + 9x - 11}{13x^2 - 3x + 5}$ to a mixed form.

Ans. $5x + \dfrac{8x^2 - 16x - 11}{13x^2 - 3x + 5}$.

4. Reduce $\dfrac{a^4 - b^4}{a^3 + b^3}$ to a mixed form.

Ans. $a - \dfrac{(a + b)b^3}{a^3 + b^3}$, or $a - \dfrac{b^3}{a^2 - ab + b^2}$.

5. Reduce $\dfrac{13a^4 - 15a^3b - 7ab^2 + b^4}{13a^3 - 6ab + b^2}$ to a mixed form.

Ans. $a + \dfrac{-15a^3b + 6a^2b - 8ab^2 + b^4}{13a^3 - 6ab + b^2}$.

6. Reduce $\dfrac{2a^2 - 3b^2}{a^2 + b^2}$ and $\dfrac{14a^2 + b^2}{7a^2 - b^2}$ to mixed forms.

Ans. $2 - \dfrac{5b^2}{a^2 + b^2}$ and $2 + \dfrac{3b^2}{7a^2 - b^2}$.

7. Reduce $\frac{12a^3+6ab-2b^2}{6a}$ to a mixed form.

Ans. $2a^2+b-\frac{b^2}{3a}$.

8. Reduce $\frac{a^3+3a^2b+3ab^2+b^3}{a^2+2ab+b^2}$ and $\frac{4a^2-8ab+4b^2}{2a-2b}$ to integral forms. *Ans.* $a+b$ and $2a-2b$.

CASE IV.

To reduce fractions to a common denominator, without altering their values.

RULE I.

Multiply each numerator by all the denominators except its own, for a new numerator; and multiply all the denominators together for a new denominator.

RULE II.

(Which gives the least common denominator.)

Take the least common multiple of all the denominators for a new denominator. Then divide the new denominator by the denominator of each fraction, and multiply the quotient by the numerator, and the product will be the new numerator of the fraction.

The reason of the first rule is, that the numerator and denominator of each fraction is multiplied by the product of all the other denominators, so that the values of the fractions remain unchanged.

And the reason of the second rule is much the same; for the numerator and denominator of each fraction is multiplied by the quotient obtained from the division of the new (or common) denominator by the denominator of the fraction.

EXAMPLES.

1. Reduce $\frac{3}{5}$ and $\frac{5}{7}$ to a common denominator.

Here $\frac{3\times 7}{5\times 7}=\frac{3}{5}$, and $\frac{5\times 5}{5\times 7}=\frac{5}{7}$, so that $\frac{21}{35}$ and $\frac{25}{35}$ are the fractions when reduced to a common denominator.

2. Reduce $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ to a common denominator.

By Rule I., multiply a and b by the product df, and we get $\frac{adf}{bdf}$ for the first new fraction; and in the same way, if we multiply c and d by the product bf, we get $\frac{bcf}{bdf}$ for the second new fraction; and in a similar way we get $\frac{bed}{bdf}$ for the third new fraction; so that $\frac{adf}{bdf}, \frac{bcf}{bdf}, \frac{bde}{bdf}$ are the new fractions, or the fractions when reduced to a common denominator.

3. Reduce $\frac{ab}{cx^3}, \frac{de}{fx^3}, \frac{gh}{kx^4}$ to the least common denominator.

The least common multiple of the denominators is easily found to be $cfkx^4$, which is to be the common denominator of the new fraction, or of the fractions when reduced to a common denominator. Dividing (by Rule II.) $cfkx^4$ by cx^3, the denominator of the first fraction, we get fkx^2 for the quotient, and multiplying this quotient by ab, the numerator of the first fraction, we get $abfkx^2$ for the new numerator of the first new fraction; in like manner, the new numerator of the second new fraction is found to be $cdekx$, and that of the third fraction is $cfgh$, so that the new fractions are $\frac{abfkx^2}{cfkx^4}, \frac{cdekx}{cfkx^4}, \frac{cfgh}{cfkx^4}$, which are easily seen to be equivalent to the given fractions.

4. Reduce $\frac{2a}{3b}, \frac{5c}{7e}, \frac{8d}{5f}$ to a common denominator.

Ans. $\frac{70aef}{105bef}, \frac{75bcf}{105bef}, \frac{168bde}{105bef}$.

5. Reduce $\frac{2a+3b}{a^2-b^2}, \frac{4a^2-5b^2}{a^3-b^3}, \frac{6abc}{a^3+b^3}$ to the least common denominator.

Ans. $\frac{(2a+3b)(a^4+a^2b^2+b^4)}{(a^3-b^3)(a^3+b^3)}, \frac{(4a^2-5b^2)(a^3+b^3)}{(a^3-b^3)(a^3+b^3)}, \frac{6abc(a^3-b^3)}{(a^3-b^3)(a^3+b^3}$

6. Reduce $\frac{2}{3}, \frac{3}{5}, \frac{7x^2}{4y^2}, \frac{9z^2}{5p^2}$ to a common denominator.

Ans. $\frac{200p^2y^2}{300p^2y^2}, \frac{180p^2y^2}{300p^2y^2}, \frac{525p^2x^2}{300p^2y^2}, \frac{540y^2z^2}{300p^2y^2}$.

7. Reduce $\frac{a+b}{a-b}, \frac{a-b}{a+b}, \frac{a^2-b^3}{a^3+b^3}$ to a common denominator.

Ans. $\frac{(a+b)^2(a^3+b^3)}{(a^2-b^2)(a^3+b^3)}, \frac{(a-b)^2(a^3+b^3)}{(a^2-b^2)(a^3+b^3)}, \frac{(a^2-b^2)(a^2-b^3)}{(a^2-b^2)(a^3+b^3)}$,

8. Reduce $\frac{x}{4}, \frac{y}{5}, \frac{3+a}{9+a}, \frac{1+x}{1-x}, \frac{2-x}{2+x}$ to a common denominator.

Ans. $\frac{(45+5a)(2x-x^2-x^3)}{20(9+a)(2-x-x^2)}, \frac{(36+4a)(2y-xy-x^2y)}{20(9+a)(2-x-x^2)}$, $\frac{(60+20a)(2-x-x^2)}{20(9+a)(2-x-x^2)}, \frac{(180+20a)(2+3x+x^2)}{20(9+a)(2-x-x^2)}$, $\frac{(180+20a)(2-3x+x^2)}{20(9+a)(2-x-x^3}$.

9. Reduce $\frac{5}{27}, \frac{13}{81}, \frac{7b}{9a^2}, \frac{5bc}{6a^3g}$ to the least common denominator. *Ans.* $\frac{30a^3g}{162a^3g}, \frac{26a^3g}{162a^3g}, \frac{126abg}{162a^3g}, \frac{135bc}{162a^3g}$.

10. Reduce 5, 6, $\frac{3a^2+5b^2}{17}, \frac{ac-be}{11m^2}$ to a common denominator. *Ans.* $\frac{935m^2}{187m^2}, \frac{1122m^2}{187m^2}, \frac{m^2(33a^2+55b^2)}{187m^2}, \frac{17ac-17be}{187m^2}$.

CASE V.

To add fractional quantities together.

RULE.

Reduce the fractions (when necessary) to a common denominator; then write the sum of the numerators (added according to their signs) over the common denominator, and the result will be the sum required.

To show the correctness of the rule, we shall take the fractions $\frac{a}{m}, \frac{b}{m}, -\frac{c}{m}$, having m for their common denominator; then we shall have $\frac{a}{m}+\frac{b}{m}-\frac{c}{m}=\frac{a+b-c}{m}$. For since $a=\frac{am}{m}$, $b=\frac{bm}{m}$, $-c=-\frac{cm}{m}$, and $a+b-c=\frac{(a+b-c)m}{m}$, we get (by Addition and Ax. IV.) $\frac{am}{m}+\frac{bm}{m}-\frac{cm}{m}=\frac{(a+b-c)m}{m}$; or, omitting m, which is clearly a common factor of these equals, we have $\frac{a}{m}+\frac{b}{m}-\frac{c}{m}=\frac{a+b-c}{m}$, agreeably to the rule; and it is clear that a similar proof is applicable in all cases of the addition of fractions, after they have been reduced to a common denominator.

EXAMPLES.

1. Add $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$; also add $\frac{a}{5}$ and $\frac{a}{6}$.

Here $\left.\begin{array}{l} 2\times 5\times 7=70 \\ 4\times 3\times 7=84 \\ 6\times 3\times 5=90 \end{array}\right\}$ are the new numerators of the first fractions, and $3\times 5\times 7=105=$ their common denominator. Hence the sum is expressed by $\frac{70}{105}+\frac{84}{105}+\frac{90}{105}=\frac{244}{105}=2+\frac{34}{105}$, as required.

Also $\left.\begin{array}{l} a\times 6=6a \\ a\times 5=5a \end{array}\right\}$ are the new numerators of the second fractions, and $5\times 6=30=$ their common denominator. Hence the sum is expressed by $\frac{6a}{30}+\frac{5a}{30}=\frac{11a}{30}=$ the sum as required.

2. Add the fractions $\frac{a}{2}, \frac{a}{3}, \frac{a}{4}, \frac{a}{5}, \frac{a}{6}$. *Ans.* $\frac{87a}{60}=a+\frac{9a}{20}$.

3. Add the fractions $\frac{a+b}{a-b}, \frac{a-b}{a+b}$ and $\frac{s+d}{2}, \frac{s-d}{2}$.

Ans. $\frac{2(a^2+b^2)}{a^2-b^2}$, and s.

Since $\frac{s+d}{2}$ is the half sum of s and d, and $\frac{s-d}{2}$ is the half difference (supposed positive) of s and d, and since their sum gives s the greater quantity, *we see that the half sum of two quantities, when added to their half difference, gives the greater quantity.* Thus, $\frac{8+3}{2}+\frac{8-3}{2}$ gives 8.

4. Add the fractions $\frac{3a-5b}{7}, \frac{11a-6b}{14}, \frac{13a-5b}{21}$.

Ans. $\frac{77a-58b}{42}=a-b+\frac{35a-16b}{42}$.

5. Add $9a$, $11a$, $6a+\frac{7b}{3c}$, $4a-\frac{9b}{6c^2}$, $5a-\frac{13b^2}{18c^3}$.

Ans. $35a+\frac{(42bc^2-27bc-13b^2)}{18c^3}$.

6. Add $\frac{1}{a+b}, \frac{1}{a^2-ab+b^2}$, and $\frac{1}{a-b}, \frac{1}{a+b}$.

Ans. $\frac{a^2-a(b-1)+b(b+1)}{a^3+b^3}$ and $\frac{2a}{a^2-b^2}$.

7. Add $\frac{3x^4-5x^3+6}{9x^2}, \frac{2x^3-5x+7}{18x}, \frac{5x^2-3}{24}$.

Ans. $\frac{47x^4-40x^3-29x^2+28x+48}{72x^2}$.

8. Add $\frac{5x^2-4x+2}{3x^3-4}, \frac{2x-5}{4x^2-1}, \frac{5x-11}{2x+1}, \frac{7}{8}$.

Ans. $\frac{324x^5-440x^4-5x^3-408x^2+832x-180}{8(4x^2-1)(3x^3-4)}$.

CASE VI.

To subtract one fractional quantity from another.

RULE.

Let the fractions be brought to a common denominator, as in Addition. Then subtract the numerator of the fraction to be subtracted, from the numerator of the other fraction; and the result, when written over the common denominator, will give the required difference.

EXAMPLES.

1. From $\frac{3}{4}$ take $\frac{5}{9}$, or reduce $\frac{3}{4}-\frac{5}{9}$.

Here $\frac{3}{4}-\frac{5}{9}=\frac{3\times 9}{36}-\frac{5\times 4}{36}=\frac{27}{36}-\frac{20}{36}=\frac{7}{36}$, as required.

2. Reduce $\frac{a^2}{3} - \frac{a^2}{5}$ and $\frac{a+b}{a-b} - \frac{a-b}{a+b}$.

Here $\frac{a^2}{3} - \frac{a^2}{5} = \frac{5a^2 - 3a^2}{15} = \frac{2a^2}{15}$, as required. And $\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{(a+b)^2}{(a-b)(a+b)} - \frac{(a-b)^2}{(a-b)(a+b)} = \frac{a^2 + 2ab + b^2}{a^2 - b^2} - \frac{a^2 - 2ab + b^2}{a^2 - b^2} = \frac{4ab}{a^2 - b^2}$, as required.

3. Reduce $\frac{3x^2 - 5}{7} - \frac{2x^2 - 9}{14}$, and $\frac{s+d}{2} - \frac{s-d}{2}$.

Ans. $\frac{4x^2 - 1}{14}$ and d. If we suppose s to be greater than d, so that $s - d$ is positive, then, since $\frac{s+d}{2}$ is the half sum of s and d, and $\frac{s-d}{2}$ is the half difference of s and d, and since $\frac{s+d}{2} - \frac{s-d}{2} = d$, *we see that the half difference of two quantities, when subtracted from their half sum, gives the less quantity.* Thus, $\frac{8+3}{2} - \frac{8-3}{2}$ gives 3.

4. Reduce $\frac{5x^3 + 8x^2 - 6}{2x - 1} - \frac{3x^2 - 7x - 2}{2x + 1}$.

Ans. $\frac{10x^4 + 15x^3 + 25x^2 - 15x - 8}{4x^2 - 1}$.

5. Reduce $\frac{a^3 + b^3}{2(a^3 - b^3)} - \frac{a^3 - b^3}{3(a^3 + b^3)}$. *Ans.* $\frac{a^6 + 10a^3b^3 + b^6}{6(a^6 - b^6)}$.

6. Reduce $9a + \frac{5x^2y}{8vw} - \left(6a + \frac{7x^2y}{8vw}\right)$.

Ans. $3a - \frac{x^2y}{4vw} = \frac{12avw - x^2y}{4vw}$.

7. Reduce $\left(\frac{a}{b} + \frac{c}{d}\right) - \left(\frac{a}{2b} - \frac{3c}{4d}\right)$.

Ans. $\frac{a}{2b} + \frac{7c}{4d} = \frac{2ad + 7bc}{4bd}$.

8. Reduce $\frac{a+x}{a-x} - \frac{2a+5x}{3a-2x}$ and $\frac{a}{a-x} - \frac{a}{a+x}$.

Ans. $\frac{a^2 - 2ax + 3x^2}{(a-x)(3a-2x)}$ and $\frac{2ax}{a^2 - x^2}$.

Remarks.—Positive quantities having the sign + either expressed or understood, and negative quantities (or quantities to be subtracted) having the sign −, it is evident that we may reduce the rules of Addition and Subtraction under the general rule of Addition, as follows; viz., collect the quantities into one sum (or whole) by adding them according to their signs; and that whether they are integral or fractional, or if some are integral and others fractional. Generally, in the addition of mixed quantities, it will be advisable to add the integral quantities into one sum, and then to add the fractional quantities into another, after they have been reduced to a common denominator; then the sum of the integral quantities and that of the fractional quantities being written together with their proper signs will be the sum required.

Thus, suppose it is required to subtract $11\frac{4}{5}$ from $17\frac{2}{3}$, or, which is the same thing, to add $17\frac{2}{3}$ and $-11\frac{4}{5}$. Here we have $17 - 11 + \frac{2}{3} - \frac{4}{5} = 6 + \frac{10}{15} - \frac{12}{15} = 6 - \frac{2}{15} = 5\frac{13}{15}$ as required.

Also, if we have to add $3a$, $5b$, $\frac{c}{d}$, and from the sum subtract $2a$, $3b$, and $\frac{c}{4d}$, we get $3a + 5b + \frac{c}{d} - 2a - 3b - \frac{c}{4d} = a + 2b + \frac{4c}{4d} - \frac{c}{4d} = a + 2b + \frac{3c}{4d}$, as required.

CASE VII.

To multiply fractional quantities together.

RULE.

Observing the rule of signs, the product of the numerators, when written over that of the denominators, in the form of a fraction, will be the required product.

Before the rule is applied, it will generally be advisable to reduce mixed quantities to improper fractions, and integral quantities to a fractional form, by writing 1 under them; also, to erase any factor which is common to the numerator of any fraction and the denominator of any other fraction; and to reduce each fraction to its lowest terms by erasing all the factors that are common to its numerator and denominator.

This rule has been proved to be true in Division.

EXAMPLES.

1. Multiply $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{7}{8}, \frac{8}{9}$ together.

Here the product is expressed by $\frac{2}{\cancel{3}}, \frac{\cancel{3}}{\cancel{4}}, \frac{\cancel{4}}{\cancel{5}}, \frac{\cancel{5}}{6}, \frac{7}{\cancel{8}}, \frac{\cancel{8}}{9} = \frac{2}{6} \cdot \frac{7}{9}$ $= \frac{1}{3} \cdot \frac{7}{9} = \frac{7}{27}$, as required.

We have drawn a line across the factors that are to be erased, since they will be common factors to the numerator and denominator of the product. We have also in the first result reduced $\frac{2}{6}$ to $\frac{1}{3}$ by erasing the factor 2 that is common to its terms; then the product of 1 and 7 gives 7 for the numerator, and that of 3 and 9 gives 27 for the denominator of the sought product.

2. To find the product of $\frac{5ab}{3cd}, \frac{3cd}{11b}, \frac{11}{a}$. Here $\frac{\cancel{5ab}}{\cancel{3cd}}, \frac{\cancel{3cd}}{\cancel{11b}}, \frac{\cancel{11}}{\cancel{a}} = 5$, as required.

3. Find the product of $\frac{x}{3} \cdot \frac{x}{5} \cdot \frac{3x^2}{7} \cdot \frac{5x^3}{6}$. *Ans.* $\frac{x^7}{42}$.

4. Find the product of $\frac{a+x}{a-x}, \frac{a^2+x^2}{(a+x)^2}, \frac{(a-x)^2}{a^2+x^2}, \frac{a^3+x^3}{a^3-x^3}$.

Ans. $\frac{a^2-ax+x^2}{a^2+ax+x^2}$.

5. Find the product of $\frac{3}{4}, \frac{4}{5}, \frac{15}{16}, \frac{a}{a+x}, \frac{a+x}{a}, \frac{(a+b)^2}{a^5+b^5}, \frac{3a-2x}{5a+4x}, \frac{3a+2x}{5a-4x}$.

Ans. $\frac{9(a+b)(9a^2-4x^2)}{16(a^4-a^3b+a^2b^2-ab^3+b^4)(25a^2-16x^2)}$.

6. Find the product of $3+\frac{2a}{x}, 3-\frac{2a}{x}, a-\frac{2b}{3c}, a+\frac{2b}{3c}, \frac{7a}{8x^2}, \frac{5x^2y}{7a^2}$.

Ans. $\frac{5y\left(9-\frac{4a^2}{x^2}\right)\left(a^2-\frac{4b^2}{9c^2}\right)}{8a}$.

7. Find the product of $\frac{5a^2-3b^2}{7ab}$, $\frac{9ab}{35a^2-21b^2}$, $\frac{5x^2}{2y^2}+8xy$, $8xy-\frac{5x^2}{2y^2}$. *Ans.* $\frac{9x^2}{49}\left(64y^2-\frac{25x^2}{4y^4}\right)$.

8. Find the product of $\frac{8(a+b)^{\frac{3}{2}}}{13mn}$, $\frac{4(a+b)^{\frac{1}{2}}}{5m^2n^2}$, $\frac{(a-b)^{\frac{1}{3}}}{6pq}$, $\frac{(a-b)^{\frac{2}{3}}}{6p^2q^2}$, $\frac{(c-d)^2}{a-b}$. *Ans.* $\frac{8(a+b)^2(c-d)^2}{585m^3n^3p^3q^3}$.

9. Find the product of $\frac{a^3+b^3}{x^3+y^3}$, $\frac{x^5+y^5}{a^7+b^7}$, $\frac{8a^3-27b^3}{13(a^3+b^3)}$, $\frac{13(x^3+y^3)}{x^5+y^5}$, $\frac{14(a^7+b^7)}{15(a+b)}$. *Ans.* $\frac{14(8a^3-27b^3)}{15(a+b)}$.

Remarks.—A compound fraction is reduced to a single one by the rule of Multiplication. Thus, if we wish to take $\frac{2}{3}$ of $\frac{4}{5}$, it is clearly the same as to multiply the numerators together for a new numerator, and the denominators together for a new denominator, and we shall have $\frac{2}{3}$ of $\frac{4}{5}=\frac{8}{15}$ for the result. Also, if we have $\frac{a}{b}\times\frac{c}{d}$, it will clearly be a compound fraction, and we shall have $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ for the correct result.

If we wish to put an integral form under the form of a fraction having a given denominator, it can be done in much the same way. Thus if we would put 6 under the form of a fraction having 13 for its denominator, we multiply 6 by $\frac{13}{13}=1$, and we shall get $6=\frac{78}{13}$, as required. In like manner, if we would put $a-x$ under the form of a fraction having $a+x$ for its denominator, we have $a-x=(a-x)\frac{a+x}{a+x}=\frac{a^2-x^2}{a+x}$, and so on in like cases. *So that we multiply the given quantity by the given denominator, and take the product for the numerator of the sought fraction, and the given denominator for its denominator.*

CASE VIII.

To divide one fractional quantity by another.

RULE.

If the dividend or divisor is mixed, reduce it to an improper fraction; or, in case either of them is integral, reduce it to the form of a fraction, having 1 *for its denominator. Then, if the numerator of the dividend is exactly divisible by that of the divisor, and the denominator of the dividend is exactly divisible by that of the divisor, take the quotient of the division of the numerator of the dividend by that of the divisor for the new numerator, and the quotient of the division of the denominator of the dividend by that of the divisor for a new denominator, and the resulting fraction will be the sought quotient. If the numerator of the dividend is not an exact multiple of that of the divisor, etc., we may invert the divisor, and then proceed as in Multiplication; which process is applicable in all cases. Observing in all instances, that it will be advisable to erase all the factors that are common to the numerators of the dividend and divisor, and also those that are common to their denominators, before the rule is applied. This rule was proved in Division.*

EXAMPLES.

1. Divide $\frac{4}{21}$ by $\frac{2}{7}$, and $\frac{a^2}{b^2}$ by $\frac{a}{b}$.

Here $4 \div 2 = 2$, $21 \div 7 = 3$, and $a^2 \div a = a$, $b^2 \div b = b$; consequently, we get $\frac{4}{21} \div \frac{2}{7} = \frac{2}{3}$, and $\frac{a^2}{b^2} \div \frac{a}{b} = \frac{a}{b}$, as required.

2. Divide $\frac{4}{15}$ by $\frac{2}{7}$, and $\frac{a^2}{c}$ by $\frac{a}{b}$.

Here $4 \div 2 = 2$, and $2 \times 7 = 14$; so that $\frac{4}{15} \div \frac{2}{7} = \frac{14}{15}$, which is the same as to invert the divisor, or change it to $\frac{7}{2}$, and then multiply, which gives $\frac{4}{15} \times \frac{7}{2} = \frac{14}{15}$, as before. In like manner, $\frac{a^2}{c} \div \frac{a}{b} = \frac{ab}{c} = \frac{a^2}{c} \times \frac{b}{a}$.

3. Divide $\frac{3(a+x)^2}{a^2 - x^2}$ by $\frac{2(a+x)}{a-x}$, and $\frac{a^3 + b^3}{a^3 - b^3}$ by $\frac{3(a+b)}{2(a-b)}$.

Ans. $\frac{3}{2}$, and $\frac{2}{3} \times \frac{a^2 - ab + b^2}{a^2 + ab + b^2}$.

4. Divide $\frac{42(a^7+b^7)}{a^4-b^4}$ by $\frac{a^8+b^8}{a^2-b^2}$, and $a+\frac{b}{c}$ by $a^2-\frac{b^2}{c^2}$.

Ans. $42 \times \frac{a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6}{(a^2-ab+b^2)(a^2+b^2)}$, and $\frac{1}{a-\frac{b}{c}}=\frac{c}{ac-b}$.

5. Divide $\frac{a+x}{a-x}$ by $\frac{a+x}{a-x}$, and $\frac{a^2+3ax+2x^2}{a^2-3ax+2x^2}$ by $\frac{a^2+ax}{ab^2-b^2x}$.

Ans. 1, and $\frac{b^2}{a} \times \frac{a+2x}{a-2x}$.

6. Divide $\frac{5ax+7x^2}{y^3-x^3}$ by $\frac{5x}{y-x}$, and $\frac{a+b}{14}$ by $\frac{a-b}{7}$.

Ans. $\frac{1}{5} \times \frac{5a+7x}{y^2+yx+x^2}$, and $\frac{1}{2} \times \frac{a+b}{a-b}$.

7. Divide $\frac{8a^3-27b^3}{16a^4-81b^4}$ by $\frac{2a-3b}{2a+3b}$, and $\frac{(a^2-b^2)^{\frac{3}{2}}}{(a^2+b^2)^{\frac{5}{2}}}$ by $\frac{(a^2+b^2)^{\frac{1}{2}}}{(a^2-b^2)^{\frac{1}{2}}}$.

Ans. $\frac{4a^2+6ab+9b^2}{8a^3-12a^2b+18ab^2-27b^3}$, and $\frac{(a^2-b^2)^2}{(a^2+b^2)^3}$.

8. Divide $\frac{35a^5x^7y^9}{34b^{11}c^{13}z^{20}}$ by $\frac{7a^3x^6y^7}{17b^5c^7z^{14}}$, and $\frac{1}{2}$ by $\frac{a+x}{a-x}$.

Ans. $\frac{5a^2xy^2}{2b^6c^6z^6}$, and $\frac{1}{2} \times \frac{a-x}{a+x}$.

9. Divide 1 by $\frac{a-x}{a+x}$, and $\frac{a^6-b^6}{d^2-e^2}$ by $\frac{a^3-b^3}{d+e}$.

Ans. $\frac{a+x}{a-x}$, and $\frac{a^3+b^3}{d-e}$.

10. Divide $\frac{2a}{3b}-\frac{4c}{5d}$ by $\frac{2a}{3b}+\frac{4c}{5d}$. *Ans.* $\frac{10ad-12bc}{10ad+12bc}$.

Remarks.—Since Division is the reverse of Multiplication, we may reduce the rule of Division to that of Multiplication, by changing the signs of the exponents of the terms of the divisor, and then proceeding as in Multiplication.

Thus, to divide $\frac{a^2}{b^2}$ by $\frac{a}{b}$, we have $\frac{a^2}{b^2} \times \frac{a^{-1}}{b^{-1}} = \frac{a^{2-1}}{b^{2-1}} = \frac{a}{b}$, the same result as found above. Also, to divide $\frac{4}{21}$ by $\frac{2}{7}$, we have $\frac{4}{21} \div \frac{2}{7} = \frac{2^2}{3.7} \times \frac{2^{-1}}{7^{-1}} = \frac{2^{2-1}}{3.7^{1-1}} = \frac{2}{3.7^0}$; or, since $7^0 = 1$, we have $\frac{4}{21} \div \frac{2}{7} = \frac{2}{3}$, as found above.

The product of any number of equal fractional quantities is called a power of one of the fractions. Thus, $\frac{2}{3} \times \frac{2}{3}$ is called the second power or square of $\frac{2}{3}$, and is denoted by $\left(\frac{2}{3}\right)^2$, or $\frac{2^2}{3^2}$; so that $\frac{2}{3} \times \frac{2}{3} = \left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$.

Also $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$ is called the cube or third power of $\frac{2}{3}$ and is denoted by $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$. And generally, by the expression $\left(\frac{a}{b}\right)^n$, or $\frac{a^n}{b^n}$, we are to understand the product arising from taking $\frac{a}{b}$, as a factor, n times, supposing n to be a positive integer. And since the product of an even number of negative quantities is positive, and the product of an odd number of negative quantities is negative, it follows that the n^{th} power of $\frac{a}{b}$ is the same as that of $-\frac{a}{b}$, when n is an even number, and that the sign of the n^{th} power of $-\frac{a}{b}$ is $-$, when n is an odd number.

Reversely, when any fraction, as $\frac{a}{b}$, is to be resolved into the product of n equal fractions, one of these equal fractions is called the n^{th} root of $\frac{a}{b}$, and is denoted by $\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$.

Thus, $\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}} = \left(\frac{2}{3}\right)^{\frac{1}{2}}$, means that two equal fractions are to be found, such that their product shall equal $\frac{2}{3}$; or, which is the same thing, that 2 is to be resolved into two equal factors, and that 3 is to be resolved into two equal factors; and that we are to use one of the factors of 2, divided by one of the factors of 3.

SECTION VIII.

OF ARITHMETICAL PROPORTIONS AND PROGRESSIONS.

(1.) The difference of two numbers or quantities of the same kind is called their *arithmetical ratio or difference.*

(2.) Four numbers or quantities of the same kind are said to be in arithmetical proportion when the difference of the first and second is equal to the difference of the third and fourth; the first and last terms being called the *extremes*, and the other terms the *means* of the proportion.

(3.) A series of numbers or quantities of the same kind that continually increase or decrease by a constant difference are said to be in continued arithmetical proportion, or to constitute an arithmetical progression; and if the successive terms increase, the progression is said to be increasing, but if the successive terms decrease, the progression is said to be decreasing.

(4.) *If four numbers or quantities of the same kind are in arithmetical proportion, the sum of the extremes is equal to the sum of the means.*

For, let a, b, c, d denote four numbers or quantities, such that we have $a - b = c - d$, then if we add $b + d$ to these equals, we shall have $a - b + b + d = c - d + b + d$, or (since $-b + b = 0$, and $-d + d = 0$) we have $a + d = b + c$, as was to be proved. Thus, if we take the numbers 12, 8, 10, and 6; since $12 - 8 = 4 =$ the arithmetical ratio of 12 to 8, and that $10 - 6 = 4 =$ the arithmetical ratio of 10 to 6, and since these ratios are equal, it follows that $12 - 8 = 10 - 6$; and we have $12 + 6 = 8 + 10 = 18$, as it ought to be, for 12, 8, 10, and 6 severally correspond to a, b, c, d.

Since a, b, c, d are connected by the equation $a+d=b+c$, *it follows that if any three of them are given, the fourth can be found.* For, suppose a, b, c are given to find d, then, since $a+d=b+c$, if we subtract a from these equals, we get $a+d-a=b+c-a$, or $d=b+c-a$.

Hence, if we take either of the extremes from the sum of the means of an arithmetical proportion, we shall get the other extreme; and if from the sum of the extremes we take one of the means, we shall get the other mean.

(5.) *If three numbers or quantities of the same kind are in arithmetical proportion, the sum of the extremes is equal to twice the mean.*

For let a, b, c be such numbers or quantities that we have $a-b=b-c$, then by adding $b+c$ to these equals, we shall have $a-b+b+c=b+b-c+c$ or $a+c=2b$, as required. Thus, since $15-11=11-7$, we get $15+7=2\times 11=22$.

Since a, b, c are connected by the equation $a+c=2b$, it will be sufficient to know any two of them in order to find the third. For if the mean b is given, and either extreme, as a, is also given, then subtracting a from the equals $a+c$ and $2b$, we get the other extreme $c=2b-a$; that is to say, *if we subtract either extreme from twice the mean, the remainder will equal the other extreme.*

Thus, if the mean is 10, and one of the extremes is 7, then we shall have $10\times 2-7=20-7=13$ for the other extreme; and it is easy to see that 13, 10, and 7 are in arithmetical proportion, as they ought to be.

Again, when the extremes a and c are given, if we divide the equals $2b$ and $a+c$ by 2, we shall have $b=\frac{a+c}{2}$; that is, *the mean equals half the sum of the extremes.* Thus, if the extremes are 11 and 17, we have $\frac{11+17}{2}=\frac{28}{2}=14=$ the mean; and it is easy to see that 11, 14, 17 are in arithmetical proportion, so that 14 is an arithmetical mean between 11 and 17, as it ought to be.

It may be observed that the common method of finding the *average* of two numbers or quantities of the same kind (which consists in taking the half sum for the average) *is the* *same as to take the arithmetical mean of the two numbers or quantities.*

Again, resuming the equation $a + c = 2b$; if we add b to both sides, we get $a + b + c = 3b$, and dividing these equals by 3, we get $\frac{a + b + c}{3} = b$.

Hence, *when three numbers or quantities of the same kind are in arithmetical proportion, the mean is equal to one third of the sum of the numbers or quantities, and it is clear that the mean is the average of the numbers or quantities.*

Hence, *we often call one third of the sum of any three numbers or quantities of the same kind their average or mean; and one fourth of the sum of four numbers or quantities of the same kind is called their average or mean; and generally, if* n *is any positive integer, the* n^{th} *part of the sum of* n *numbers or quantities of the same kind is called their average or mean.*

Thus, if 20.23, 20.24, 20.26, 20.28, represent four values of a certain quantity, as found by observation or experiment; then dividing the sum of these numbers by 4, we get $\frac{20.23 + 20.24 + 20.26 + 20.28}{4} = \frac{81.01}{4} = 20.2525 =$ the average of all the quantities, supposing the above numbers to have been correctly found to four decimal places; if, however, they have been correctly found only to two decimal places, we must take 20.25 for the average of the quantities. *And it is clear that the average ought to be taken as the most probable value of the quantity as determined from the four observations or experiments.*

It is plain that we may proceed in a similar way in all cases to find the most probable values of quantities as determined by observation or experiment, supposing equal skill and care to have been used in each instance, and supposing any result that differs much from the others to be rejected.

(6.) If a denotes the first term of an arithmetical progression, and d the difference of any two of its successive terms, then by the definition of an arithmetical progression, if the progression is increasing, we shall have the progression expressed by the series a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$,

$a + 5d, \ldots . a + (n - 1)d$, (a), in which n denotes the number of the term $a + (n - 1)d$.

It is easy to see that we may represent any term of the series (a) *by* $a + (n - 1)d$*;* for if we put $n = 1$, $a + (n - 1)d$ becomes a, the first term of the series; and if we put $n = 2$, $a + (n - 1)d$ becomes $a + d$, and if $n = 3$, $a + (n - 1)d$ becomes $c + 2d$, and so on to any extent; hence, since any term of (a) results from $a + (n - 1)d$ by putting n equal to the number of the term, $a + (n - 1)d$ *is called the general term of the series.*

(**7.**) Again, if in (a) we put $-d$ for $+d$, we shall have the decreasing arithmetical progression a, $a - d$, $a - 2d$, $a - 3d$, $a - 4d$, $a - 5d, \ldots . a - (n - 1)d$, (a^1), whose *general term* is expressed by $a - (n - 1)d$.

(**8.**) It is easy to see that we may write the series (a) and (a^1) in one series, by writing a, $a \pm d$, $a \pm 2d$, $a \pm 3d$, $a \pm 4d$, $a \pm 5d, \ldots . a \pm (n - 1)d$, (A); observing that for $\pm$ we must use $+$ when the series is increasing, and for $\pm$ we must use $-$ when the series is decreasing.

(**9.**) It is easy to see that in the series (A) we have to consider a, which is called the first term, and $a \pm (n - 1)d$, which is called *the last term*, and n which is supposed to denote *the number of terms*, together with $\pm d$ which is called the *arithmetical ratio, or the common difference of any two successive terms;* and, in addition to these, *it is often necessary to consider the sum of all the terms.*

(**10.**) If we represent the last term by l, we shall have $l = a \pm (n - 1)d$, (1), for the equation that connects the first term, last term, common difference, and number of terms with each other; observing that for $\pm$ we must use $+$ when the series is increasing, and $-$ must be used for $\pm$ when the series is decreasing.

(**11.**) *If we know* a, *the first term, the common difference* $\pm d$, *and the number of terms* n, *it is easy to find* l, *the last or* n^{th} *term from* (1); *for it requires us to multiply the common difference* $\pm d$ *by the number of terms less one, and to add the first term to the product (according to their signs) for the last term.*

Thus, if the first term $a = 2$, the common difference

$\pm d = 3$, and the number of terms $n = 7$; (1) gives $l = 2 + (7 - 1) \times 3 = 2 + 6 \times 3 = 20$ = the last or seventh term, which is clearly correct, for the series is 2, 5, 8, 11, 14, 17, 20, 23, etc., whose seventh term is 20, as it ought to be.

Again, if we put $a = 40$, $\pm d = -5$, $n = 8$, we have, from (1), $l = 40 - (8 - 1) \times 5 = 40 - 7 \times 5 = 5$ = the last term, as required; for the series is 40, 35, 30, 25, 20, 15, 10, 5, 0, -5, -10, -15, etc., whose eighth term is 5.

(**12.**) Since the last term of the series (A) is represented by l, it is easy to see that the last term but one will be expressed by $l \mp d$, and the last term but two will be expressed by $l \mp 2d$, and so on, to the n^{th} term, which will be expressed by $l \mp (n - 1)d$. Hence, if we add the sum of the series (A) to the sum of the same series when taken in a reverse order, we shall have $a + (a \pm d) + (a \pm 2d) + \ldots . + (a \pm (n - 1)d) + l + (l \mp d) + (l \mp 2d) + \ldots . + (l \mp (n - 1)d) = na + nl$; since the terms that involve d destroy each other on account of the opposition of signs, and that there are n terms in the series (A), so that a and l must each be taken n times in the sum. Hence, if s denotes the sum of the series (A), then $2s$ will equal the sum of the series (A) together with the sum of the same series taken in reverse order; consequently we shall have $2s$ equal to $na + nl = n(a + l)$, and dividing these equals by 2, we have $s = \frac{n(a + l)}{2}$, (2).

(**13.**) (2) is the equation that connects s, n, a, and l; and shows that *in order to find the sum of a series of numbers in arithmetical progression, we must take one half of the product of the sum of the extremes (or of the first and last terms added according to their signs) and the number of terms.*

Thus, if $a = 13$, $l = 119$, $n = 40$, we have $a + l = 132$, and $s = \frac{n(a + l)}{2}$ becomes $s = \frac{40 \times 132}{2} = 2640$, as required.

Again, if $a = 3$, $\pm d = 4$, $n = 21$, we get by (1) $l = a \pm (n - 1)d = 3 + 20 \times 4 = 83$; and then by (2) we shall have $s = \frac{n(a + l)}{2} = \frac{21 \times (3 + 83)}{2} = 903$, which is the sum of 21 terms of the series whose first term is 3 and ratio of increase 4.

For another example, we shall find the sum of the series

12, 10, 8, 6, 4, 2, 0, -2, -4, -6, -8, whose first term is 12 and last term is -8, and number of terms 11. Here $a=12$, $l=-8$, $n=11$, and by (2) we shall have $s=\frac{n(a+l)}{2}=\frac{11(12-8)}{2}=22$; which is easily seen to be correct, by actually adding the terms of the given series according to their signs.

(**14.**) If a and l together with n are known, then we may find $\pm d$, and thence determine the terms of the series that are situated between the first term a and last term l, *which is called the method of finding means between given extremes, or intercalating deficient terms.*

For, since (1) gives $l=a\pm(n-1)d$, we easily find $\pm d=\frac{l-a}{n-1}$, (3), which is evidently true, from the law of formation of the terms of an arithmetical progression. Since n denotes the number of terms in the series, $n-1$ is one less than the number of terms, or one more than the number of terms that are situated between the extremes; hence it follows from (3), that *if we divide the difference between the extremes by the number of terms to be intercalated increased by* 1, *the quotient will be the common difference of any two successive terms.*

Thus, if $a=7$, $l=22$, $n=6$, we have $\pm d=\frac{l-a}{n-1}=\frac{22-7}{6-1}=\frac{15}{5}=3$, and $n-2=$ the number of means to be found $=4$. Hence, since $\pm d=3$, we have $a\pm d=7+3=10$, $a\pm 2d=13$, $a\pm 3d=16$, $a\pm 4d=19$, for the terms that are to be intercalated between 7 and 22; and we shall have 7, 10, 13, 16, 19, 22 for the series, when it is completed by intercalating the terms that have been found.

For another example, we shall suppose that seven terms are to be intercalated between 12 and -12. Here $a=12$, $l=-12$, and $n-1=7+1=8$; and we shall have $\pm d=\frac{a-l}{n-1}=\frac{12+12}{8}=3$. Hence, as in the last example, we get 12, 9, 6, 3, 0, -3, -6, -9, -12 for the series as completed by intercalating the seven terms, as required.

(**15.**) Because a, $\pm d$, l, n, and s are so connected with the arithmetical progression that its terms may be made to depend upon them, they are often called *its conditions or terms.* And since a, l, n, and $\pm d$ are connected by (1), and s, n, a, and l are connected by (2), it is evident that any two of the above five terms may be found from the solution of equations (1) and (2), in terms of the other three; so that only three of the above terms can be assumed at will; in other words, only three of the above five terms can be regarded as independent of each other.

(**16.**) It is easy to see from (A), that any three successive terms of an arithmetical progression are in arithmetical proportion; so that we may regard an arithmetical progression as being a continued arithmetical proportion.

(**17.**) *It follows, also, from* (A), *if we take any three terms of an arithmetical progression, such that the intermediate term is equidistant from the other two, that the three terms will be in arithmetical proportion, so that the sum of the extremes is equal to twice the mean.* Thus, if we add the second and eighth terms of (A), the sum will equal twice the fifth term. *It is also clear that if we take four terms of an arithmetical progression, such that the intermediate terms are equidistant from the corresponding extreme terms, that the sum of the extreme terms equals the sum of the two mean terms.*

Thus, if we add the second and ninth terms of (A) for the extremes, and the fifth and sixth terms which are equidistant from them for the means, we shall find that the sum of the extremes equals the sum of the means.

(**18.**) Again, if we have the arithmetical proportion $a - b = b - c$, then, by the definition of an arithmetical progression, a, b, c will be three successive terms of an arithmetical progression. Thus, since $20 - 16 = 16 - 12$, it follows that 20, 16, 12 are three successive terms in an arithmetical progression, whose ratio of decrease is 4; and it is easy to continue the progression to any extent.

(**19.**) But if we have the arithmetical proportion $a - b = c - d$, then a, b, c, d can not be in arithmetical progression if $a - b$ is not equal to $b - c$, and the proportion is said to be discontinued; so that a, b, c, d are said to be in discontinued arithmetical proportion. Thus, 12, 10, 6, 4 are in discontinued arithmetical proportion.

(**20.**) If, however, we have $a - b = c - d$, and at the same time $a - b = b - c$, we shall of course have $a - b = b - c = c - d$; and a, b, c, d will be in continued proportion, or they will be four successive terms of an arithmetical progression. Thus, 12, 10, 8, 6 are in continued arithmetical proportion, and they are four successive terms of an arithmetical progression, whose ratio of decrease is 2.

(**21.**) Finally, it may be proper to notice that in an arithmetical proportion, *the first and third terms are sometimes called the antecedents, the first term being called the first antecedent, and the third term being called the second antecedent; also the second and fourth terms are called the consequents, the second term being called the first consequent, and the fourth term the second consequent.* Thus, in the proportion $12 - 10 = 24 - 22$, 12 is the first antecedent, and 10, that is subtracted from it, is the first consequent; 24 is the second antecedent, and 22, which is taken from it, is the second consequent.

QUESTIONS FOR EXERCISE.

1. Find a third arithmetical proportional to 13 and 16.

Ans. $16 \times 2 - 13 = 19$.

2. Find a fourth arithmetical proportional to 27, 33, 78.

Ans. $33 + 78 - 27 = 84$.

3. Find an arithmetical mean between 62 and 92.

Ans. $\frac{62 + 92}{2} = 77$.

4. Find an arithmetical mean between 62 and -92.

Ans. $\frac{62 - 92}{2} = -15$.

5. Find an arithmetical mean between a and $\pm c$.

Ans. $\frac{a \pm c}{2}$.

6. The first term of an arithmetical progression is 5, the ratio of increase 3; to find the eleventh term.

Ans. $5 + 10 \times 3 = 35$.

7. The first term of an arithmetical progression is 25, and the last term is 125, and the number of terms is 50; to find the sum of all the terms. *Ans.* $\frac{25 + 125}{2} \times 50 = 3750$.

8. The first term of an arithmetical progression is 100, the ratio of decrease 7; to find the nineteenth term.

Ans. $100 - 18 \times 7 = -26$.

9. A man traveled 7 miles the first day, 14 miles the second day, 21 miles the third day, and so, in increasing arithmetical progression; to find how many miles he traveled the tenth day, and the whole distance traveled.

Ans. He traveled 70 miles the tenth day, and the whole distance traveled was 385 miles.

10. A man has traveled 600 miles, of which he traveled 10 miles the first day and 50 miles the last day; to find how many days he was on his journey, supposing his daily travel increased in arithmetical progression.

Ans. If n stands for the required number of days, we get $n \times \frac{10 + 50}{2} = 30n = 600$; and dividing these equals by 30, there results $n = 20 =$ the required number of days.

11. Given, the increasing arithmetical progression, 4, 10, 16, 22, 28, 34, etc., whose ratio of increase is 6; to insert two arithmetical means between 4 and 10, also to insert two arithmetical means between 10 and 16, and so on.

Subtracting 4 from 10 we get 6, which is to be divided by the number of means to be inserted (between every two terms) increased by one, and we have $\frac{6}{2 + 1} = \frac{6}{3} = 2$, which is the common difference between the two means, and between them and the terms that correspond to them among the terms between which they are to be inserted. Hence, if we add 2 to 4, and 2 to the sum, we get the first two means; and in the same way by adding 2 to 10, and 2 to the sum, we have the two means to be inserted between 10 and 16, and so on.

The means being inserted as required, the answer is 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, etc. Hence, we see if we insert the same number of arithmetical means between every two adjacent terms of an arithmetical progression, that the result will also be an arithmetical progression.

SECTION IX.

OF GEOMETRICAL PROPORTIONS AND PROGRESSIONS.

(**1.**) *The quotient of one number divided by another, or of one quantity divided by another of the same kind, is called the geometrical ratio or relation of the number or quantity that is taken for the dividend to the number or quantity that is taken for the divisor; the dividend being called the antecedent, and the divisor the consequent of the ratio*, and the antecedent and consequent are called the terms of the ratio.

Thus, if A and B stand for two numbers or quantities of the same kind, then the fractional expression $\frac{A}{B}$ *represents the geometrical ratio of* A *to* B; A *being the antecedent and* B *the consequent of the ratio*, and A and B *the terms of the ratio.*

(**2.**) If A and B are two numbers or quantities of the same kind, and C and D are two numbers or quantities of the same kind with each other (or with A and B, should the nature of the case require it), then if the ratios $\frac{A}{B}$ and $\frac{C}{D}$ are equal to each other, we shall have $\frac{A}{B}=\frac{C}{D}$, (1), which is called a *geometrical proportion;* A being called the *first antecedent* and B the *first consequent*, and C is called the *second antecedent* and D the *second consequent.*

It is clear that $\frac{A}{B}$ *is number, and not quantity*, and in like manner $\frac{C}{D}$ *is number and not quantity;* and (1) signifies that A contains B as often as C contains D, or that A is the same part of B that C is of D.

Thus, if A stands for 4 dollars and B for 2 dollars, and C for 6 bushels of wheat and D for 3 bushels of wheat, we shall have $\frac{4 \text{ dollars}}{2 \text{ dollars}}=$ the number 2, since 4 dollars contain 2 dollars twice; also $\frac{6 \text{ bushels}}{3 \text{ bushels}}=$ the number 2; so that 4 dollars contain 2 dollars just as often as 6 bushels of wheat contain 3 bushels of wheat, which is in accordance with the true meaning of (1), *which merely signifies an equality of geometrical ratios.*

(**3.**) If A, B, C are three numbers or quantities of the same kind, such that we have $\frac{A}{B} = \frac{B}{C}$, (2), then A, B, C are said to be in continued geometrical proportion; in which it will be observed that the first consequent equals the second antecedent.

(**4.**) It is customary to write (1) and (2) in the forms $A : B :: C : D$, (3), and $A : B :: B : C$, (4), or in the forms $A : B = C : D$, (5), and $A : B = B : C$, (6).

Because A and D occupy the extreme places in (3) and (5), we call them *the extremes of the proportion*, and since B and C are between A and D, we call them *the means of the proportion;* in like manner A and C are called the extremes in (4) or (6), and B is called *the geometrical mean between* A *and* C. Also A, B, C, D in (3) or (5) are called *geometrical proportionals*, A and D being *the extremes* and B and C *the means*, and D is called *a fourth proportional to* A, B, *and* C; and A, B, C, in (4) or (6), are called continued proportionals, A and C being *the extremes* and B *the mean proportional*, and C is called *a third proportional to* A *and* B.

(3) or (5) is read by saying A *is to* B as C *is to* D, or we say, *as* A *is to* B *so is* C *to* D; and it is clear that the meaning is the same as to say that A *contains* B *as often as* C *contains* D, *or that* A *is the same part of* B *that* C *is of* D; (4) and (6) are read in a similar way, by saying A *is to* B as B *is to* C, or as A *is to* B *so is* B *to* C.

It may be observed that the colon (:) which is written between A and B in (3), is used to signify that A and B are numbers or quantities of the same kind, and that A which precedes the colon *is to be divided by* B *which follows it;* and the use of the colon is to be understood in like manner *in all cases.*

The double colon (::), which is used in (3) and (4), signifies that the quotient of the division of A by B is equal to the quotient of the division of C by D, these quotients being abstract numbers, and not quantity; and the same thing (if we please) may be expressed by =, as in (5) and (6); yet it may perhaps, for the sake of expressing the species of equality intended in geometrical proportion, be advisable to use the double colon (::) instead of =.

It follows clearly, from what has been done, that for $A : B :: C : D$ we may use the equation $\frac{A}{B} = \frac{C}{D}$, and *vice versa.*

(5.) Resuming (1), and multiplying its two members by any whole number m, we shall have the equation $\frac{mA}{B} = \frac{mC}{D}$ (2), as is evident, since $\frac{A}{B}$ is equal to $\frac{C}{D}$.

It is clear from (2) that the integral part of the quotient $\frac{mA}{B}$ is the same as that of $\frac{mC}{D}$, and that whether A, B, C, D are of rational or irrational forms, so that when we have $\frac{A}{B} = \frac{C}{D}$, the multiple mA of A can not contain B *a greater or less integral number of times than the equimultiple* mC *of* C *contains* D.

Reversely, if we have A, B, C, and D, such that by taking any equimultiples of A and C by any whole number m, we find that the multiple mA of A can not contain B a greater or less integral number of times than mC, the equimultiple of C, contains D, *then we shall have the equation* $\frac{A}{B} = \frac{C}{D}$.

For if $\frac{A}{B}$ is not equal to $\frac{C}{D}$, one of them must be the greater; let therefore $\frac{A}{B}$ be greater than $\frac{C}{D}$, and suppose p denotes the excess of $\frac{A}{B}$ over $\frac{C}{D}$, and we shall have the equation $\frac{A}{B} = \frac{C}{D} + p$, (3), where p is evidently a number, since A and B are supposed to be numbers or quantities of the same kind, and that C and D are either numbers or quantities of the same kind; and it is clear that p is less than unity, and that it may be rational or irrational, according to the nature of the case.

Now, if we take the integer m, such that the inequality $m > 1 \div p$ shall have place, then will the inequality $pm > 1$ also have place.

If we multiply the two members of (3) by m, we shall have $\frac{mA}{B} = \frac{mC}{D} + mp$; consequently, since $mp = pm$ is not less than unity, mA contains B a greater integral number of times than mC contains D, which is against the hypothesis; therefore $\frac{A}{B}$ is not greater than $\frac{C}{D}$.

In the same manner it may be shown, that $\frac{C}{D}$ is not greater than $\frac{A}{B}$; of course we must have $\frac{A}{B} = \frac{C}{D}$, as was to be proved. Hence, we may use the conditions, that the integral part of $\frac{mA}{B}$ is not greater than that of $\frac{mC}{D}$, and that the integral part of $\frac{mC}{D}$ is not greater than that of $\frac{mA}{B}$, as tests of the proportionality of A, B, C, D, supposing m to stand for any positive whole number. *This method of testing the proportionality of quantities will often be much more simple than to use* (1), *especially in the application of the doctrine of proportion to geometrical investigations.*

(6.) If we multiply the two members of (1) by the integer m, and divide the products by the integer n, we shall evidently have $\frac{mA}{nB} = \frac{mC}{nD}$; or, which is the same, if we have the proportion A : B :: C : D, then we shall also have the proportion $mA : nB :: mC : nD$. *In words, if we multiply the antecedents of any proportion by any integer, and the consequents by any integer, the multiple of the first antecedent will be to the multiple of its consequent as the multiple of the second antecedent is to the multiple of its consequent.*

It is easy to see that in the equation $\frac{mA}{nB} = \frac{mC}{nD}$ it is not necessary to suppose m and n to be positive integers; for when (1) has place, $\frac{mA}{nB} = \frac{mC}{nD}$ will also have place, whether

m and n are integral or fractional, rational or irrational, positive or negative. Hence, if we have the proportion $A : B :: C : D$, then we shall also have the proportion $mA : nB :: mC : nD$, without regarding the nature of the numbers that are represented by m and n; and it is clear that, if we please, we may put $m = 1$ or $n = 1$, or both of them equal to 1, and return to the proportion $A : B :: C :: D$.

Consequently, from the proportion $A : B :: C : D$ we get the proportions $mA : nB :: mC : nD$, $mA : B :: mC : D$, $A : nB :: C : nD$, *which are true without any reference to the nature of the numbers that are represented by m and n.*

(7.) If we multiply A and B in (1) by m, we shall get $\frac{mA}{mB} = \frac{C}{D}$; or, which is the same, if we have the proportion $A : B :: C : D$, then we shall have the proportion $mA : mB :: C : D$; and in like manner we shall have $A : B :: nC : nD$; or, more generally, $mA : mB :: nC : nD$; these proportions being true without reference to the nature of the numbers that are represented by m and n.

In words, *if in any proportion either antecedent and its consequent are multiplied by the same numerical expression, the proportion will not be affected; neither will it be affected if we multiply the remaining antecedent and its consequent by any numerical expression.*

(8.) Since $\frac{A}{B} = \frac{mA}{mB}$, we have $A : B :: mA : mB$. Hence, *one of two quantities of the same kind has the same ratio to the remaining one of the two as the product of the first and any numerical expression has to the product of the second and the same numerical expression.*

(9.) From the equation $\frac{mA}{nB} = \frac{mC}{nD}$ (which is the same as the proportion $mA : nB :: mC : nD$) we see that if mA is greater than nB, then mC must be greater than nD; and if mA equals nB, mC must equal nD; also, if mA is less than nB, mC must be less than nD. Hence, *in any proportion, if the first antecedent is greater than its consequent, the second antecedent is greater than its consequent; and if the first* *antecedent equals its consequent, the second antecedent equals its consequent; and if the first antecedent is less than its consequent, the second antecedent is less than its consequent.*

(**10.**) If, in the equation $\frac{mA}{nB}=\frac{mC}{nD}$, we suppose A, B, C, D to be quantities of the same kind, then it is evident that if we have mA greater than mC, we must also have nB greater than nD, for otherwise the equation will not hold true; and if mA equals mC, nB must equal nD; also, if mA is less than mC, nB must be less than nD. Hence, *if we have any proportion, as* $mA : nB :: mC : nD$, *whose terms are all of the same kind, then, if the first antecedent is greater than the second antecedent, the first consequent is greater than the second; and if the first antecedent equals the second, the first consequent equals the second; also, if the first antecedent is less than the second, the first consequent is less than the second.*

(**11.**) If, as in (1), we have $\frac{A}{B}=\frac{C}{D}$, and if we also have the ratio $\frac{C}{D}=$ to the ratio $\frac{E}{F}$, then, since things which equal the same thing must equal each other, we shall of course have $\frac{A}{B}=\frac{E}{F}$; or, which is the same thing, if we have the proportions $A : B :: C : D$, and $C : D :: E : F$, we get $A : B :: E : F$. Hence, *if two ratios are equal to the same ratio, they are equal to each other; or if an antecedent and its consequent in two proportions are the same, then the remaining antecedents and consequents will constitute a proportion. And in like manner it is easy to see that ratios which are equal to equal ratios, are equal to each other.*

(**12.**) Since by (1) we have $\frac{A}{B}=\frac{C}{D}$, if we divide these equals by any positive integer n, we shall evidently get $\frac{A}{nB}=\frac{C}{nD}$; and if we divide A and nB by A, we shall get $\frac{A}{nB}=\frac{1}{\frac{nB}{A}}$, and in the same way we shall get $\frac{C}{nD}=\frac{1}{\frac{nD}{C}}$;

consequently, we must have the equation $\frac{\frac{1}{nB}}{A} = \frac{\frac{1}{nD}}{C}$. Now, if the multiple nB of B contains A a greater or less integral number of times than the equimultiple nD of D contains C, the preceding equation evidently can not hold true.

Hence, since nB can not contain A a greater or less integral number of times than nD contains C, it follows from (5) that we shall have the equation $\frac{B}{A} = \frac{D}{C}$, which is also immediately evident from the equation $\frac{\frac{1}{nB}}{A} = \frac{\frac{1}{nD}}{C}$.

Hence, if we have the equation $\frac{A}{B} = \frac{C}{D}$, we shall also have the equation $\frac{B}{A} = \frac{D}{C}$; in other terms, if we have the proportion A : B :: C : D; then we shall also have the proportion B : A :: D : C.

Consequently, the terms of a proportion are proportional *by inversion, or when taken inversely;* that is, *the second is to the first as the fourth is to the third.*

(**12.**) If, in the equation $\frac{A}{B} = \frac{C}{D}$, we suppose A, B, C, D to be quantities of the same kind, then, by multiplying A and B by the positive integer m, we shall clearly have the equation $\frac{mA}{mB} = \frac{C}{D}$, which is the same as $\frac{\frac{mA}{C}}{\frac{mB}{D}} = \frac{1}{1} = 1$; consequently, we must have $\frac{mA}{C} = \frac{mB}{D}$, or $\frac{A}{C} = \frac{B}{D}$, which is also evident from the equation $\frac{mA}{mB} = \frac{C}{D}$, which shows that the multiple mA of A can not contain C a greater or less number of times than the equimultiple mB of B contains D; and of course by (5) we must have $\frac{A}{C} = \frac{B}{D}$.

Hence, when we have the proportion A : B :: C : D, whose terms are all of the same kind, then we shall also have the proportion A : C :: B : D.

In words, when the terms of a proportion are all of the same kind, *they will be proportional when taken alternately;* that is, *the first is to the third as the second to the fourth.*

The same things being supposed; since $\frac{A}{C} = \frac{B}{D}$, we shall have $\frac{mA}{nC} = \frac{mB}{nD}$ or $mA : nC :: mB : nD$, which is true without reference to the nature of the numbers represented by m and n. Also, from $\frac{A}{C} = \frac{B}{D}$, we get $\frac{mA}{mC} = \frac{nB}{nD}$ or $mA : mC :: nB : nD$, which is also true without reference to the nature of the numbers represented by m and n; and if we please we may put either m or n or both of them equal to unity in the preceding proportions, and they will continue to be true.

(**14.**) Let A, B, C, D be four quantities, such that A, B are of one kind and C, D of one kind, which may be of the same kind with A and B, if the case requires it.

Then suppose A and C to be multiplied by any positive integer m, and B and D to be multiplied by any positive integer n, and the multiples will be expressed by mA, nB, mC, nD.

If, now, we find for all the integers that m and n can represent, that when mA is greater than nB, mC is greater than nD; and when mA is equal to nB, mC is equal to nD; also, when mA is less than nB, mC is less than nD; then we shall have the proportion A : B :: C : D, or, which is the same $\frac{A}{B} = \frac{C}{D}$.

For, by (5), if the proportion A : B :: C : D does not have place, we can suppose A and C to be multiplied by some positive integer m, so that mA shall contain B a greater or less integral number of times than mC contains D; for otherwise the proportion A : B :: C : D will have place, as was shown in (5).

If we suppose that mA contains B a greater integral number of times than mC contains D, we may evidently put

$\frac{mA}{B} = \frac{mC}{D} + p$, (1), where p is evidently not less than unity; since mA contains B a greater integral number of times than mC contains D.

If, now, we suppose that mC contains D an integral number of times represented by q, then clearly mA will contain B at least as many integral times as there are units in $q + 1$. Hence, if we represent $q + 1$ by n, and divide both sides of (1) by n, we shall get the equation $\frac{mA}{nB} = \frac{mC}{nD} + \frac{p}{n}$, (2).

Since mA contains B, n times at least, it will contain nB at least once, and of course mA is not less than nB; but since mC contains D a less number of times than there are units in n, it can not contain nD once; and consequently mC is less than nD. Hence, when mA is not less than nB, we have mC less than nD, which is against the hypothesis; which is, that when mA is not less than nB, mC is not less than nD.

In a similar way it may be shown if we suppose mC contains D a greater number of times than mA contains B, that the hypothesis will be contradicted.

Consequently, we must suppose that the proportion A : B :: C : D or the equation $\frac{A}{B} = \frac{C}{D}$ has place, as was to be proved.

Hence, *the first and third of four quantities being multiplied by any positive integer whatever, and the second and fourth being multiplied by any positive integer whatever; if it is found, when the first multiple is greater, equal to, or less than the second, that the third multiple is in like manner greater, equal to, or less than the fourth; then the first quantity has the same ratio to the second that the third has to the fourth.*

This proposition is substantially the same as the famous Definition V. of Book V. of Euclid; see R. Simson's Euclid, Book V., Def. V.

(15.) If we have any number of quantities of one kind, as A, B, C, D, and as many others, E, F, G, H, of one kind (which may be of the same kind as the first, should the case require it), then if we have the equations $\frac{A}{B}=\frac{E}{F}$, $\frac{B}{C}=\frac{F}{G}$, $\frac{C}{D}=\frac{G}{H}$, it will follow that we must have $\frac{A}{D}=\frac{E}{H}$.

For, since A, B, C are one kind of quantities, we can divide A and B by C, and we shall get $\frac{A}{B}=\frac{\frac{A}{C}}{\frac{B}{C}}$; and in a similar way we have $\frac{E}{F}=\frac{\frac{E}{G}}{\frac{F}{G}}$.

Hence, since $\frac{A}{B}=\frac{E}{F}$, we must have $\frac{\frac{A}{C}}{\frac{B}{C}}=\frac{\frac{E}{G}}{\frac{F}{G}}$; consequently, since $\frac{B}{C}=\frac{F}{G}$, we must have $\frac{A}{C}=\frac{E}{G}$. Hence, we have the equations $\frac{A}{C}=\frac{E}{G}$, and $\frac{C}{D}=\frac{G}{H}$; which (as before) give $\frac{A}{D}=\frac{E}{H}$, as required.

It is evident, from what has been done, that the same reasoning and conclusion will hold when there are five quantities of each of the above kinds, and when there are six of each kind, and so on, to any (the same) number of each kind.

Since the equations used are equivalent to proportions, *we say, if we have two ranks of quantities, such that the quantities in each rank are of one kind, and the number of quantities in one rank is the same as the number of quantities in the other rank; and if the first quantity in the first rank has the same ratio to the second that the first quantity of the second rank has to the second (of that rank), and if the second quantity in the first rank has the same ratio to the third that the second quantity of the second rank has to the third (of that rank), and so on in order for all the quantities of the two ranks; then we infer from what has been done (which is called* "ex æquali"), *that the first quantity of the first rank has the same ratio to the last of that rank, as the first of the second rank has to the last of that rank.*

(**16.**) Let A, B, C denote three quantities of one kind, and D, E, F three quantities of one kind (which may also be of the same kind as A, B, C, if required), such that we have the equations $\frac{A}{B}=\frac{E}{F}$, $\frac{B}{C}=\frac{D}{E}$, then we shall also have the equation $\frac{A}{C}=\frac{D}{F}$.

For if we represent any two positive integers by m and n, we evidently get from the supposed equations $\frac{mA}{mB}=\frac{nE}{nF}$, and $\frac{mB}{nC}=\frac{mD}{nE}$, or for this equation (by inversion of proportion, as has been shown in (12), we may write $\frac{nC}{mB}=\frac{nE}{mD}$. If we now suppose that mA is greater than nC, we shall of course have $\frac{mA}{mB}$ greater than $\frac{nC}{mB}$, and since $\frac{mA}{mB}=\frac{nE}{nF}$, and $\frac{nC}{mB}=\frac{nE}{mD}$, we must have $\frac{nE}{nF}$ greater than $\frac{nE}{mD}$, consequently mD must be greater than nF; so that when mA is greater than nC, mD is greater than nF. In like manner, if mA equals nC, we have mD equal to nF; and if mA is less than nC, we have mD less than nF. Hence we have the four multiples mA, nC, mD, nF such that when the first is greater, equal to, or less than the second, the third is in like manner greater, equal to, or less than the fourth; consequently, from (14) we have the proportion A : C :: D : F, or, which is the same, we have the equation $\frac{A}{C}=\frac{D}{F}$, as was to be shown.

Again, let A, B, C, D be four quantities of one kind, and E, F, G, H four quantities of one kind, such that we have the equations $\frac{A}{B}=\frac{G}{H}$, $\frac{B}{C}=\frac{F}{G}$, $\frac{C}{D}=\frac{E}{F}$; then we shall have the equation $\frac{A}{D}=\frac{E}{H}$.

For, by what has just been proved, the first two of the supposed equations give the equation $\frac{A}{C}=\frac{F}{H}$, which, and the last of the supposed equations, give in the same way the equation $\frac{A}{D}=\frac{E}{H}$, as required.

It is easy to see that if we have two ranks of quantities, such that there are as many quantities in one rank as in the other, and that the quantities in each rank are all of one kind; then if conditions or equations like to those supposed in the cases of three and four quantities in a rank have place, it will follow that the first of the first rank divided by the last of the first rank, will give the same quotient as the first of the second rank divided by the last of the second rank. For when there are five quantities in each rank, they can be reduced as in the case of four quantities in a rank to the case of three quantities in a rank; and in the case of six quantities in a rank the same reductions will have place, and so on, to any extent.

Hence, *if we have the same number of quantities in two ranks; such that the first quantity in the first rank is to the second quantity in that rank as the last quantity but one in the second rank is to the last quantity in that rank; and the second quantity in the first rank is to the third quantity in that rank, as the last quantity but two in the second rank is to the last quantity but one in that rank, and so on, for all the quantities of the two ranks, taking them in a cross order; then we shall have the first quantity of the first rank is to the last quantity of that rank as the first quantity of the second rank is to the last quantity of that rank; as is evident, since the proportions stated are virtually the same as the equations which we have used in the proof and conclusion of our proposition.*

Remark.—*The proposition here proved is usually quoted by the words* "ex æquali in proportione perturbata;" *or* "ex equo perturbato." See Simson's Euclid, Book V., Prop. XXIII.

(**17.**) *If the antecedents of one proportion are the same as the consequents or antecedents of any other proportion, then shall the remaining terms in the proportions constitute a proportion; such that the two terms of one of the proportions shall be the antecedents and the two remaining terms of the other proportion shall be the consequents.*

For let the proportions be A : B :: C : D and B : E :: D : F; then, placing them in two ranks, we have A, B, E for one rank, and C, D, F for the other rank.

Hence, by (15) (which is usually cited by the words "ex equali"), since the quantities in the two ranks are proportional when taken two by two in order, we get the proportion A : E :: C : F, as required. In the proof, the equal terms are the consequents in one of the proportions, and the antecedents in the other; if, however, they should be the consequents or antecedents in both proportions, then, by (12), we can invert the terms of one of the proportions, and then the proof of the proposition will be the same as before.

(**18.**) *If the extremes of one proportion are the same with the means of another proportion, or if the extremes or means of two proportions are the same, then shall the remaining terms of the two proportions constitute a proportion whose means shall be the remaining terms of one of the proportions, and extremes the remaining terms of the other proportion.*

For let A : B :: E : F, B : C :: D : E be the two proportions; then, writing them down in two ranks, we have A, B, C for one rank, and D, E, F for the other rank of quantities. Hence, since the quantities in the two ranks are proportional when taken two by two in each rank in a cross order, we have by (16) (or "ex equo perturbato"), A : C :: D : F, as was to be shown. If the extremes or means are the same in the given proportions, then, by (12), invert the terms of one of them, and proceed as before.

(**19.**) If we have the equation $\frac{A}{B}=\frac{C}{D}$, we shall also have the equation $\frac{A+B}{B}=\frac{C+D}{D}$. For $\frac{A+B}{B}=\frac{A}{B}+1$, since B is contained once in B; and, in the same way, $\frac{C+D}{D}=\frac{C}{D}+1$; consequently, since by supposition $\frac{A}{B}=\frac{C}{D}$, we shall have $\frac{A}{B}+1=\frac{C}{D}+1$, and thence $\frac{A+B}{B}=\frac{C+D}{D}$, as was to be proved. In other terms, if we have the proportion A : B :: C : D, we shall also have the proportion A + B : B :: C + D : D. In like manner, by inverting the terms of the proportion A : B :: C : D by (12), we shall have the proportion B : A :: D : C; hence (as before) we shall have the proportion A + B : A :: C + D : C.

Hence, we say (*by composition*) *that the sum of the first two terms of any proportion is to either of the two terms as the sum of the last two terms is to that term of the last two terms that corresponds to the term that is taken for the second term of the proportion.*

(**20.**) If we have the equation $\frac{A}{B} = \frac{C}{D}$, or the proportion A : B :: C : D, then, if A is greater than B, we shall also have the equation $\frac{A - B}{B} = \frac{C - D}{D}$, or the proportion A − B : B :: C − D : D. For $\frac{A - B}{B} = \frac{A}{B} - 1$, and $\frac{C - D}{D} = \frac{C}{D} - 1$; consequently, since $\frac{A}{B}$ is by supposition equal to $\frac{C}{D}$, we shall have the equation $\frac{A - B}{B} = \frac{C - D}{D}$, or the proportion A − B : B :: C − D : D. From the proportion A − B : B :: C − D : D, we get by (19) A − B + B : A − B :: C − D + D : C − D, or (since − B + B = 0, and − D + D = 0) we have A : A − B :: C : C − D, or by (12) A − B : A :: C − D : C. If we have B greater than A, we may write the equation $\frac{A}{B} = \frac{C}{D}$ in the form $\frac{B}{A} = \frac{D}{C}$, as appears from (12); hence, as before, we shall have B − A : A :: D − C : C, or B − A : B :: D − C : D.

Hence (*by division*), we say that the difference of the first two terms of any proportion is to either of the first two terms as the difference of the last two terms is to that term of the two which corresponds to the term that is taken for the second term of the proportion.

(**21.**) From the equation $\frac{A}{B} = \frac{C}{D}$ we can deduce the equation $\frac{A + B}{A - B} = \frac{C + D}{C - D}$, when A is greater than B; or, which comes to the same, from the proportion A : B :: C : D we shall have the proportion A + B : A − B :: C + D : C − D, when A is greater than B. For from (19) we get the proportion A + B : B :: C + D : D; and from (20) we have A − B : B :: C − D : D; hence, by (17), since the consequents of these proportions are equal, the antecedents will constitute a proportion, and we shall have A + B : A − B :: C + D : C − D. In like manner, when B is greater than A, since for the equation we may by (12) put $\frac{B}{A} = \frac{C}{D}$, we shall get A + B : B − A :: C + D : D − C.

Hence (*mixedly, or uniting composition and division*), we say that the sum of the first two terms of any proportion is to their difference as the sum of the last two terms is to their difference.

(**22.**) From the equation $\frac{A}{B} = \frac{C}{D}$, or proportion A : B :: C : D, when A is greater than B, we have the proportion A : A − B :: C : C − D; and when B is greater than A, we shall have B : B − A :: D : D − C. Since this proposition has in fact been proved in (20), it will not be necessary to repeat the proof in this place.

Hence, we say (*by conversion*) *that the greater of the first two terms of any proportion is to the difference of the first two terms as the greater of the last two terms is to the difference of the last two terms. And it is easy to see that the less of the first two terms is to their difference as the less of the last two terms is to their difference.*

(**23.**) We shall now suppose that A, B, C, D are quantities of one kind, and that we have the equation $\frac{A}{B} = \frac{C}{D}$, or the proportion A : B :: C : D; then if A is greater than C, it is easy to see that B will be greater than D, and we shall have the proportion A − C : B − D :: A : B. For by alternating the terms of the given proportion by (13), we get A : C :: B : D, hence, by (20), we shall get A − C : A :: B − D : B, and alternating this proportion by (13) we get A − C : B − D :: A : B, as required.

Hence, *if a whole quantity is to a whole quantity as a quantity taken from the first quantity is to a quantity taken from the second quantity; then the first remainder is to the second remainder as the first whole quantity is to the second whole quantity.*

(**24.**) Suppose that we have the proportion $A : B :: C : D$, such that its terms are all of one kind; then if A is the greatest term, it is easy to show that D is the least term, and we shall have $A + D$ greater than $B + C$. For, by (22), $A : A - B :: C : C - D$; consequently, since A is greater than C, we must have $A - B$ greater than $C - D$; and adding $B + D$ to these unequals, we have $A + D$ greater than $B + C$. Again, if we suppose that B is the greatest term of the given proportion, then inverting its terms by (12), we shall have $B : A :: D : C$, and we shall have (as before) C for the least term, and $B + C$ will be greater than $A + D$.

Hence, *in any proportion whose terms are of one kind, the sum of the greatest and least terms is always greater than the sum of the other two.*

(**25.**) If A, B, C are three quantities of one kind, and if $A = B$, then we shall clearly have $\frac{A}{C} = \frac{B}{C}$, or by (12) $\frac{C}{A} = \frac{C}{B}$. Also, if we have the equation $\frac{A}{C} = \frac{B}{C}$ or $\frac{C}{A} = \frac{C}{B}$, it is clear that we must have $A = B$.

Hence (since the equations used are virtually proportions), *we say that equal quantities have the same ratio to the same quantity; and that the same quantity has the same ratio to equal quantities.*

And, reciprocally, *quantities which have the same ratio to the same quantity are equal to each other; and those quantities to which the same quantity has the same ratio, are equal to each other.*

(**26.**) If A, B, C are three quantities of one kind, and if A is greater than B, it is clear that we shall have the ratio $\frac{A}{C}$ greater than the ratio $\frac{B}{C}$. It is also clear that the ratio $\frac{C}{A}$ is less than the ratio $\frac{C}{B}$. Also, if the ratio $\frac{A}{C}$ is greater than the ratio $\frac{B}{C}$, it is plain that A is greater than B; and if $\frac{C}{A}$ is less than $\frac{C}{B}$, it is also evident that A must be greater than B.

Hence, *of unequal quantities, the greater has a greater ratio to the same quantity than the less has; and the same quantity has a greater ratio to the less quantity than it has to the greater.*

Reciprocally, *if one of two quantities has a greater ratio to another quantity than the remaining quantity has, then the remaining quantity is the less of the two; and if the same quantity has a greater ratio to one of two quantities than it has to the other, then the quantity to which it has the greater ratio is the less of the two.*

(27.) We shall now suppose that we have $\frac{A}{B}=\frac{C}{D}=\frac{E}{F}=\frac{G}{H}=$ etc., where we shall suppose that A, B, C, D, etc., are (either numbers or) quantities of one sort, then we shall have

$$\frac{A+C+E+G+\text{etc.}}{B+D+F+H+\text{etc.}}=\frac{A}{B}=\frac{C}{D}=\frac{E}{F}=\text{etc.}$$

For if m and n are any two positive integers, we shall get from the supposed equations the equations $\frac{mA}{nB}=\frac{mC}{nD}=\frac{mE}{nF}=\frac{mG}{nH}=$ etc., which show that when mA is greater than nB, we shall have mC greater than nD, mE greater than nF, and so on; also, when mA equals nB, we have mC equal to nD, $mE=nF$, and so on; and when mA is less than nB, mC is less than nD, mE is less than nF, and so on. Hence, when mA is greater, equal to, or less than nB, we shall have $mA+mC+mE+\text{etc.}=m(A+C+E+\text{etc.})$, greater, equal to, or less than $n(B+D+F+\text{etc.})$; hence, by (14), we shall have $\frac{A+C+E+G+\text{etc.}}{B+D+F+H+\text{etc.}}=\frac{A}{B}=\frac{C}{D}=\frac{E}{F}=$ etc.

Hence, *when (numbers or) quantities of the same kind are proportionals, we say that the sum of all the antecedents is to the sum of all the consequents, as any antecedent is to its consequent.*

(28.) If we have $\frac{A}{B}=\frac{C}{D}$, and $\frac{E}{B}=\frac{F}{D}$, then by adding these equations we have $\frac{A+E}{B}=\frac{C+F}{D}$.

Consequently, if we call A, B, C, D, etc., the first, second, third, etc., quantities, successively, *we may say that if the first has to the second the same ratio that the third has to the fourth, and the fifth to the second the same ratio that the sixth has to the fourth, then the sum of the first and fifth is to the second as the sum of the third and sixth is to the fourth.* In like manner, if the seventh is to the second as the eighth is to the fourth, *we shall have the sum of the first, fifth, and seventh is to the second as the sum of the third, sixth, and eighth is to the fourth;* and so on for any number of quantities that are connected with the second and fourth by equations, as above.

(29.) OF THE COMPOSITION AND RESOLUTION OF RATIOS.

Because ratios are numbers, it is evident that they may be added or subtracted, multiplied or divided, like numbers.

The product of any number of ratios is said to be a ratio that is compounded of all the ratios that enter as factors into the product. Thus, $\frac{A}{B} \times \frac{C}{D} \times \frac{E}{F} \times \frac{G}{H} \times$ etc., is a ratio that is compounded of the ratio of A to B, and of the ratio of C to D, and of the ratio of E to F, and so on.

A ratio which is compounded of two equal ratios is called a *duplicate ratio*, which is clearly equal to the square of either of the equal ratios.

Thus, if we have $\frac{A}{B} = \frac{C}{D}$, then $\frac{A}{B} \times \frac{C}{D}$ is a duplicate ratio, which is evidently the same as $\left(\frac{A}{B}\right)^2$ or $\left(\frac{C}{D}\right)^2$.

A ratio that is compounded of three equal ratios is called a *triplicate ratio*, which is manifestly the same as the cube or third power of either of the equal ratios. Thus, if we have $\frac{A}{B} = \frac{C}{D} = \frac{E}{F}$, then $\frac{A}{B} \times \frac{C}{D} \times \frac{E}{F}$ is a triplicate ratio, and it is clearly the same as either of the expressions $\left(\frac{A}{B}\right)^3$, $\left(\frac{C}{D}\right)^3$, $\left(\frac{E}{F}\right)^3$.

In like manner, a ratio that is compounded of four equal ratios is called a *quadruplicate ratio*, and a ratio that is compounded of five equal ratios is termed a *quintuplicate ratio*, and a ratio that is compounded of six equal ratios is termed a *sextuplicate ratio;* and universally, if n denotes any positive integer, then a ratio that is compounded of n equal ratios may be said to be of the n^{th} order, if we consider either of the equal ratios as being of the first order; and it is clear that the ratio of the n^{th} order is the same as the n^{th} power of either of the equal ratios.

Thus, if we have the n equal ratios $\frac{A}{B}=\frac{C}{D}=\frac{E}{F}=$ etc., then $\frac{A}{B}\times\frac{C}{D}\times\frac{E}{F}\times$ etc., to n factors, is a ratio of the n^{th} order, which is evidently the same as either of the expressions $\left(\frac{A}{B}\right)^n$, $\left(\frac{C}{D}\right)^n$, $\left(\frac{E}{F}\right)^n$, etc.

Reciprocally, if a ratio is resolved into two equal factors or ratios, then either of the equal ratios is named a *subduplicate ratio*, which is manifestly the same as the square root of the given ratio.

Thus, if we find that $\frac{A}{B}$ equals the product of the equal ratios $\frac{C}{D}$, $\frac{E}{F}$, then $\frac{C}{D}$ or $\frac{E}{F}$ is called a subduplicate ratio, which is clearly the same as the second or square root of the given ratio or as $\sqrt{\frac{A}{B}}$, which is called the subduplicate of the ratio $\frac{A}{B}$.

Also, if a ratio is resolved into three equal ratios, then either of the equal ratios is called a *subtriplicate ratio*, which is clearly the same as the third or cube root of the given ratio. Thus, if we find that $\frac{A}{B}$ equals the product of the three equal ratios $\frac{C}{D}$, $\frac{E}{F}$, $\frac{G}{H}$, then either of these equal ratios is a *subtriplicate ratio*, which is plainly the same as the third or cube root of the given ratio, or $\sqrt[3]{\frac{A}{B}}$, which is also called the subtriplicate ratio of the given ratio.

And when a ratio is resolved into four equal ratios, either of the ratios is called a *subquadruplicate ratio;* and if a ratio is resolved into five equal ratios, either of the equal ratios is called a *subquintuplicate ratio;* and if it is resolved into six equal ratios, either of the equal ratios is called a *subsextuplicate ratio;* and generally, if a ratio is resolved into n equal ratios, we may say that either of the equal ratios is the $\frac{1}{n}$th ratio of the given ratio, which is plainly the same as the n^{th} root of the given ratio.

Thus, if $\frac{A}{B}$ equals the product of the n equal ratios $\frac{C}{D}$, $\frac{E}{F}$, $\frac{G}{H}$, etc., then either of the equal ratios may be called the $\frac{1}{n}$th ratio of $\frac{A}{B}$, which is clearly expressed by $\sqrt[n]{\frac{A}{B}}$, the n^{th} root of the given ratio.

If a ratio that is compounded of three equal ratios is resolved into two equal ratios, either of the two equal ratios is called a sesquiplicate ratio. Thus, $\sqrt{\left(\frac{A}{B}\right)^3}$ is the sesquiplicate ratio of $\frac{A}{B}$.

If we compound the ratios $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$, etc., we shall have $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times$ etc., for the compound ratio.

Because A, B, C, etc., are (numbers or) quantities of the same sort, it is clear that we may divide A and B by C, so that we have $\frac{A}{B} = \frac{\frac{A}{C}}{\frac{B}{C}}$. Consequently, we shall have $\frac{A}{B} \times \frac{B}{C} = \frac{\frac{A}{C}}{\frac{B}{C}} \times \frac{B}{C} = \frac{A}{C}$, by erasing the number $\frac{B}{C}$ that is common to the dividend and divisor in the fractional product. In like manner we have $\frac{A}{C} \times \frac{C}{D} = \frac{A}{D}$, and $\frac{A}{D} \times \frac{D}{E} = \frac{A}{E}$, and so on to any extent; so that the compound ratio $\frac{A}{B} \times \frac{B}{C}$ is reduced to the ratio $\frac{A}{C}$, and the compound ratio $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D}$ is reduced to the ratio $\frac{A}{D}$, and the compound ratio $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E}$ is reduced to the ratio $\frac{A}{E}$, and so on.

Hence, if A, B, C, D, etc., represent any (numbers or) quantities of the same sort, *we say that the ratio of the first to the last is compounded of the ratio of the first to the second, and of the ratio of the second to the third, and of the ratio of the third to the fourth, and so on to the last;* see Def. A in R. Simson's Euclid, Book V.

(**80.**) *Ratios that are compounded of equal ratios are equal to each other.*

For, let the equal ratios be expressed by the equations $\frac{A}{B} = \frac{C}{D}$, $\frac{E}{F} = \frac{G}{H}$, $\frac{K}{L} = \frac{M}{N}$, and so on; then, since the products of equal numbers must be equal, we have $\frac{A}{B} \times \frac{E}{F} \times \frac{K}{L} \times$ etc. $= \frac{C}{D} \times \frac{G}{H} \times \frac{N}{M} \times$ etc.; that is, ratios which are compounded of equal ratios are equal, as was to be shown.

Reciprocally, *equal ratios divided by equal ratios or by the products of equal ratios, must give equal ratios for the quotients.*

For, let $\frac{A}{B} = \frac{C}{D}$ denote the equal ratios to be divided, whether they are simple or compound ratios, and $\frac{E}{F} = \frac{G}{H}$, the equal ratios by which they are to be divided, whether they are simple or compound.

Then, since equal numbers divided by equal numbers must give equal numbers for the quotients, we shall have $\frac{A}{B} \div \frac{E}{F} = \frac{C}{D} \div \frac{G}{H}$, or, since to divide by the numbers $\frac{E}{F}$, $\frac{G}{H}$ is clearly the same as to multiply by their reciprocals $\frac{F}{E}$, $\frac{H}{G}$, we shall have $\frac{A}{B} \times \frac{F}{E} = \frac{C}{D} \times \frac{H}{G}$; that is, equal ratios divided by equal ratios give equal ratios for the quotients, as required.

(**31.**) We will now show how, from knowing any three terms of a proportion, to find the remaining term.

Suppose that we have the proportion A : B :: C : D, or, which is the same, the equation $\frac{A}{B} = \frac{C}{D}$, (1). Then, if we know A, B, and D, we of course know the quotient $\frac{A}{B}$, and since $\frac{C}{D}$ expresses the number of times that C contains D, if we multiply D by $\frac{C}{D}$, we shall of course have $D \times \frac{C}{D} = C$. Now, since $\frac{A}{B} = \frac{C}{D}$, we shall have $D \times \frac{C}{D} = D \times \frac{A}{B} = C$, which gives the value of C, since D and the quotient $\frac{A}{B}$ are known.

Hence, *we see that the third term of a proportion equals the fourth term multiplied by the ratio of the first to the second term.*

Again, by (12), we may take the equation $\frac{B}{A} = \frac{D}{C}$ instead of (1); hence (as before), if we know A, B, and C, we shall have D expressed by $D = C \times \frac{B}{A}$.

Consequently, *if we multiply the third term of a proportion by the ratio of the second term to the first term, the product will equal the fourth term.*

In a similar way, it appears that *the first term of a proportion equals the product of the second term by the ratio of the third to the fourth term; and the second term equals the first term multiplied by the ratio of the fourth to the third term.*

(**32.**) We will now adapt what has been done to a continued proportion which consists of three terms. Thus, if in the proportion A : B :: C : D we suppose that C = B, we shall have A : B :: B : D, which is a continued proportion of three terms.

Instead of the proportion we may use the equation $\frac{A}{B}=\frac{B}{D}$, (2), or, by (12), we may use the equation $\frac{B}{A}=\frac{D}{B}$, (3).

Since, from equation (1), we have $D = C \times \frac{B}{A}$, if in this we put B for C, we get $D = B \times \frac{B}{A}$, (4).

(4) *shows that the third term of a continued proportion equals the mean term multiplied by the ratio of the mean to the first term.* And it may be shown, in a similar way, *that the first term equals the mean term multiplied by the ratio of the mean to the third term.*

To get the mean term, we compound the ratio of the first to the mean term with the ratio of the mean to the third term, and we get $\frac{A}{B} \times \frac{B}{D} = \frac{A}{D}$; or since (2) gives $\frac{A}{B}=\frac{B}{D}$, we get $\frac{B}{D} \times \frac{B}{D} = \frac{A}{D}$ or $\left(\frac{B}{D}\right)^2 = \frac{A}{D}$, and by taking the square root of both sides of this equation we have $\frac{B}{D} = \sqrt{\frac{A}{D}}$; or, since $B = D \times \frac{B}{D}$, we have $B = D\sqrt{\frac{A}{D}}$, (5), for the required mean term.

(5) *shows that the mean term equals the third term multiplied by the square root of the ratio of the first to the third term.*

It may be shown, in like manner, *that the mean term equals the first term multiplied by the square root of the ratio of the third to the first term.*

Since $B = A \times \frac{B}{A}$, and that $D = B \times \frac{B}{A}$, it follows that the terms A, B, D of our proportion will become A, $A \times \frac{B}{A}$, $A \times \frac{B}{A} \times \frac{B}{A}$, or A, $A \times \frac{B}{A}$, $A \times \left(\frac{B}{A}\right)^2$.

It is manifest, from what has been done, that any one of the three terms of the proportion can easily be found from the other two terms.

(33.) OF GEOMETRICAL PROGRESSIONS.

Numbers or quantities which are such that the first is to the second as the second to the third, and the second to the third as the third to the fourth, and the third to the fourth as the fourth to the fifth, etc., are said to be in continued geometrical proportion, or in geometrical progression.

If the terms increase from the first term, the proportion (or progression) is said to be increasing, but if the terms decrease from the first term, the progression is said to be decreasing. The ratio of the second term of the progression to the first term is called the ratio of increase or decrease of the progression. When the ratio is greater than unity, the progression is clearly increasing, and when it is less than unity, the progression is decreasing; and it is evident if the ratio is equal to unity, that the progression will be neutral, or that its terms will neither increase nor decrease, or that they will equal each other.

If A, B, C, D, E, etc., represent the successive terms of the continued proportion (or progression) whose first term is A, we shall signify that they are in continued proportion by writing them as follows, viz., $A : B :: B : C :: C : D :: D : E :: E : F ::$ etc., (1), which is in accordance with the usual manner of writing the terms of a continued geometrical proportion.

(34.) For A, B, C, D, E, F, etc., we may write A, $A \times \frac{B}{A}$, $A \times \left(\frac{B}{A}\right)^2$, $A \times \left(\frac{B}{A}\right)^3, \ldots . A \times \left(\frac{B}{A}\right)^{n-1}$; supposing n to denote the number of the last term.

For, since A, B, C are in continued geometrical proportion, we shall have, from (32), A, B, C expressed by A, $A \times \left(\frac{B}{A}\right)$, $A \times \left(\frac{B}{A}\right)^2$. And because B, C, D are in continued geometrical proportion, we shall have $D = C \times \frac{C}{B}$; and since $\frac{C}{B} = \frac{B}{A}$, we shall get $D = C \times \frac{B}{A}$; and because $C = A \times \left(\frac{B}{A}\right)^2$, we get $D = A \times \left(\frac{B}{A}\right)^2 \times \frac{B}{A} = A \times \left(\frac{B}{A}\right)^3$. In like manner, because C, D, E are in continued geometrical proportion, we shall have $\mathrm{E} = \mathrm{D} \times \frac{\mathrm{D}}{\mathrm{C}} = \mathrm{D} \times \frac{\mathrm{B}}{\mathrm{A}} = \mathrm{A} \times \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^4$, and so on; the law of continuation of the successive terms being evident.

Hence, collecting the results, our continued proportion or geometrical progression becomes A, $\mathrm{A} \times \frac{\mathrm{B}}{\mathrm{A}}$, $\mathrm{A} \times \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^2$, $\mathrm{A} \times \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^3$, $\mathrm{A} \times \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^4$, $\mathrm{A} \times \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^5$, ... $\mathrm{A} \times \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^{n-1}$, (2), as required.

(2) shows that the second term of the progression is equal to the first term A multiplied by the ratio $\frac{\mathrm{B}}{\mathrm{A}}$, and that the third term equals the first term A multiplied by the second power or square of the ratio $\frac{\mathrm{B}}{\mathrm{A}}$, and that the fourth term equals the first term multiplied by the third power or cube of the ratio; *and, generally, that the term whose number is denoted by n is equal to the first term* A *multiplied by the power of the ratio* $\left(\frac{\mathrm{B}}{\mathrm{A}}\right)$, *whose exponent is the number* $n - 1$.

It is also evident, from (2), if the ratio $\frac{\mathrm{B}}{\mathrm{A}}$ is greater than unity, that the series will be increasing, and if the ratio is less than unity, the series will manifestly decrease; also, if $\frac{\mathrm{B}}{\mathrm{A}}$ equals unity, the terms of the series are evidently equal to each other.

It may not be improper here to observe that the first term A of the geometrical progression (2) is frequently said to be to the third term $\mathrm{A} \times \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^2$ in the duplicate ratio of the first term to the second, and that the first term is to the fourth term in the triplicate ratio of the first to the second, and that the first term is to the fifth in the quadruplicate ratio of the first to the second, and so on; and it is clear that the meaning is the same as to say (as we have done) that the third term equals the first term multiplied by the square of the ratio of the second to the first term, and that the fourth term equals the first term multiplied by the cube of the ratio of the second to the first term, and so on.

(**25.**) It is easy to see, from (2), that the ratio of the term whose number is n to the first term, is expressed by $\left(\frac{B}{A}\right)^{n-1}$; so that if we extract the $(n-1)^{th}$ root of this ratio, we shall get $\frac{B}{A}$, which is the ratio of the second term to the first. Hence, if we know only the first and n^{th} term, we take the $(n-1)^{th}$ root of the ratio of the n^{th} term to the first term for the ratio of the second term to the first term.

Then we multiply the first term by the ratio, and the product will equal the second term, and the second term multiplied by the ratio gives the third term, and the third term multiplied by the ratio gives the fourth term, and so on; so that all the terms of (2) can be found from the first and n^{th} terms.

Instead of multiplying the first term by the ratio for the second term, etc., we may divide the n^{th} term by the ratio, and the quotient will clearly equal the term whose number is $n-1$, and dividing this term by the ratio, the quotient will equal the term whose number is $n-2$, and so on.

Instead of finding the ratio of the n^{th} term to the first term, we may find the ratio of the first term to the n^{th} term, and then extracting the $(n-1)^{th}$ root of the ratio thus found, we get the ratio of the first to the second term. Hence, if we divide the first term by the ratio of the first to the second term, the quotient will equal the second term, and if we divide the second term by the same ratio, the quotient will equal the third term, and so on, for the successive terms. Also, if we multiply the n^{th} term by the ratio of the first term to the second, the product will equal the term whose number is $n-1$, and if we multiply this term by the same ratio, the product will equal the term whose number is $n-2$, and so on, for the preceding terms until we arrive at the first term.

(**26.**) From what has been done, it is easy to see how to insert any number of geometrical means between any two given numbers or quantities of the same sort.

For, let a denote the first and b the second of the given numbers or quantities, and suppose that the number m equals the number of means that are to be inserted between a and b, then when the means are inserted, the whole number of terms in the geometrical progression thus found will be expressed by $m + 2$, whose first term will be a and last term b, whose number will be expressed by $m + 2$ when we count the terms from a to b. Consequently, for $n - 1$, in what has been previously shown, we must use $m + 1$, and we shall have $\frac{b}{a}$ for the ratio of the last term of the progression to its first term, whose $(m + 1)^{th}$ root will be expressed by $\left(\frac{b}{a}\right)^{\frac{1}{m+1}}$, which expresses the ratio of the second term of the progression to the first term. Hence the terms of the progression will be expressed by a, $a\left(\frac{b}{a}\right)^{\frac{1}{m+1}}$, $a\left(\frac{b}{a}\right)^{\frac{2}{m+1}}$, $a\left(\frac{b}{a}\right)^{\frac{3}{m+1}}$, ... $a\left(\frac{b}{a}\right)^{\frac{m}{m+1}}$, b (3), which can evidently be found by dividing b, the last term, by the ratio $\left(\frac{b}{a}\right)^{\frac{1}{m+1}}$ for the last term but one, and dividing the last term but one by the same ratio, the quotient will be the last term but two, and so on.

It is also easy to see that (3) can be found by dividing a by $\left(\frac{a}{b}\right)^{\frac{1}{m+1}}$, (which expresses the ratio of the first to the second term), for the second term, and dividing the second term by the same ratio for the third term, and so on; also, if we multiply the last term by the ratio $\left(\frac{a}{b}\right)^{\frac{1}{m+1}}$, the product will equal the last term but one, and this term multiplied by the same ratio will equal the last term but two, and so on, which is in accordance with what has been previously shown.

An example or two will serve to show the use of (3), and to make what has been said more evident.

For the first example we shall suppose that we are required to insert two geometrical means between 4 and 108.

Here we have $a = 4$, $b = 108$, $m = 2$, $m + 1 = 3$, and $\frac{b}{a} = \frac{108}{4} = 27$, $\left(\frac{b}{a}\right)^{\frac{1}{m+1}} = (27)^{\frac{1}{3}} = 3$. Hence (3) becomes 4, 12, 36, 108; and 12 and 36 are the two means which have been inserted between 4 and 108, as required.

For the second example we shall suppose that it is required to insert three geometrical means between 4375 and 7.

In this example, we have $a = 4375$, $b = 7$, $m = 3$, $m + 1 = 4$, and $\frac{b}{a} = \frac{7}{4375} = \frac{1}{625}$, $\left(\frac{b}{a}\right)^{\frac{1}{m+1}} = \left(\frac{1}{625}\right)^{\frac{1}{4}} = \frac{1}{5}$. Hence (3) becomes 4375, 875, 175, 35, 7; and 875, 175, 35 are the three means which have been inserted between 4375 and 7, as required.

For the third and last example, we shall suppose that six means are to be inserted between c and cm'^7.

Here $a = c$, $b = cm'^7$, $m = 6$, $m + 1 = 7$, and $\frac{b}{a} = m'^7$, $\left(\frac{b}{a}\right)^{\frac{1}{m+1}} = (m'^7)^{\frac{1}{7}} = m'$. Hence, c, cm', cm'^2, cm'^3, cm'^4, cm'^5, cm'^6, cm'^7, is the required progression.

It is evident, from what has been done, if we insert the same number of geometrical means between every two successive terms of any geometrical progression, that the series thus formed will be a geometrical progression.

(87.) If (for simplicity) we put r for the ratio, $\frac{\text{B}}{\text{A}}$ of the second term of the series (2) to the first term, then the series (2) will become $\text{A}, \text{A}r, \text{A}r^2, \text{A}r^3, \text{A}r^4, \ldots \text{A}r^{n-1}$, (4); and if we represent the sum of the terms of the series by S, we shall have $\text{S} = \text{A} + \text{A}r + \text{A}r^2 + \text{A}r^3 + \text{A}r^4 + \ldots + \text{A}r^{n-1}$, (5).

If we multiply both sides of (5) by r, we shall have $\text{S}r = \text{A}r + \text{A}r^2 + \text{A}r^3 + \text{A}r^4 + \text{A}r^5 + \ldots + \text{A}r^n$, and if we subtract (5) from this, we get $\text{S}r - \text{S} = \text{A}r^n - \text{A}$, or $\text{S}(r - 1) = \text{A}r^n - \text{A}$, which gives $\text{S} = \frac{\text{A}(r^n - 1)}{r - 1}$, (6). If we put $l = \text{A}r^{n-1}$, (a), then we get $\text{A}r^n = lr$, which reduces (6) to $\text{S} = \frac{lr - \text{A}}{r - 1}$, ($b$).

From (b) we derive the following rule for finding the sum of a series of numbers or quantities of the same sort that are in geometrical progression.

RULE.

Multiply the last term by the ratio, and from the product subtract the first term; then divide the remainder by the ratio less one, and the quotient will be the sum required.

If r is sensibly less than unity, and n indefinitely great, then lr will become indefinitely small, and may therefore be rejected, which will reduce (b) to $S = \frac{A}{1-r}$, (c).

(c) *shows that if r is sensibly less than one, the sum of the series continued to infinity is expressed by the first term divided by one, less the ratio.*

Since A, l, r, n and S are connected by the equations (a) and (b), it is evident when any three of them are known that the remaining two can be found by the equations (a) and (b).

We will now show the use of (a), (b), and (c) by a few examples.

EXAMPLES.

1. The first term of a geometrical progression is 6 and the ratio 2; it is required to find the ninth term and the sum of the nine terms.

Ans. The 9th term $= Ar^{n-1} = 6 \times 2^8 = 1536 = l$, and by ($b$) or the rule we have $S = \frac{lr - A}{r-1} = \frac{1536 \times 2 - 6}{2-1} = 3066.$

2. The first term of an infinite number of terms in geometrical progression is 3 and the ratio is $\frac{1}{2}$; required, the value of S. *Ans.* $S = \frac{A}{1-r} = \frac{3}{1-\frac{1}{2}} = 6.$

3. The first term of a geometrical progression is 2 and the ratio is 5; to find the seventh term and the sum of the seven terms.

Ans. The seventh term is 31250, and $S = 39062$ is the sum of the seven terms.

4. The first term of a geometrical progression is 8000 and the ratio is $\frac{1}{5}$; to find the fifth term and the sum of the five terms.

Ans. The fifth term is $12\frac{4}{5}$ and $S = 9996\frac{4}{5}$ is the sum of the five terms.

5. To find the sum of the infinite geometrical progression whose first term is 81 and ratio $\frac{1}{4}$.

Ans. $S = \dfrac{81}{1 - \frac{1}{4}} = 108$, as required.

(38.) Supposing A and B to stand for any two given numbers or quantities of the same sort, and that n denotes any given positive integer, we shall show how to reduce the compound ratio $\left(\frac{A}{B}\right)^n$ to the single ratio of two numbers or quantities of the same sort.

Thus, when $n = 2$, we have $\left(\frac{A}{B}\right)^2 = \frac{A}{B} \times \frac{A}{B}$, and if we put $\frac{A}{B} = \frac{B}{C}$, we get $\left(\frac{A}{B}\right)^2 = \frac{A}{B} \times \frac{B}{C} = \frac{A}{C}$, as required; and since $\frac{A}{B} = \frac{B}{C}$, we have $C = B \times \frac{B}{A}$, as is evident from (32); consequently, since B and A are given, C is found; and the terms A and C of the single ratio $\frac{A}{C}$ are known.

Again, if $n = 3$, we have $\left(\frac{A}{B}\right)^3 = \frac{A}{B} \times \frac{A}{B} \times \frac{A}{B}$, (1), to be reduced. By putting $\frac{A}{B} = \frac{B}{C} = \frac{C}{D}$, we get $C = B \times \frac{B}{A}$, $D = C \times \frac{C}{B} = B \times \frac{B}{A} \times \frac{B}{A} = B \times \left(\frac{B}{A}\right)^2$, which gives the value of D.

From the supposed equations $\frac{A}{B} = \frac{B}{C}$ and $\frac{B}{C} = \frac{C}{D}$, we get $\frac{A}{B} \times \frac{A}{B} = \frac{B}{C} \times \frac{C}{D}$, which reduces (1) to $\left(\frac{A}{B}\right)^3 = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{A}{D}$, as required.

And universally, if we put $B \times \left(\frac{B}{A}\right)^{n-1} = P$, we shall know the value of P, since A, B and n are given; and we shall have $\left(\frac{A}{B}\right)^n = \frac{A}{P}$, as required.

Hence, we see that if we have the equation $\left(\frac{A}{B}\right)^n = \left(\frac{C}{D}\right)^n$, and put $Q = D \times \left(\frac{D}{C}\right)^{n-1}$, we shall have the equation $\frac{A}{P} = \frac{C}{Q}$; so that the equation which involves two compound ratios is reduced to an equation which involves the simple ratios of A to P, and of C to Q.

In like manner, if $\left(\frac{A}{B}\right)^n = \left(\frac{E}{F}\right)^m$, and we put $R = F\left(\frac{F}{E}\right)^{m-1}$, we shall get $\frac{A}{P} = \frac{E}{R}$, observing that m is supposed to be a positive integer.

(**39.**) We shall now reduce the compound ratio $\frac{A}{B} \times \frac{C}{D}$ to the single ratio of two numbers or quantities of the same kind; supposing, of course, that A, B, C, D are given numbers, or quantities such that A and B are of the same kind, and that C and D are of the same kind with each other.

Thus, put $\frac{C}{D} = \frac{B}{E}$, and by (31) we have $E = B \times \frac{D}{C}$, which gives the value of E; consequently, we shall have $\frac{A}{B} \times \frac{C}{D} = \frac{A}{B} \times \frac{B}{E} = \frac{A}{E} =$ the required single ratio of two numbers or quantities of the same sort.

In a similar way we can reduce any compound ratio which is compounded of the ratios of given numbers or quantities to the single ratio of two given numbers or quantities. Hence, if we have the equation $\frac{A}{B} \times \frac{C}{D} = \frac{E}{F} \times \frac{G}{H}$, we see how to reduce the equation to the form $\frac{A}{L} = \frac{E}{K}$, in which L and K are found as before; so that the equation is reduced to the equality of the ratios of A to L, and of E to K. And we may proceed in a similar way for any number of ratios that may be compounded in the two members of the equation.

(**40.**) If we have a proportion such that either antecedent and its consequent are numbers, then shall the product of the extremes equal that of the means, and *vice versa.*

For let m and n be two numbers, and M and N two numbers or quantities of the same sort, such that we have the proportion $m : n :: M : N$, or its equivalent the equation $\frac{m}{n} = \frac{M}{N}$, (1). Then, since $\frac{M}{N}$ denotes the number of times that M contains N, we shall have $M = N \times \frac{M}{N}$, or since (1) gives $\frac{M}{N} = \frac{m}{n}$, we have $M = N \times \frac{m}{n}$; and if we multiply these equals by the number n, we shall have $M \times n = N \times m$, as required.

Reciprocally, if we have $M \times n = N \times m$, such that m and n are numbers, we shall have $\frac{m}{n} = \frac{M}{N}$, (2).

For since $m = n \times \frac{m}{n}$, and $M = N \times \frac{M}{N}$, the equation is easily changed to $Nn \times \frac{M}{N} = Nn \times \frac{m}{n}$, and omitting Nn, which is common to both sides of the equation, we must have $\frac{M}{N} = \frac{m}{n}$, or (which is the same) $\frac{m}{n} = \frac{M}{N}$, or its equivalent the proportion $m : n :: M : N$, as required.

Hence, if the first and second terms of a proportion are numbers (or are regarded as such), it follows that the product of the means divided by the first term gives the fourth term, which is in accordance with the common method of working the statement of a question in the Rule of Three.

Remarks.—1. It results from (31), that a fourth proportional to three numbers or quantities can always be found by multiplying the third by the ratio of the second to the first.

2. Similarly, it follows from (32), that a third proportional to two numbers or quantities equals the second multiplied by its ratio to the first.

(41.) If we have the proportion $A : B :: C : D$, or the equation $\frac{A}{B} = \frac{C}{D}$; then supposing n to represent any positive or negative real number, we shall clearly (from Ax. II.) get $\left(\frac{A}{B}\right)^n = \left(\frac{C}{D}\right)^n$, since $\frac{A}{B}$ and $\frac{C}{D}$ are equal numbers.

Hence, if $\left(\frac{A}{B}\right)^n$ is given, we can easily find $\frac{C}{D}$; and thence, if C is known, D can be found. Thus, if $n = 3$, and 27 is put for $\left(\frac{A}{B}\right)^3$, we shall have $\left(\frac{C}{D}\right)^3 = 27 = 3^3$, whose cube root gives $\frac{C}{D} = 3$, and, of course, $D = \frac{C}{3}$.

If A, B, C, D are numbers (or are regarded as such), then the equation $\left(\frac{A}{B}\right)^n = \left(\frac{C}{D}\right)^n$ can be written in the form $\frac{A^n}{B^n} = \frac{C^n}{D^n}$, which gives the proportion $A^n : B^n :: C^n : D^n$; consequently, since $A^n \times D^n = B^n \times C^n$, we have $D^n = B^n \times C^n \div A^n$, or $D = \sqrt[n]{\frac{B^n \times C^n}{A^n}} = \frac{B \times C}{A}$, a result obtained according to the common method of proceeding.

It may be added that if A, B, C, D are not numbers (nor are regarded as such), the proportion $A^n : B^n :: C^n : D^n$ can not exist, since $\left(\frac{A}{B}\right)^n = \left(\frac{C}{D}\right)^n$ can not be written in the form $\frac{A^n}{B^n} = \frac{C^n}{D^n}$.

Again, if we have the proportions $A : B :: C : D$, $E : F :: G : H$, $K : L :: M : N$, etc., or their equivalents, the equations, $\frac{A}{B} = \frac{C}{D}$, $\frac{E}{F} = \frac{G}{H}$, $\frac{K}{L} = \frac{M}{N}$, etc.; then, by compounding these equal ratios, we shall (from Ax. II.) have $\frac{A}{B} \times \frac{E}{F} \times \frac{K}{L} \times \text{etc.} = \frac{C}{D} \times \frac{G}{H} \times \frac{M}{N} \times \text{etc.}$

Hence, if A, B, C, D, etc., are numbers (or are regarded as such), it is clear that the preceding equation can be written in the form $\frac{AEK \text{ etc.}}{BFL \text{ etc.}} = \frac{CGM \text{ etc.}}{DHN \text{ etc.}}$, which gives the proportion AEK etc. : BFL etc. :: CGM etc. : DHN etc., which can be immediately obtained from the given proportions by taking the product of their corresponding terms. But if A, B, C, D, etc., are neither numbers (nor considered as such), then the preceding proportion can not exist, because the equation $\frac{A}{B} \times \frac{E}{F} \times$ etc. $= \frac{C}{D} \times \frac{G}{H} \times$ etc. can not be written in the form $\frac{AE \text{ etc.}}{BF \text{ etc.}} = \frac{CG \text{ etc.}}{DH \text{ etc.}}$.

(42.) ABRIDGED PROPORTIONS OR VARIATIONS.

1. If, in the proportion $A : B :: C : D$, A and B are variable, while C and D are invariable, then it is clear that the ratio of A to B will be constant or invariable. Hence, we say that A varies as B directly, which we express (according to custom) by writing $A \propto B$ directly.

2. Since the proportion $A : B :: C : D$ gives $A = B \times \frac{C}{D}$, if B and C are invariable, while A and D are variable, then it is clear that, if A increases, D must decrease, and *vice versa.* Hence, we may say that as A increases, D must decrease, and reciprocally, or that A varies inversely as D, which we may express by $A \propto D$ inversely or reciprocally.

3. Since $A = B \times \frac{C}{D}$, it follows that A varies as the product $B \times \frac{C}{D}$, which may be expressed by the form $A \propto B \times \frac{C}{D}$, directly.

(43.) HARMONICAL PROPORTIONS AND PROGRESSIONS.

1. Three numbers or quantities (of the same kind) are said to be in harmonical proportion or progression when the first is to the third as the difference of the first and second is to the difference of the second and third. And the third term is called a third harmonical proportional to the first and second; also, the second term is called a harmonical mean between the first and third terms, noticing that the three terms are sometimes called *harmonicals.*

Thus, if A, B, C are the harmonicals, we must have either $A : C :: A - B : B - C$, or $A : C : B - A : C - B$; and since the terms are of the same kind, these proportions by (13) may be written in the forms $A : A - B :: C : B - C$, and $A : B - A :: C : C - B$.

Hence, if by (19) and (20) we apply composition to the first of these proportions, and division to the second, they will be reduced to the single proportion $2A - B : A :: B : C$, or to the equivalent equation $C = B \times \frac{A}{2A - B}$; consequently, if A, B, C are numbers (or are regarded as such), we shall have $C = \frac{AB}{2A - B}$.

Hence, *if the second of three harmonicals is multiplied by the ratio of the first to twice the first diminished by the second, the product will give the third term.*

If the harmonicals are numbers, the equation $C = \frac{AB}{2A - B}$, being written in the form $\frac{1}{C} = \frac{2A - B}{AB} = \frac{2}{B} - \frac{1}{A}$, gives $\frac{1}{A} - \frac{1}{B} = \frac{1}{B} - \frac{1}{C}$; and if B, C, D, E, etc., are also harmonicals, we shall also have $\frac{1}{B} - \frac{1}{C} = \frac{1}{C} - \frac{1}{D}$, and $\frac{1}{C} - \frac{1}{D} = \frac{1}{D} - \frac{1}{E}$, and so on.

Hence, since the numbers A, B, C, D, etc., on the suppositions here made, are said to be in harmonical progression, it follows that their reciprocals, $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D}, \frac{1}{E}$, etc., are in arithmetical progression. Conversely, it is easy to show that the reciprocals of a series of numbers in arithmetical progression will be a series of numbers in harmonical progression. Thus, by taking the reciprocals of the arithmeticals 2, 5, 8, we get the harmonicals $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}$, or their equivalents $\frac{40}{80}, \frac{16}{80}, \frac{10}{80}$; consequently, since we have $40 : 10 :: 40 - 16 : 16 - 10$, it follows that the reciprocals of the arithmeticals are in harmonical progression.

Again, resuming the proportion $2A - B : A :: B : C$, and by alternation by (13), writing it in the form $2A - B : B :: A : C$; then by composition, according to (19), we shall get the proportion $2A : B :: A + C : C$, or the equivalent equation $\frac{2A}{B} = \frac{A + C}{C}$.

From the preceding equation we easily get $B = 2A \times$

$\frac{C}{A + C}$; which, if A, B, C are numbers, may be written in the form $B = \frac{2AC}{A + C}$.

Hence, *the mean of three harmonicals equals twice the first multiplied by the ratio of the third to the sum of the first and third.*

2. Four numbers or quantities of the same kind are in harmonical proportion when the first is to the fourth as the difference of the first and second is to that of the third and fourth; where it is to be noticed that the fourth term is called a fourth harmonical proportional to the first three terms.

Thus, if A, B, C, D are in harmonical proportion, then we must either have $A : D :: A - B : C - D$, or $A : D :: B - A : D - C$. Proceeding as in the case of three harmonicals, these proportions are easily reduced to the single proportion $2A - B : A :: C : D$, or to the equation $D = C \times \frac{A}{2A - B}$, which may be written in the form $D = \frac{AC}{2A - B}$, when the harmonicals are numbers, or are regarded as such.

Hence, *the fourth proportional to three given harmonicals equals the third term multiplied by the ratio of the first to twice the first less the second term.*

(44.) EXAMPLES.

To illustrate what has been said concerning ratios and proportions.

1. To find the ratios of 45 to 9 and 40 to 8.

Because $\frac{45}{9} = 5$ and $\frac{40}{8} = 5$, the ratios are equal, and we shall have $\frac{45}{9} = \frac{40}{8}$; consequently, by the definition of Geometrical Proportion, we have the proportion $45 : 9 :: 40 : 8$.

2. To compare the ratios of 17 to 3 and 23 to 4.

Because the ratio of 17 to 3 is expressed by $\frac{17}{3} = 5.666$, and that of 23 to 4 by $\frac{23}{4} = 5.75$, it follows that 17 has a less ratio to 3 than 23 has to 4, which we may (according to custom) express by the form $17 : 3 < 23 : 4$.

3. To compare the ratios of 25 to 6 and 29 to 7.

Because the ratios are expressed by $\frac{25}{6} = 4\frac{1}{6}$, and $\frac{29}{7} = 4\frac{1}{7}$, it follows that 25 has a greater ratio to 6 than 29 has to 7, which may be expressed by the form $25 : 6 > 29 : 7$.

4. Supposing A, B, and C to be three numbers or quantities of the same kind, then it is proposed to compare the ratios $\frac{A}{B}$ and $\frac{A+C}{B+C}$.

1st. Because $\frac{A+C}{B+C} = \frac{A}{B} \times \frac{1+\frac{C}{A}}{1+\frac{C}{B}}$, it is clear that the ratios will be equal, and each equal to unity, when $A = B$; noticing that such ratios, in which each antecedent equals its consequent, are called *ratios of equality.*

2d. If A is less than B, then A, B, and C, being positive, since $\frac{C}{A}$ is greater than $\frac{C}{B}$, it is clear, from the above form, that the ratio $\frac{A+C}{B+C}$ is greater than the ratio $\frac{A}{B}$.

3d. Similarly, if A is greater than B, it is manifest that the ratio $\frac{A+C}{B+C}$ is less than the ratio $\frac{A}{B}$.

Remarks.—When the antecedent is greater than the consequent, the ratio is sometimes called a ratio of *greater inequality;* and when the antecedent equals the consequent, it is called a *ratio of equality;* also, when the antecedent is less than the consequent, the ratio is called a ratio of *lesser inequality.*

5. To find a fourth proportional to 6, 12, and 15; also, a third proportional to 4 and 6.

From (31), we shall have $15 \times \frac{12}{6} = 15 \times 2 = 30 =$ the fourth proportional; and from (32) we shall have $6 \times \frac{6}{4} = 6 \times 1.5 = 9 =$ the third proportional.

6. To find a mean proportional between 2 and 8, and between 4 and 9.

From (32) it follows that the sought mean proportionals are expressed by $8\sqrt{\frac{2}{8}} = \frac{8}{2} = 4$, and by $9\sqrt{\frac{4}{9}} = 9 \times \frac{2}{3} = 6$.

7. If 25 yards of a certain kind of cloth cost £30 10*s.*, it is proposed to find how much 60 yards of the same kind of cloth is worth.

If we denote the required value by x, then the question, properly stated, gives the proportion 25 yds. : 60 yds. :: £30 10*s.* : x. Hence, since 60 yds. ÷ 25 yds. $= 2\frac{2}{5} = 2.4$, we shall get $x =$ £30 10*s.* × 2.4 = £73 4*s.*

If 25 yds. and 60 yds. are treated as numbers, our proportion becomes 25 : 60 :: £30 10*s.* : $x = \frac{\text{£}30\ 10s. \times 60}{25} =$ £73 4*s.*, which mode of proceeding is in conformity to the common method of working the statement of a question in the Rule of Three.

8. If 36 men eat a certain quantity of provisions in 6 weeks, then how long will it take 48 men to eat the same quantity?

Because 48 men will clearly eat the provisions in less time than 36 men, or (which is the same), since more requires less, this question clearly falls under what is commonly called the Rule of Three Inverse.

If x stands for the required time, the question gives the proportion 36 m. : 48 m. :: x : 6 w.; which, by inverting the terms (according to (12)), becomes 48 m. : 36 m. :: 6 w. : x; consequently, since 36 m. ÷ 48 m. $= \frac{36}{48} = \frac{3}{4} = 0.75$, we shall get $x = 6$ w. × 0.75 = 4.5 weeks.

If 48 m. and 36 m. are regarded as numbers, our proportion becomes 48 : 36 :: 6 w. : x, which gives $x = \frac{36 \times 6\text{ w.}}{48} = 4\frac{1}{2}$ weeks.

9. Given, the proportion $a + x : b + x :: 3 : 2$, to find x.

Dividing according to (20), we get $a - b : b + x :: 1 : 2$, or the equivalent equation $\frac{b+x}{a-b}=\frac{2}{1}=2$; and subtracting $\frac{b}{a-b}$ from these equals, we shall get $\frac{x}{a-b}=2-\frac{b}{a-b}$, or because x is $a-b$ times as great as $\frac{x}{a-b}$, we shall get $x=\left(2-\frac{b}{a-b}\right)\times(a-b)=2a-2b-b=2a-3b$, as required.

If we put $2a-3b$ for x in the given proportion, it becomes $3a-3b:2a-2b::3:2$; which is equivalent to the identical proportion $3:2::3:2$, as it clearly ought to be.

10. To find a third harmonical proportional to 12 and 8.

From (43) we get $8\times\frac{12}{24-8}=6=$ the third harmonical; and it is easy to perceive that 12, 8, and 6 are in harmonical proportion or progression.

11. To find a harmonical mean between 3 and 6.

From (43) we have $6\times\frac{6}{3+6}=4$; and 3, 4, and 6 are clearly in harmonical proportion.

12. To find a fourth harmonical proportional to 5, 2, and 16.

From (43) we get $16\times\frac{5}{10-2}=10$, for the required proportional, and it is easy to perceive that 5, 2, 16, and 10 are in harmonical proportion.

(45.) LIMITS OF VARIABLE RATIOS.

If the antecedent and consequent of a variable ratio are such that (without actually changing the signs of any of their terms) they can be varied, so that the ratio can be made to differ from a certain fixed ratio, by a difference which is less than any given difference, then we shall call the fixed ratio *a limit of the variable ratio.*

Thus, if we take the ratio $\frac{a+x}{b+x}$, on the supposition that a and b are constant or invariable, while x is variable, then it is clear that the ratio will be variable. Supposing x to diminish until it differs from naught (0) by a difference less than any given difference, then it is clear that the ratio will differ from the ratio $\frac{a}{b}$ by a difference which will be less than any given difference; consequently, $\frac{a}{b}$ is a limit to the proposed ratio; and it is clear that the limit can be immediately obtained by putting $x = 0$.

Again, since $\frac{a+x}{b+x} = \frac{1 + \frac{a}{x}}{1 + \frac{b}{x}}$, it is evident that the ratios $\frac{a}{x}$ and $\frac{b}{x}$ will be diminished indefinitely, and differ from 0 by differences less than any given differences, when x is unlimitedly great, since a and b are supposed to be finite; consequently, when x is unlimitedly great, it is clear that $\frac{1}{1} = 1$ is the limit of the proposed ratio. And it is clear that this ratio can be immediately obtained from the proposed ratio by omitting a and b, on account of their minuteness in comparison to x, when x is unlimitedly great, which reduces the ratio to $\frac{x}{x} = 1$.

It is clear, from what has been done, that the given ratio has two limits; one of which results from making $x = 0$, or an infinitesimal, and the other from making x infinite, or unlimitedly great.

It is hence easy to perceive how we are to proceed in any case to find the limits of any variable ratio.

To illustrate what has been done, take the following

EXAMPLES.

1. To find the limits of the ratio $\frac{4 + 9x}{2 + 3x}$, which correspond to x an infinitesimal and x infinite.

Putting 0 for x, the first limit is $\frac{4}{2} = 2$; and putting $x = \infty$, by rejecting 4 and 2 in comparison to $9x$ and $3x$, the second limit is $\frac{9x}{3x} = 3$.

2. To find the limits of $\frac{a + x^4}{b + x^3}$, that correspond to x an infinitesimal and x infinite.

Putting 0 for x, we get $\frac{a}{b}$ for the first limit; and putting ∞ for x, the second limit is $\frac{x^4}{x^3} = x$, which is unlimitedly great, since x is infinite.

3. To find the limits of $\frac{7 + x}{3 + x^2}$, which correspond to x, an infinitesimal and infinite.

Putting 0 for x, the first limit is $\frac{7}{3}$; and putting $x = \infty$, the second limit is $\frac{x}{x^2} = \frac{1}{x}$, which is infinitesimal, since $\frac{1}{x}$ is infinitesimal, because x is infinite.

4. To find the limits of $\frac{3 + 5x + 7x^2}{9 - 11x + 14x^2}$, that correspond to x an infinitesimal and x infinite.

Putting 0 for x, the first limit is $\frac{3}{9} = \frac{1}{3}$. And putting $x = \infty$, it is clear that $3 + 5x$ must be rejected in comparison to $7x^2$, and that $9 - 11x$ must be rejected in comparison to $14x^2$; consequently, $\frac{7x^2}{14x^2} = \frac{1}{2}$, is the second limit.

5. To find the limits of $\frac{ax + bx^{\frac{3}{2}}}{5cx + dx^{\frac{5}{2}}}$, that correspond to x infinitesimal and x infinite.

Erasing the factor x from the dividend and divisor, we get the equivalent expression $\frac{a + bx^{\frac{1}{2}}}{5c + dx^{\frac{3}{2}}}$. Hence, putting $x = 0$ and $x = \infty$, we get $\frac{a}{5c}$ and $\frac{b}{dx}$ = an infinitesimal (x being infinite), for the required values.

6. To find the limits of $\frac{3 + x}{4}$ and $\frac{4}{5 - 2x}$, that correspond to x infinitesimal and x infinite.

Ans. $\frac{3}{4}$, and infinite; $\frac{4}{5}$, and $-$ an infinitesimal.

7. To find the limits of $\frac{a + bx - cx^2 - dx^3}{c + bx + gx^2 - hx^3}$, that correspond to x infinitesimal and x infinite. *Ans.* $\frac{a}{c}$ and $\frac{d}{h}$.

8. To find the limits of $\frac{3x^{-1} + 2x^{-2}}{8x^{-1} - 5x^{-2}}$, that correspond to x infinitesimal and x infinite.

Since the expression is equivalent to $\frac{2 + 3x}{-5 + 8x}$, the answers are $-\frac{2}{5}$ and $\frac{3}{8}$.

(46.) To find the true value of a ratio or fractional expression, whose dividend and divisor are reduced to 0, by assigning certain particular values to some of their terms; so that the expression becomes of the form $\frac{0}{0}$.

Thus, if we take $\frac{a^2 - x^2}{a(a-x)}$; then, by putting a for x, it becomes $\frac{a^2 - a^2}{a(a - a)}$, or, since $a^2 - a^2 = 0$, $a - a = 0$, and that $a \times 0 = 0$, it becomes $\frac{0}{0}$.

To find the true value; we observe that $\frac{a^2 - x^2}{a(a - x)} = \frac{a + x}{a} \times \frac{a - x}{a - x} = \frac{a + x}{a}$, when the useless factor of the dividend and divisor is erased, as it evidently ought to be, since it makes no part of the value. Hence, if we put a for x in $\frac{a + x}{a}$, we get $\frac{2a}{a} = 2$ for the required value of $\frac{a^2 - x^2}{a(a - x)}$, when a is put for x.

Again, by putting $x = a + y$, we get $x^2 = a^2 + 2ay + y^2$, and thence $\frac{a^2 - x^2}{a(a - x)}$ is reduced to $\frac{2ay + y^2}{ay} = 2 + \frac{y}{a}$; consequently (since $a = x$ reduces y to 0), by putting $y = 0$, we get 2 for the true value of the expression; which is the same as found before.

And it is clear that we may proceed in a similar manner in all analogous cases.

Hence, *if by either of the preceding methods we reduce the dividend and divisor of the given expression to their lowest terms, then, by proceeding with the reduced expression according to the conditions of the question, we shall get the required value.*

EXAMPLES.

1. To find the value of $\frac{a^3-x^3}{a(a^2-x^2)}$, when a is put for x.

Dividing the dividend and divisor by $a-x$, the reduced expression is $\frac{a^2+ax+x^2}{a(a+x)}$; consequently, putting a for x in this, the required value $=\frac{3a^2}{2a^2}=\frac{3}{2}$.

2. To find the value of $\frac{a^2-x^2}{(a+x)^2}$, when a is put for x.

Since the reduced expression $=\frac{a-x}{a+x}$, the required value $=\frac{a-a}{a+a}=\frac{0}{2a}=0$.

3. To find the value of $\frac{x^2-a^2}{x^3-ax^2-a^2x+a^3}$, when a is put for x.

The reduced expression being $\frac{1}{x-a}$, the required value $=$ $\frac{1}{a-a}=\frac{1}{0}=$ infinity, since 1 contains the infinitessimal 0 an unlimited number of times.

4. To find the value of $\frac{ax^2-2acx+ac^2}{bx^2-2bcx+bc^2}$, when c is put for x.

Because $x^2-2cx+c^2$ is a factor of the dividend and divisor, the expression equals $\frac{a}{b}$, and does not depend on any particular value of x. See page 247, Vol. I., of the Calcul. Differential, etc., of Lacroix.

5. To find the value of $\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$, when a is put for x.

Putting $x = a + y$, the expression becomes $\frac{(2ay + y^2)^{\frac{3}{2}}}{y^{\frac{3}{2}}} = (2a + y)^{\frac{3}{2}}$; consequently, putting $y = 0$, the answer $= (2a)^{\frac{3}{2}}$, which is manifestly correct, since $\frac{(x^2 - a^2)^{\frac{3}{2}}}{(x - a)^{\frac{3}{2}}} = \left(\frac{x^2 - a^2}{x - a}\right)^{\frac{3}{2}} = (x + a)^{\frac{3}{2}}$.

(47.) OF THE CHARACTERS 0 AND ∞, AND THEIR USES.

1. A number or quantity that is contained an unlimitedly great number of times in the number or quantity that is taken for the unit in any calculation, is called *infinitely small*, or *an infinitesimal*, which is (generally) denoted by 0, called *naught*, *zero*, or *cipher;* noticing that the naught may be taken with either of the signs + or —, according to the nature of the case. Again, a number or quantity that contains the unit of number or quantity an unlimitedly great number of times is said to be *infinite*, and is frequently represented by ∞, called *the sign of infinity;* noticing that the sign of infinity may be taken with either of the signs + or —, accordingly as the number or quantity represented, is regarded as positive or negative.

It may be added, that any number or quantity that is contained in the unit (of number or quantity), or which contains it a limited number of times, is said to be *finite*.

2. It is clear (from Ax. VII., and from the nature of the case) that an infinitesimal which is connected with a finite number or quantity by + or — must be rejected, and that a finite or infinite number or quantity which is connected with an infinitely greater number or quantity by + or — must also be rejected. Thus, if we take the expression $3 + 2x + 4x^2$, then, if x is an infinitesimal, it must be reduced to 3, and if x is infinite (since x^2 will clearly be infinitely greater than x) it must be reduced to $4x^2$.

3. Supposing x to be an infinitesimal of the first order, then x^2, which is (clearly) infinitely less than x, x^3 infinitely less than x^2, and so on, together with all the infinitesimals that have finite ratios to these, will clearly be infinitesimals of the second, third, etc., orders; also, if x is an infinite of

the first order, then x^2, x^3, etc., together with all infinites that have finite ratios to these, will clearly be infinites of the second, third, etc., orders.

4. If 0 and 0 stand for two infinitesimal numbers or quantities of the same kind, then it is clear (from what has been done) that they may be supposed to be connected together by an equation of the form $0 \times a = 0$, such that if the infinitesimals are of the same order, a shall be a finite number or quantity, which may be constant or variable, according to the nature of the case; noticing that if the dividend in the expression for a is an infinitesimal of a higher order than the divisor, a must be an infinitesimal of a certain order; and that if the dividend is of a lower order of infinitesimals than the divisor, then a must be an infinite of a certain order. Similarly, if two infinite numbers or quantities of the same kind are denoted by ∞ and ∞, then they may be connected by the equation $\infty \times \text{A} = \infty$, or $\text{A} = \frac{\infty}{\infty}$, such that A is finite when the infinites are of the same order; noticing that A is an infinitesimal of a certain order when the dividend is of a less order than the divisor, and *vice versa.*

(**48.**) From what has been done, we are brought to the consideration of the following most important

PRINCIPLE.

Supposing the quantity X to be expressed in terms of the (number or) quantity x, if x is changed into $x + h$, it is clear that X will be changed into the corresponding quantity $\text{X} + \text{H}$; in which H results from changing x into $x + h$, so that H depends on h; now if x and h are such that they may be considered as being indeterminate or arbitrary, the quotient $\frac{\text{H}}{h}$ shall (in part at least) be independent of h; so that when h and H are small, the quotient shall be nearly invariable.

For, put $h = 2h_1$, and use $\text{X} + \text{H}_1$ to represent the value of X resulting from the substitution of $x + h_1$ for x in X; then if we substitute $x + h_1$ for x in $\text{X} + \text{H}_1$, and represent the change in H_1 (resulting from the substitution of $x + h^1$

for x in it) by H'_1, $X + H_1$ will become $X + 2H_1 + H'_1$; since X becomes $X + H_1$, and that H_1 becomes $H_1 + H'_1$.

Because $h = 2h_1$, and that X becomes $X + H$ by changing x into $x + h$, and that X becomes $X + 2H_1 + H'_1$ by changing x into $x + h_1$ and then changing x in the result into $x + h_1$, it is clear that we must have $H = 2H_1 + H'_1$; because in each process x has been changed into $x + h$, and that the same value of X must correspond to the same value of $x + h$. Dividing the equals H and $2H_1 + H'_1$ by the equals h and $2h_1$, we get the equation $\frac{H}{h} = \frac{2H_1 + H'_1}{2h_1} = \frac{H_1}{h_1} + \frac{H'_1}{2h_1}$, (1); in which, it is clear, if we put $h_1 = \frac{h}{2}$ for h, that the quotient $\frac{H}{h}$ will become the quotient $\frac{H_1}{h_1}$; and it is also clear that $\frac{H'_1}{2h_1}$ equals the variation of $\frac{H_1}{2h_1}$, which results from changing x (in it) into $x + h_1$. Because $\frac{H'_1}{2h_1}$ is the variation of $\frac{H_1}{2h_1}$ that results from putting $x + h_1$ for x, it is plain that $\frac{H_1}{h_1}$ must have a part or a term, which we shall denote by A, such that it shall be different from any term or part of $\frac{H'_1}{2h_1}$; and since (1) must clearly be an identical equation, or that the terms of the two members must be identical, so that it may be satisfied independently of h or h_1, it follows that A must be common to $\frac{H}{h}$ and $\frac{H_1}{h_1}$; and since $\frac{H}{h}$ is changed into $\frac{H_1}{h_1}$ by putting $h_1 = \frac{h}{2}$ for h, it is clear that A is independent of h or h_1; consequently, *the quotient* $\frac{H}{h}$ *is in part independent of* h.

Again, because $\frac{A}{2}$ is a part or a term of the quotient $\frac{H_1}{2h_1}$, it results, if A contains x, that the variation $\frac{H'_1}{2h_1}$ of $\frac{H_1}{2h_1}$ must contain a term of the form Bh_1, such that B is independent of h_1; noticing that Bh_1 results from the substitution of $x + h_1$ for x in A.

Hence, it follows that the right member of (1) must contain the term $Bh_1 = \frac{B}{2}h$, and of course, since the equation must be identical, the first member $\frac{H}{h}$ must contain a corresponding term, which we shall represent by A_1h, in which A_1 is independent of h; consequently (agreeably to what has been shown), if h_1 is put for h, $\frac{H_1}{h_1}$ must contain the term A_1h_1.

Because $\frac{H_1}{h_1}$ contains the term A_1h_1, if A_1 contains x, it follows that $\frac{H'_1}{2h^1}$ must contain a term of the form $\frac{A'_1h^2_1}{2} = \frac{A'_1h^2}{8}$, since A_1 becomes of the form $A_1 + A'_1h_1 +$ etc., in which A'_1 is supposed not to contain h_1; consequently, since (1) must be an identical equation, it follows that $\frac{H}{h}$ must contain a corresponding term, which may be expressed by A_2h^2; in which A_2 is supposed not to contain h.

It may be shown, in like manner, if A_2 contains x, that $\frac{H}{h}$ must contain a term of the form A_3h^3, and so on; consequently, $\frac{H}{h}$ must be expressed by the form $\frac{H}{h} = A + A_1h + A_2h^2 + A_3h^3 +$ etc., (2); in which A, A_1, A_2, etc., are supposed to contain x, and to be independent of h.

It is clear that we may assign particular values to x and h in (2), provided A, A_1, A_2, etc., do not any of them become infinite; for (2) is clearly true when A, A_1, A_2, etc., are finite, or one or more of them reduced to 0.

If one or more of the quantities A, A_1, A_2, A_3, etc., as A_3, becomes infinite for any particular value of x, it is clear that (2) is true no further than to the term A_3h^3, or to the term which first becomes infinite.

If A does not contain x, it is clear that (2) will be reduced to $\frac{H}{h} = A$, which shows the quotient $\frac{H}{h}$ to be invariable.

If A, A_1, A_2, etc., contain x, and A is not very small in comparison to A_1, A_2, etc., then if h is small in comparison to unity, it is clear that the quotient $\frac{H}{h}$ will not differ much from A, when different values are assigned to h; but if A_1, A_2, A_3, etc., are very great in comparison to A, it will be necessary to make h extremely small in comparison to unity (so that the terms A_1h, A_2h^2, etc., in (2) may not sensibly affect A), in order that the quotient may be nearly invariable when h is varied.

Hence, if $\frac{H}{h}$ is treated as a ratio, it results *that h may be taken so small, that* H *shall vary very nearly as h.*

It may be noticed, that if the two members of (2) are multiplied by h, there results $H = Ah + A_1h^2 + A_2h^3 + \text{etc.}$; consequently, if $X' = X + H$, we get $X' = X + Ah + A_1h^2 + A_2h^3 + \text{etc.}$, (3); in which X' represents the value of X, that corresponds to $x + h$. If in X' we change x into $x + k$, and represent the corresponding change in X' by K, X' will be changed into $X' + K$, and, according to what has been shown, the quotient $\frac{K}{k}$ will be very nearly invariable when k is very small. If h is very small, it is clear, from (3), that we shall have $\frac{H}{h} = \frac{K}{k}$, (4), very nearly; consequently, if h and k have like signs, H and K must also have like signs, and conversely.

Remarks.—1. It may not be improper here to notice that any number or quantity whose value is supposed to remain the same during any calculation, is called a *constant;* while any number or quantity whose value is supposed to change is called a *variable;* where it may be observed that a constant is usually expressed by one of the first letters of the alphabet, while a variable is expressed by one of the last letters.

2. If any variable, as X, depends on another as x, then X is called a *function* of x, and X is sometimes called the *dependent variable* and x the *independent variable.* If the form of the expression of X in terms of x is known or given, then X is said to be an *explicit* function of x; but if the form of the expression of X in terms of x is not given, then X is said to be an *implicit* function of x. Thus, in $X = ax + x^2$,

X is an explicit function of x, while x is an implicit function of X.

To show the use of what has been done, take the following

EXAMPLES.

Ex. 1. Given, $\frac{H}{h} = 2x + h$, to determine whether it has been derived from a function of x, by the process indicated in (1).

By changing h into h_1 we get $\frac{H_1}{h_1} = 2x + h_1$; and, changing x in this into $x + h_1$, we get $\frac{H'_1}{2h_1} = \left[\frac{2(x + h_1) + h_1}{2}\right] - \left(\frac{2x + h_1}{2}\right) = h_1$; consequently, we shall have $\frac{H_1}{h_1} + \frac{H'_1}{2h_1} = 2x + 2h_1 = 2x + h$, and of course, since $\frac{H}{h} = \frac{H_1}{h_1} + \frac{H'_1}{2h_1}$, the given expression has been derived from a function of x; indeed, we have $(x + h)^2 - x^2 = 2xh + h^2 = H$, and thence get $\frac{H}{h} = 2x + h$, so that the given expression may have been derived from x^2.

Ex. 2.—Let s and d denote the sum and difference of two numbers or quantities of the same kind, to find them.

If x represents the less of the two, it is plain that $x + d$ will represent the greater; consequently, if a and b are particular values of x, we have a, $a + d$, and b, $b + d$ for the corresponding numbers or quantities, and $2a + d$, $2b + d$ will be their corresponding sums.

Hence, considering $2a + d$, $2b + d$ as different particular values of X, we have $2a - 2b = 2(a - b) = H$; and since $a - b = h$ equals the difference of the corresponding values of x, we shall get $\frac{H}{h} = \frac{2(a - b)}{a - b} = 2$. Because s is the value of the sum that corresponds to x, and $2a + d$ that which corresponds to $x = a$, we get $s - (2a + d) = H$, and $x - a = h$; consequently, since $\frac{H}{h} = 2$, we get $2 = \frac{s - 2a - d}{x - a}$, or, if we multiply these equals by $x - a$, we get $2x - 2a = s - 2a - d$, or $x = \frac{s - d}{2}$, and $x + d = \frac{s - d}{2} + \frac{2d}{2} = \frac{s + d}{2}$.

Hence, *the half sum added to the half difference gives the greater of the two; and the half difference subtracted from the half sum gives the less of the two.*

It may be noticed that the method of doing this question is substantially the same as the Rule of Double Position in Arithmetic. See Rutherford's edition of Hutton's Mathematics, p. 79.

Ex. 3.—Given 5, the square root of 25, and 5.1961, the approximate square root of 27, to find the approximate square root of 26.

Here, using 25, 26, and 27 as different values of x, and their square roots as the corresponding values of X, we shall get $\frac{\sqrt{27}-\sqrt{25}}{2}=\frac{\sqrt{26}-\sqrt{25}}{1}$ very nearly, or $\frac{0.1961}{2}=$ $0.0980=\sqrt{26}-5$; consequently, $\sqrt{26}=5.0980$ nearly.

Since $\sqrt{26}=5.099$, we perceive that our process gives the root correctly to two decimal places.

Ex. 4.—Given $\sqrt{289}=17$, and $\sqrt{291}=17.058722$, to find $\sqrt{290}$.

By proceeding as in the preceding question, we get $\sqrt{290}=17.02936$ very nearly; since $\sqrt{290}=17.02938$, it appears that our root does not differ from the true root by more than two units in the fifth decimal place; so that we have found the root more correctly than in the preceding question.

The reason why we have found the root more correctly in this than in the preceding question clearly is, that the numbers 289, 290, 291 are nearer to a ratio of equality than the numbers 25, 26, 27.

Remarks.—Hence we perceive the reason for correcting (numbers or) quantities found nearly from tables, such as logarithmic, astronomical, etc., tables, and then taking proportional parts for the corrections. Thus, having 2.414973 and 2.416641, the logarithms of 260 and 261, by proceeding as in the preceding questions, we get 2.415307 very nearly for the logarithm of 260.2, which agrees with the logarithm as found directly from the tables.

Ex. 5.—To find a number whose square added to three times the number makes 20.

Let x represent the number, then, using 3.2 and 3.3 as particular values of x, we get $(3.2)^2 + 3 \times 3.2 = 19.84$ and $(3.3)^2 + 3 \times 3.3 = 20.79$ for the corresponding particular values of X; consequently, we get $\frac{H}{h} = \frac{9.5}{1}$; and since x must correspond to $X = 20$, we also have $\frac{H}{h} = \frac{0.16}{x - 3.2}$. Equating the values of $\frac{H}{h}$, we get $9.5 = \frac{0.16}{x - 3.2}$, or $x - 3.2 = \frac{0.16}{9.5} = 0.0168$, which gives $x = 3.2168$ nearly; for $(3.2168)^2 + 3 \times 3.2168 = 19.998 +$, which differs but little from 20. If 3.2168 and 3.217 are used as particular values of x, we shall find x more nearly, and so on to any extent.

Remark.—Since this question virtually requires such a value of x as satisfies the equation $x^2 + 3x = 20$, we perceive how to apply our process to the solution of equations.

Ex. 6.—Supposing X to have unlike signs for the particular values a and b of x, and that b is greater than a, we propose to show that there is at least one value of x which is greater than a and less than b, such that the corresponding value of X equals 0.

Representing the values of X which correspond to a and b by A and B, and supposing x to increase from a to b by the successive additions of the very small positive (numbers or) quantities h, i, k, l, etc., and that H, I, K, L, etc., are the very small variations of the corresponding values of X, then we shall clearly have $a + h + i + k +$ etc. $= b$, (1), and $A + H + I + K +$ etc. $= B$, (2).

If we subtract A from the two members of (2), we get $H + I + K + L +$ etc. $= B - A$, (3), in which (since A and B are supposed to have unlike signs) B and $-A$ have like signs; consequently, the first member of the equation equals the absolute sum of $-A$ and B. Hence, if we form the successive sums H, $H + I$, $H + I + K$, $H + I + K + L$, and so on, it is clear that $-A$ must either equal one of these sums, or be greater than one of two that are successive, and less than the remaining one of the two. Hence, by taking h, i, k, etc., sufficiently small, we shall clearly have $H + I + K + \dots + T' = -A$, or $A + H + I + K + \dots T' = 0$, while the sum of the remaining terms of the first member of (3) equals B.

Because $X = A + H + I + \dots + T' = 0$ corresponds to $x = a + h + i + \dots + t'$, which is clearly between a and b, or greater than a and less than b, it is evident that a value of x can be found as proposed.

Remarks.—1st. If $b - a$ is very small, while A and B are very great, it is evident from (3) that H, I, K, etc., can not be very small, unless h, i, k, etc., are extremely small.

2d. If the particular values a and b of x can be found, such that the corresponding values A and B of X have unlike signs, it results (from what has been done) that there is at least one particular value of x between a and b which will reduce X to 0, or which will satisfy the equation $X = 0$. Hence, *if we define the roots of an equation to be those values of the unknown letter* (*or the independent variable*) *which satisfy the equation*, we perceive how we may (often) proceed in order to find them. Thus, if we put $x^2 - 7x + 5$ for X, and use 6 and 7 for a and b, we get -1 and 5 for the corresponding values of A and B; consequently, the equation $x^2 - 7x + 5 = 0$ has a root, which is greater than 6 and less than 7. Also, if we use 0.8 and 1 for a and b, we get 0.04 and -1 for the corresponding values of A and B; consequently, the equation $x^2 - 7x + 5 = 0$ has another root, which lies between 0.8 and 1.

To find these roots more nearly, we remark that $B - A$ is the variation of $x^2 - 7x + 5$, which results from the variation $b - a$ of x, and that to find the root x we have $x^2 - 7x + 5 = 0$ for the corresponding value of $x^2 - 7x + 5$; consequently, $0 - A$ is the variation of $x^2 - 7x + 5$, which results from changing x into $x - a$; so that we shall have $\frac{B - A}{b - a} = \frac{-A}{x - a}$ nearly.

If, in this equation, we put -1, 5, 6, and 7, severally for A, B, a and b, we shall get $6 = \frac{1}{x - 6}$; or we shall have

$x = 6\frac{1}{6} = 6.16 +$ for the value of the first of the preceding roots, correct to one decimal place. If 6.1 and 6.2 are put for a and b in $x^2 - 7x + 5$, the corresponding values of A and B will be found to be $- 0.49$ and 0.04; consequently, if we put $- 0.49$, 0.04, 6.1, and 6.2 for A, B, a, and b, and proceed as before, we shall have $6.192 +$ for a still more correct value of the root; and by continuing the preceding process sufficiently far, the root may be found to any required number of decimal places.

If 0.04, $- 1$, 0.8, and 1 are severally used for A, B, a, and b, we shall in like manner get $0.807 +$ for the second of the preceding roots.

3d. Taking the equation of the n^{th} degree, $x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} + \ldots . + An - 1\ x + An = 0$, it may be noticed, if n is an odd number, that the equation must have at least one real root, which has a sign contrary to An, the absolute term of the equation; and that if n is an even number, and the absolute term is negative, the equation must have at least two real roots, one of which is positive and the other negative.

For if n is odd, and we put $x = 0$, the function $x^n + A_1x^{n-1} +$ etc., is reduced to An; and if we give x a contrary sign to An, and suppose x to be very great, it is clear that the sign of the function $x^n + A_1x^{n-1} +$ etc., will be the same as that of its first term x^n, which will be the same as the sign of x, because n is an odd number; consequently, the sign of the function will be contrary to that of An.

Hence, because the function $x^n + A_1x^{n-1} + A_2x^{n-1} + \ldots . + An$ changes its sign between $x = 0$ and x very great and of a contrary sign to An, it follows, from what has been shown, that the given equation has at least one real root which lies between $x = 0$ and x very great and of a contrary sign to An; consequently, if An is negative, the equation has a positive root, and if An is positive, the equation has a negative root.

Again, if n is an even number and An is negative, then, by putting $x = 0$, the function will be reduced to An, a negative result; but if we give x the same sign as An, or a sign which is contrary to it, and make x very great, the sign of the function will be the same as that of its first term x^n, which will be positive, because n is an even number; consequently, because the function $x^n + A_1x^{n-1} + A_2x^{n-2} + \dots + An$ changes its sign between $x = 0$ and x very great and of the same sign as An, and between $x = 0$ and x very great and of a contrary sign to An, it follows that the given equation has a pair of roots, one of which is positive and the other negative. Thus, the equation $x^3 + 3x^2 - 4 = 0$, has a positive root; the equation $x^5 - 7x + 5 = 0$, has a negative root; and the equation $x^4 + 4x^3 - 5 = 0$, has a pair of roots, one of which is positive and the other negative.

Ex. 7.—Let X be such a function of x that when a, $a + h$, $a + h + k$ are severally put for x, it shall become A′, 0 (or naught), and K; then supposing h and k to be positive and very small, we propose to show that A′ and K will have unlike signs.

Let H express the variation of $X = A'$ which results from changing $x = a$ into $x + h = a + h$; then, by the question, we shall have $A' + H = 0$, or $H = -A'$, so that H and A′ have unlike signs. Again, since K is clearly the variation of $X = A' + H = 0$, which results from changing $x + h = a + h$ into $x + h + k = a + h + k$, and that h and k are very small, it follows (from (4), given at p. 229) that we shall have $\frac{H}{h} = \frac{K}{k}$ very nearly; consequently, since h and k have like signs, H and K must also have like signs, and since H and A′ have unlike signs, K and A′ must also have unlike signs, which is in accordance with what we proposed to show.

Remarks.—1st. If the quotients $\frac{H}{h}$ and $\frac{K}{k}$ are finite, it is evident (because A′ is the value of X just before it is reduced to 0, and K its value just after) that the sign of the function X is changed when it passes through 0, in consequence of the (supposed) increase of its independent variable x. Hence, supposing x to increase from being less than any root of the equation $X = 0$, and to pass through the successive roots (or to become equal to them) as it is increased; then if the quotients $\frac{H}{h}$ and $\frac{K}{k}$ are finite for all the roots, it follows from what has been done, that X and $\frac{H}{h}$ or $\frac{K}{k}$ will have unlike signs just before x passes through each real root, and the same sign just after the passage.

Because X and $\frac{H}{h}$ or $\frac{K}{k}$ have like signs just after the passage of x through a root of the equation $X = 0$, and since they must have unlike signs just before x passes through the next successive root of the equation, it follows that $\frac{H}{h}$ and $\frac{K}{k}$ must change their signs between every two successive real roots of the equation $X = 0$; consequently, the equation $\frac{H}{h} = 0$ or $\frac{K}{k} = 0$ must have a root, which lies between every two successive roots of the equation $X = 0$.

Since (by (2) at p. 228) we have $\frac{H}{h} = A + A_1h + A_2h^2 +$ etc., in which A, A_1, A_2, etc., are independent of h and finite; it is clear when h is very small (or when x is increased by the successive additions of the very small numbers or quantities h, k, etc., as is here supposed), that the sign of $\frac{H}{h}$ must be the same as that of A, *which is called the first derived function* of X; consequently, since $\frac{H}{h}$ changes its sign with A, it follows that one root of the equation $A = 0$ must lie between every two successive roots of the equation $X = 0$.

If a and b are two particular values of x, and A and B the corresponding values of X, then if A and B have unlike signs, it is clear, from what has been shown, that an odd number of the roots of the equation $X = 0$ must lie between a and b; but if A and B have like signs, then an even number of the roots or no (real) roots of the equation $X = 0$ can lie between a and b.

Resuming the equation of the n^{th} degree, $x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + An = 0$, from p. 234; then, since when n is an odd number, and An negative, the equation must have one positive root, it follows, if it has any more positive roots, that it must have an even number of them; and since the function $x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + An$ has the same sign when $x = 0$, that it has when we put $-x$

for x, and make x unlimitedly great, it follows that the equation has either an even number of negative roots or no negative roots.

If n is an odd number and An positive, it may be shown in a similar way that the equation must have an odd number of negative roots, and that it has either an even number of positive roots or no such roots.

If n is an even number and An negative, since it has been shown that the equation must have a positive and negative root, it follows that the equation may have any odd number of positive or negative roots, according to the nature of the case.

If n is an even number and An positive, then, since the sign of the function $x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + An$ is the same when $x = 0$ that it is when x is unlimitedly great, it follows that the equation has either an even number of positive roots or no such roots; and because the sign of the function, when $-x$ is put for x and x is unlimitedly great, is the same as the sign of An, it follows, as before, that the equation has either an even number of negative roots or no negative roots.

2d. If $\frac{H}{h}$ or $\frac{K}{k}$ is an infinitesimal, or reduced to 0, when x passes through a root of the equation $X = 0$, then the equation must have equal roots, for otherwise it is clear, from what has been done, that $\frac{H}{h}$ or $\frac{K}{k}$ can not be reduced to 0.

Because X and $\frac{H}{h}$ or $\frac{K}{k}$ are reduced to 0 by the same value of x, it is clear that the equations $X = 0$ and $A = 0$ must exist at the same time, and that any equal roots must enter once oftener into the first of these equations than into the second.

Hence, if we find the greatest common divisor of X and A, and put it equal to 0, and find the roots of the resulting equation, they will clearly be the equal roots of the equation $X = 0$; and by increasing the number of equal roots thus found by one, we shall clearly get the number of times that each equal root enters into the equation $X = 0$.

Thus, if we take the equation $x^3 - x^2 - 8x + 12 = 0$, and change x into $x + h$, we shall get $(x + h)^3 - (x + h)^2 - 8$

$(x+h)+12 = x^3 - x^2 - 8x + 12 + (3x^2 - 2x - 8)h + (3x-1)h^2 + h^3$; consequently, $3x^2 - 2x - 8$, the coefficient of the simple power of h, is clearly the value of A, the first derived function of the function $x^3 - x^2 - 8x + 12$. Since $x^3 - x^2 - 8x + 12 = x^3 - 2x^2 + x^2 - 8x + 12 = x^2(x-2) + x(x-2) - 6(x-2) = (x^2 + x - 6)(x-2)$, and $3x^2 - 2x - 8 = 3x(x-2) + 4(x-2) = (3x+4)(x-2)$, it is plain that $x^3 - x^2 - 8x + 12$ and $3x^2 - 2x - 8$ have $x - 2$ for a common factor; consequently, they will both be reduced to 0 by putting $x - 2$ equal to 0 or $x = 2$, so that $x = 2$ is a common root of the equations $x^3 - x^2 - 8x + 12 = 0$ and $3x^2 - 2x - 8 = 0$.

Since the root 2 is contained once more in the equation $x^3 - x^2 - 8x + 12 = 0$ than in $3x^2 - 2x - 8 = 0$, it is plain that $x^3 - x^2 - 8x + 12$ must be divisible by $(x-2)^2 = x^2 - 4x + 4$; performing the division, we get $x + 3$ for the quotient, so that we shall have $x^3 - x^2 - 8x + 12 = (x-2)(x-2)(x+3) = 0$, which, being satisfied by putting $x = 2$, $x = 2$, or $x = -3$, it follows that the equation $x^3 - x^2 - 8x + 12 = 0$ has 2, 2, and -3 for its roots.

The equal roots of any equation may be found in a similar way, and then the remaining roots may be found as in the example.

3d. Resuming the equation $A' + H = 0$, whose root is expressed by $a + h$, which is such that the value, a, of x reduces X to the corresponding very small quantity A′, and that $x = a + h$ reduces X to 0; then it is clear that h must be very small.

If we suppose that $\frac{H}{h}$ or $\frac{K}{k}$ are finite when $a + h$ is put for x in them, then, since $\frac{H}{h} = A + A_1h + A_2h^2 +$ etc., it is clear that A the first derived function of X will be finite; consequently, since $H = Ah + A_1h^2 + A_2h^3 +$ etc., the equation $A' + H = 0$ will be reduced to $A' + Ah + A_1h^2 +$ etc. $= 0$, (1), in which A′ is supposed to be very small, while A is not supposed to be very small.

If A_1, A_2, etc., are not very great in comparison to A, then, on account of the supposed minuteness of h, the terms A_1h^2, A_2h^3, etc., of (1) may be rejected, and (1) will be reduced to $A' + Ah = 0$ very nearly, or $Ah = -A'$; and dividing by A, we get $h = -\frac{A'}{A}$, (2). Hence, we may suppose that, in order to find one of the unequal roots $a + h$ of $X = 0$, a must be so assumed that h, as found from (2), shall be very small; then, if we add the value of h thus found to a according to their signs, and represent the sum by a, we shall have a more correct value of a; then, using the corrected value of a, we may from (2) get a new value of h, and thence obtain a still more correct value of a; and so on, to any extent; where it may be noticed, that this process is generally called *Newton's Method of Successive Substitutions.*

If we change h in (1) into $h + k$, it will become $A' + A(h + k) + A_1(h + k)^2 + A_2(h + k)^3 + \text{etc.} = 0$, or, expanding the powers of $h + k$, and retaining only those terms which involve the simple power of k, on account of its supposed minuteness relatively to h, we shall get $(A + 2A_1h + 3A_2h^2 + \text{etc.}) \times k = -(A' + Ah + A_1h^2 + A_2h^3 + \text{etc.})$, very nearly; consequently, dividing by the coefficient of k, we get

$$k = -\frac{A' + Ah + A_1h^2 + \text{etc.}}{A + 2A_1h + 3A_2h^2 + \text{etc.}} \text{ nearly, (3).}$$

Hence, having found h from (2) or in any other way, if we substitute its value for h in (3), and obtain a value of k, which is much less (numerically) than that of h, the approximation to the root $(a + h)$ will clearly be correct; and if we take the sum of h and k (added according to their signs) for a new value of h, we may get a new value of k from (3), and so on, to any extent.

If the equation $X = 0$ has no equal roots, yet if A is very small at the same time that $X = A'$ is very small, it is clear that the equations must have roots which are nearly equal to each other; and it is clear that in cases of this kind we can often find h more readily and correctly from (3) than from (2).

To illustrate what has been done, take the following

EXAMPLES.

Ex. 1.—To find the roots of the equation $x^2 - 7x + 5 = 0$.

Putting $x + h$ for x in the equation, it becomes $(x + h)^2$

$-7(x+h)+5 = x^2 - 7x + 5 + (2x-7)h + h^2 = 0$, or omitting h^2 on account of its supposed minuteness, we shall have $(2x-7)h = -(x^2 - 7x + 5)$ nearly, which divided by $2x-7$ gives $h = -\frac{x^2-7x+5}{2x-7}$, which corresponds to (2). Putting 1 for x in the equation, we get $h = -0.2$, and of course $1 - 0.2 = 0.8$ is a more correct value of x; consequently, putting 0.8 for x in the equation, we get $h = -\frac{0.04}{-5.4} = \frac{0.4}{54} = 0.0074$, which added to 0.08 gives 0.8074 for one of the roots of the equation; which is correct in all its figures.

To get the other root of the equation, we put 6 for x in the expression for h, and thence get $h = -\frac{36-42+5}{12-7} = 0.2$; consequently, we have 6.2 for a more correct value of x.

Substituting 6.2 for x in the value of h, we shall get $h = -\frac{0.04}{5.4} = -0.0074$; consequently, we have $6.2 - 0.0074 = 6.1926$ for the remaining value of x, which is correct to four decimal places.

Ex. 2.—To find the roots of $x^3 - 7x + 7 = 0$.

Putting $x = a + h$, the equation becomes $(a+h)^3 - 7(a+h) + 7 = a^3 - 7a + 7 + (3a^2 - 7)h + 3ah^2 + h^3 = 0$, and changing h into $h + k$, and retaining only the simple powers of k, the equation becomes $a^3 - 7a + 7 + (3a^2 - 7)h + 3ah^2 + h^3 + (3a^2 - 7 + 6ah + 3h^2) \times k = 0$ very nearly, or $(3a^2 - 7 + 6ah + 3h^2) \times k = -(a^3 - 7a + 7 + (3a^2 - 7)h + 3ah^2 + h^3)$ nearly, or dividing by the coefficient of k, we shall have $k = -\frac{a^3 - 7a + 7 + (3a^2 - 7)h + 3ah^2 + h^3}{3a^2 - 7 + 6ah + 3h^2}$ nearly, which corresponds to (3), A′ being represented by $a^3 - 7a + 7$, A by $3a^2 - 7$, A_1 by $3a$, and A_2 by 1.

We will now ascertain whether $a^3 - 7a + 7$ and $3a^3 - 7$ (which correspond to A′ and A_1) can simultaneously become very small, or whether the given equation has roots which are nearly equal to each other. To the end in view, we put $3a^2 - 7$ equal to 0, and thence get $3a^2 - 7 = 0$ or $a^2 = \frac{7}{3} = 2.333$; and extracting the square roots of these equals, we get $a = \pm 1.5 = \pm \frac{3}{2}$ nearly, so that we have $a = \frac{3}{2}$ or $a = -\frac{3}{2}$ nearly.

Putting $\frac{3}{2}$ for a in $a^3 - 7a + 7$ it becomes $\frac{27}{8} - \frac{21}{2} + 7 = \frac{27}{8} - \frac{84}{8} + \frac{56}{8} = -\frac{1}{8}$, which being small, the given equation has roots which are nearly equal to each other, and which do not differ much from $\frac{3}{2}$.

Hence, putting $\frac{3}{2}$ for a in the expression for k, it will become

$$k = -\frac{-\frac{1}{8} - \frac{h}{4} + \frac{9}{2}h^2 + h^3}{-\frac{1}{4} + 9h + 3h^2} = -\frac{-1 - 2h + 36h^2 + 8h^3}{-2 + 72h + 24h^2};$$

in which we must assume h such that the numerical value of k shall be small in comparison to that of h.

To find the nearly equal roots; we observe that since h must be small, and as the coefficient of the simple power of h is small in comparison to that of h^2, it will clearly answer our purpose to assume $-1 + 36h^2 = 0$, which gives $h^2 = \frac{1}{36}$, or, extracting the square root, we get $h = \frac{1}{6}$ or $h = -\frac{1}{6}$; consequently, the equation is reduced to $k = -\frac{-2h + 8h^3}{-2 + 72h + 24h^2} = -\frac{-1h + 4h^3}{-1 + 36h + 12h^2}$. Substituting $\frac{1}{6}$ for h in this equation, we get $k = \frac{1}{36}$; also, substituting $-\frac{1}{6}$ for h in the same equation, we get $k = \frac{1}{45}$; consequently, we have $\frac{1}{6} + \frac{1}{36} = \frac{7}{36}$ and $-\frac{1}{6} + \frac{1}{45} = -\frac{13}{90}$, which are to be used for the new values of h in the above expression for k. Putting $\frac{7}{36}$ for h, we get $-\frac{181}{75276}$ for the new value of k, and thence $\frac{7}{36} - \frac{181}{75276} = \frac{14456}{75276} = 0.1920 +$, which is correct in all its figures, is the new value of h; substituting this new value of h in the expression for k, we get .00002147 for the new value of k, and adding this to 0.1920 we get 0.19202147, which is correct in all its figures, for the new value of h; consequently, if we add this to $\frac{3}{2} = 1.5$, we shall have 1.69202147 for one of the nearly equal roots of the equation.

To get the remaining one of the nearly equal roots, we must put $-\frac{13}{90} = -0.144$ nearly, in the expression for k, and we shall get 0.000893 nearly for the new value of k, and of course we shall have $-0.144 + 0.000893 = -0.143107$ for the new value of h very nearly. Substituting this value of h in the expression for k, we get $k = 0.00000286$ for the new value of k, and thence $-0.143107 + 0.00000286 = -0.14310414$ is the new value of h; and adding this to $a = 1.5$ we get $1.5 - 0.14310414 = 1.35689586$, which is correct in all its figures, for the sought root.

We will now proceed to find the remaining root of the given equation; thus, because the coefficients of h^2 and h^3 in the numerator of k are large in comparison to the coefficient of h and -1, if we regard h as being somewhat great, we may clearly assume $8h^3 + 36h^2 = 0$ or $h = -\frac{9}{2}$; consequently, we shall have $k = -\frac{-1-2h}{-2+72h+24h^2} = -\frac{1}{20} = -0.05$, which is numerically very small in comparison to h, as it ought to be.

Hence, we shall have $a + h + k = 1.5 - 4.5 - 0.05 = -3.05$ nearly, for an approximate value of the sought root.

We will now proceed to find the remaining figures of the root, after the manner of Horner, as exhibited in the scheme (E), Ex. 4, p. 115.

Thus, writing the coefficients in their order, and supplying that of the deficient term by 0, we have the following work, which we hope will be easily understood from the examples given at the page cited (above), etc.

1 + 0	− 7	+ 7	− 3.048917339
− 3	+ 9	− 6	
− 3	+ 2	+ 1	
− 3	+ 18	− 0.814464	
− 6	+ 20.3616	+ 0.185536	
− 3	.3632	− 0.166382592	
− 9.04	+ 20.7248	+ 0.019153408	
− .04	0.073024	− 0.018791228169	
− 9.08	+ 20.797824	+ 0.000362179831	
− .04	+ 0.073088	− 0.000208874650971	
− 9.128	+ 20.870912	+ 0.000153305180029	
− .008	+ 0.00823041		
− 9.136	+ 20.87914241		
− .008	+ .00823122		
− 9.1449	+ 20.88737363		
− .0009	+ 0.0000914671		
− 9.1458	+ 20.8874650971		
− .0009	.0000914672		
− 9.14671	+ 20.8875565643		
− .00001			
− 9.14672			
− 0.00001			
− 9.14673			

If one or more of the first figures of a root of an equation are known, it is clear from this example that the elegant method of Horner enables us to find the remaining figures with great facility.

The reason for our method of extracting the root is evident from the consideration that it consists in a succession of transformations of the given equation, united in one scheme, by putting down only the coefficients with their proper signs in the successive transformations.

Thus, having found − 3 for the first part of the root, we put $y - 3$ for x in the given equation, and it will become $y^3 - 9y^2 + 20y + 1 = 0$, where the coefficients agree with those in the scheme. To get the next transformation, we observe that, on account of the minuteness of y, we may take $20y + 1 = 0$ for the preceding equation nearly, which gives $y = -\frac{1}{20} = -0.05$ nearly; but as this will be found to be numerically too great, we put $y = -0.04$.

To get the second transformation, we put $y' - 0.04$ for y in the preceding transformed equation, and thence get $y'^3 - 9.12y'^2 + 20.7248y' - 0.814464 + 1 = y'^3 - 9.12y'^2 + 20.7248y' + 0.185536 = 0$, as required. Dividing -0.185536 by 20.7248, we get -0.008 for the quotient; then, putting $y'' - 0.008$ for y' in the preceding transformed equation, we proceed to get the third transformed equation, and so on, until a sufficient number of the figures of the root have been found. It may be noticed, that after four transformations, we have obtained the last four figures of the root, by dividing -0.000153305180029 by 20.8875565643 the coefficient of y'''; and it is clear that we may proceed in a similar way in all cases, after more than half of the number of decimal places in the root have been found.

4th. If a stands for a root of the equation $X = 0$, and $a + h$ for any value of the independent variable x, it is clear from (3) (given at p. 229), that the value of X which corresponds to $a + h$, will be expressed by $X = Ah + A_1h^2 + A_2h^3 +$ etc. Because the right member of this equation is exactly divisible by $h = x - a$, it follows that X must be exactly divisible by $x - a$, or that the quotient $\frac{X}{x-a}$ must be exact. Hence, because a may represent any root of the equation $X = 0$, *it follows that, if we subtract the roots of the equation separately from the independent variable x, the remainders will be factors of* X.

Since the quotient $\frac{X}{x-a}$ is of less dimensions in x by a unit than in X, and that it contains all the roots of the equation $X = 0$ except a, it follows that, having found the root a, of the equation $X = 0$, we may find the remaining roots from the more simple equation $\frac{X}{x-a} = 0$.

Thus, because the equation $x^3 - 3x^2 - x + 3 = 0$ is satisfied by putting 1 for x, we get $\frac{x^3 - 3x^2 - x + 3}{x - 1} = x^2 + 4x$

$+ 3$; and since the equation $x^2 + 4x + 3 = 0$ is satisfied by putting -1 for x, we get $\frac{x^2 + 4x + 3}{x + 1} = x + 3$; and by putting -3 for x, the equation $x + 3 = 0$ is also satisfied; consequently, 1, -1, and -3 are the roots of the equation; and we shall have $x^3 - 3x^2 - x + 3 = (x - 1) \times (x + 1) \times (x + 3)$.

Again, if a is a root of the equations $X = 0$, and $A = 0$, then the above general value of X will become $X = A_1h^2 + A_2h^3 +$ etc.; and because the right member of this equation is exactly divisible by $h^2 = (x - a)^2$, it follows that the quotient $\frac{X}{(x-a)^2}$ must be exact.

Hence, because X is exactly divisible by $(x - a)^2 = (x - a).(x - a)$, the equation $X = 0$ has two roots, whose common value is a.

If a is a root of the equations $X = 0$, $A = 0$, $A_1 = 0$, $A_2 = 0$, etc., it may be shown, in like manner, that the equation $X = 0$ has as many equal roots whose common value is a, as there are successive equations which have a for a common root.

Thus, if X is represented by $x^4 - x^3 - 3x^2 + 5x - 2$, and $1 + h$ is put for x, the function is reduced to $3h^3 + h^4$, which shows that the equation $x^4 - x^3 - 3x^2 + 5x - 2 = 0$ has three equal roots, whose common value is 1; and since -2 is the remaining root of the equation, we get $x^4 - x^3 - 3x^2 + 5x - 2 = (x - 1)^3 \times (x + 2)$.

Similarly, the equation $x^5 + 2x^4 + x^3 = (x + 1)^2 \times x^3 = 0$ has three equal roots, whose common value is 0; and two other equal roots, whose common value is -1.

5th. If we represent an equation of the n^{th} degree in which the coefficients and the absolute term are real positive or negative (numbers or) quantities, by $Ax^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} + \ldots + A_{n-1}x + A_n = 0$; then, if the roots of the equation are all real, there will be n roots, or the number of roots equals the degree of the equation.

For if a, b, c, d, etc., represent the roots, then, since $x - a$, $x - b$, $x - c$, $x - d$, etc. (as has been shown), must be factors of the first member of the equation, it is clear that there must be n such factors, or that our proposition is true; and we shall have $Ax^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} + \ldots +$

$A_{n-1}x + A_n = A \times [(x-a) \times (x-b) \times (x-c) \times (x-d) \times$ etc. to n factors].

Again, if the roots of the equation are not all real; then, if n is an odd number (as has been shown), the equation must have an odd number of real roots, and of course it must have an even number of what are called *impossible or imaginary roots.* Also, if n is an even number; if the equation has real roots, then (as has been shown) there must be an even number of such roots, and of course there must be an even number of impossible or imaginary roots. Hence, *whether the equation is of an odd or even degree, if it has impossible roots, then it must have an even number of such roots.*

Now, if p and q represent real numbers or quantities, we may clearly suppose that impossible or imaginary roots occur *in pairs* of the forms $p + q\sqrt{-1}$ and $p - q\sqrt{-1}$; consequently, it is clear that we shall have $Ax^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n = A \times [(x-a) \times (x-b) \times (x-c) \times$ etc. $\times\ (x - (p + q\sqrt{-1})) \times (x - (p - q\sqrt{-1})) \times (x - (p' + q'\sqrt{-1})) \times (x - (p' - q'\sqrt{-1})) \times$ etc., to n factors]; observing, that a, b, c, etc., are supposed to stand for the real roots, and that p, q, p', q', etc., in the imaginary roots, stand for real numbers or quantities, and that p or p', etc., may be 0, should the nature of the case require it.

To make what has been said more clear, we will add the following

Remarks.—1. If, in $Ax^n + A_1x^{n-1} + A_2x^{n-2} +$ etc., $x + h$ is put for x, it will become $A(x+h)^n + A_1(x+h)^{n-1} + A_2(x+h)^{n-2} + \ldots + A_{n-1}(x+h) + An = Ax^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + An + [nAx^{n-1} + (n-1)A_1x^{n-2} + (n-2)A_2x^{n-3} + \ldots + A_{n-1}] \times h +$ etc., as is evident from the expansions of the powers of $x + h$, according to the ascending powers of h.

Because $nAx^{n-1} + (n-1)A_1x^{n-2} + (n-2)A_2x^{n-3} + \ldots + A^{n-1}$, the coefficient of the simple power of h, is the first derived function of $Ax^n + A_1x^{n-1} + A_2x^{n-2} +$ etc., it is clear that the first derived function can be found by the following

RULE.

Multiply each term of the given function by the index of *the independent variable in the term, and then diminish the index of the variable by unity, or subtract* 1 *from it; then the sum of the terms thus obtained will be the first derived function.*

Thus, the first derived function of $2x^4 - 5x^3 + 6x^2 - 7x + 20$ is $8x^3 - 15x^2 + 12x - 7$; and that of $x^3 - 3x - 1$ is $3x^2 - 3 = 3(x^2 - 1)$; observing that $x^0 = 1$.

2. It follows from what has heretofore been shown, that if the roots of the equation $Ax^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + An = 0$ are all real, the roots of the equation (obtained by putting the first derived function equal to 0) $nAx^{n-1} + (n-1)A_1x^{n-2} + \ldots + A_{n-1} = 0$ must also all be real, because a root of this equation must lie between every two roots of the preceding equation.

Similarly, if we put the first derived function of $nAx^{n-1} + (n-1)A_1x^{n-2} + \ldots + A_{n-1}$ equal to 0, it may be shown that the roots of the equation thus obtained must also all be real, and so on, until we arrive at an equation of the first degree, which will clearly have only one root.

Because a root of the equation $nAx^{n-1} + (n-1)A_1x^{n-2} + \ldots + A_{n-1} = 0$ lies between every two roots of the equation $Ax^n + A_1x^{n-1} + A_2x^{n-2} +$ etc. $= 0$, it follows that this equation has one more root than the preceding equation.

Similarly, the equation $nAx^{n-1} + (n-1)A_1x^{n-2} +$ etc. $= 0$ has one more root than $n(n-1)x^{n-2} + (n-1)(n-2)x^{n-3} +$ etc. $= 0$, and so on, until an equation of the first degree is obtained.

Hence, because the equation $Ax^n + A_1x^{n-1} +$ etc. $= 0$, and the derived equations, make n equations, it follows that $Ax^n + A_1x^{n-1} +$ etc. $= 0$ *must have n roots, or that the number of roots equals the degree of the equation;* and it is clear that the same conclusion is true if the equation has two or more equal roots.

It may be added, if the given equation has two or more equal roots, *that they will enter once oftener into the given equation than into the first derived equation,* as is evident from what has been done.

3. Resuming the equation $Ax^n + A_1x^{n-1} +$ etc. $= 0$; then if the equation is such that two or more of the roots of one or more of the derived equations are impossible, it clearly

follows from what has been done that the (given) equation must have impossible roots, because impossible roots can not lie between real roots.

Thus, if we take the equation $2x^3 + 3x - 8 = 0$, we get $6x^2 + 3 = 0$ for its first derived equation; and since the derived equation gives $x^2 = -\frac{1}{2}$, which is impossible (because the square of any real number or quantity is positive), it follows that the given equation has a pair of impossible roots.

Similarly, if we take the equation $2x^4 - 20x + 19 = 0$, we get $8x^3 - 20 = 0$ for its first derived equation; and since the first derived equation gives $x^3 = 2.5$, which has only one real root, it follows that the given equation has at least one pair of impossible roots.

Again, let the equation $x^4 + 8x^2 + 16x - 440 = 0$ be given to determine the nature of its roots.

Because the absolute term of the equation is negative, it follows, from what has heretofore been proved, that two of the roots must be real, one of them being positive, and the other negative.

To determine the nature of the remaining roots, we take the first derived equation $4x^3 + 16x + 16 = 0$, or its equivalent $x^3 + 4x + 4 = 0$; and then we take the first derived equation of this equation, which gives $3x^2 + 4 = 0$, or $x^2 = -\frac{4}{3}$; which, being an impossible result, it follows that the two remaining roots of the given equation are impossible.

4. If the roots of any one of the equations which are derived from $Ax^n + A_1x^{n-1} + A_2x^{n-2} +$ etc. $= 0$ are real, yet if one of them does not lie between every two (or coincide with one or both) roots of the equation from which the equation is immediately derived, it is clear that the given equation must have imaginary roots.

Thus, the equation $x^3 - 3x - 6 = 0$ has a pair of impossible or imaginary roots.

For by taking its first derived equation, we get $3x^2 - 3 = 0$ or $x^2 - 1 = 0$, whose roots are 1 and -1.

By putting 1 and -1 successively, for x in $x^3 - 3x - 6$, we get -8 and -4 for the results; consequently, since these results have like signs, it follows from what has heretofore been shown, that the given equation has no real root which lies between 1 and -1; consequently, it must have a pair of imaginary roots. It may be added, that the equation has one real root, which lies between 2 and 3.

5. To show how to apply what has been done, to the determination of the real roots of equations, we shall take the following

EXAMPLES.

Ex. 1.—To find the real roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$.

The equation, and the derived equations or their equivalents, give the equations $x^3 - 6x^2 + 11x - 6 = 0$, $3x^2 - 12x + 11 = 0$, $x - 2 = 0$. Because the last of the preceding equations is satisfied by putting 2 for x, it follows that 2 is its root; consequently, if the roots of $3x^2 - 12x + 11 = 0$ are real, 2 must lie between them.

If we use a and b to represent the roots of $3x^2 - 12x + 11 = 0$, we shall have $3x^2 - 12x + 11 = 3(x - a) \times (x - b)$, which shows that a and b must have like signs; consequently, since one of the roots must be greater than 2, it must be positive, and of course the other root must also be positive.

Putting 1.5 for x in $3x^2 - 12x + 11 = 0$, we get the negative result -0.25, and putting 3 for x, we get the positive result 2; consequently, the roots of the equation are real, and 2 lies between them.

Because the less of the roots of the preceding equation is easily shown to lie between 1.4 and 1.5, and the greater to lie between 2.5 and 2.6, if the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ are all real, it is easy to perceive that one of them must lie between 1.4 and 2.6; consequently, putting 2 for x, the equation is satisfied, and 2 is one of its roots; and it is easy to show that 1 and 3 are the remaining roots; consequently, $x^3 - 6x^2 + 11x - 6 = (x - 1) \times (x - 2) \times (x - 3)$.

Ex. 2.—To find the roots of the equation $x^4 - 6x^2 - 8x - 3 = 0$.

Here we have the equations $x^4 - 6x^2 - 8x - 3 = 0$, $x^3 - 3x - 2 = 0$, and $x^2 - 1 = 0$, which is satisfied by putting 1 or -1 for x.

Since -1 is a root of $x^3 - 3x - 2 = 0$, it is clear that the given equation ought to have three roots, whose common value is -1; and as $\frac{x^4 - 6x^2 - 8x - 3}{(x + 1)^3} = x - 3$, it results that 3 is the remaining root.

Ex. 3.—To find the roots of the equation $x^4 + 12x^3 + 54x^2 + 108x + 81 = 0$.

Since the derived equations are equivalent to $x^3 + 9x^2 + 27x + 27 = 0$, $x^2 + 6x + 9 = 0$, and $x + 3 = 0$, it is clear that the given equation is the same as $(x + 3)^4$, which has four equal roots, whose common value is -3.

Ex. 4.—To find the roots of the equation $x^3 + 11x^2 - 102x + 181 = 0$.

Because the absolute term is positive, it is clear that one of the roots must be negative. Taking the derived equations, we have $3x^2 + 22x - 102 = 0$, and $3x + 11 = 0$ or $x = -\frac{11}{3}$; consequently, $-\frac{11}{3}$ must lie between the roots of $3x^2 + 22x - 102 = 0$, one of which is positive and the other negative.

It is easy to show that one of the roots of $3x^2 + 22x - 102 = 0$ lies between -11 and -10, and that the other lies between 3.22 and 3.23; since the substitutions of 3.22 and 3.23 for x in the given equation give results with opposite signs, it follows that the equation has one root which lies between 3.22 and 3.23; consequently, the equation must have a pair of roots which are nearly equal to each other. Hence, substituting 3.21 for x in the given equation, we find that the result has a contrary sign to that obtained from the substitution of 3.22; consequently, the equation has another root, which lies between 3.21 and 3.22.

Since 3.21 and 3.22 are the first figures of two of the roots, these roots will easily (by Horner's method) be found to be 3.21312, and 3.22952, very nearly.

It follows, from what has heretofore been shown, that 11, the coefficient of x^2 in the given equation, is the negative of the sum of the roots; consequently, $-11 - 3.21312 - 3.22952 = -17.44264$ is the remaining root nearly.

Ex. 5.—To find the roots of $5x^3 - 6x + 2 = 0$.

Here the first derived equation being equivalent to $x^2 =$

0.4, we get 0.63 + and $-$ 0.63 $-$ for its roots; consequently, one of the roots of the equation must lie between these numbers; hence, we easily get 0.3 for the first figures of this root. And because one of the remaining roots must be less than $-$ 0.63 $-$, and the other greater than 0.63 +, we soon find $-$ 1.2 and 0.8 for the first figures of these roots. Hence, the roots can easily be found by Horner's method.

Another solution. Substitute $\frac{1}{y}$ for x in the given equation, and it will be readily reduced to $2y^3 - 6y^2 + 5 = 0$; and as the first derived equation gives $y^2 - 2y = 0$, whose roots are 0 and 2, it follows that one of the roots of the equation in y must be less than 0 or negative; another root must lie between 0 and 2, and the remaining root must be greater than 2. The first figures of the root which lies between 0 and 2 are soon found to be 1.2; and $-$ 0.8, 2.6 are the first figures of the remaining roots. Hence, the roots are easily found by Horner's method, or by any of the common methods of approximation.

Because $x = \frac{1}{y}$, it follows that if we divide 1 by the roots, or take the *reciprocals* of the roots, we shall get the roots or values of x which satisfy the given equation.

Remark.—It will be noticed that the roots of the equation in x, which are numerically less than 1, are in the equation in y represented by roots which are numerically greater than 1. It will also be observed that the positive root which is without the limits 0.63 + and $-$ 0.63 $-$ in the equation in x, is represented by a root in the equation in y, which is within the limits 2 and 0.

It is hence clear that it will often be useful to proceed in like manner to find the reciprocals of the roots of equations.

Ex. 6.—Supposing the roots of $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$ to be real, it is required to find how many of them lie between 0 and 3.

Here the first, second, and third derived equations become $x^3 - 6x^2 + 7x + 1 = 0$, $3x^2 - 12x + 7 = 0$, and $x - 2 = 0$.

If we put 0 and 3 for x in the given equation, the results are $-$ 8 and $-$ 5; which being affected by the same sign, it follows that an even number of the roots must lie between 0 and 3.

And putting 0 and 3 for x in the first derived equation, the results are 1 and -5; which having opposite signs, it follows that an odd number of the roots of the (first derived) equation lie between 0 and 3.

Also, putting 0 and 3 for x in the second derived equation, the results are 7 and -2; which having opposite signs, it follows that only one root of the (second derived) equation lies between 0 and 3.

Hence, because the root of the second derived equation lies between two roots of the first derived equation, it is clear that there is only one root of this equation which lies between 0 and 3; consequently, since this root lies between two roots of the given equation, it follows that the given equation has only two roots, which lie between 0 and 3. Indeed, the roots of the given equation are -0.732050, 0.763932, 2.732050, and 5.236068, nearly; two of which lie between 0 and 3.

Hence we perceive how we may proceed in order to find the number of real roots of a given equation, which may lie between two given numbers.

(**49.**) We will now proceed to complete the indicated expansion of X' (see (3), p. 229), and to show its use in the development of functions.

1. Resuming the equation, we have $X' = X + Ah + A_1h^2 + A_2h^3 + A_3h^4 + \text{etc.}$, (1); in which X, A, A_1, A_2, etc., are supposed to be functions of x, and X' is the value of X which results from changing x into $x + h$, supposing x and h to be indeterminates. If the value of X which corresponds to $x + h + k$ is represented by X'', it is clear, from (1), that we shall have $X'' = X + A(h + k) + A_1(h + k)^2 + A_2(h + k)^3 + A_3(h + k)^4 + \text{etc.}$, or developing the powers of $h + k$ according to the ascending powers of k, we shall get $X'' = X + A(h + k) + A_1(h^2 + 2hk + \text{etc.}) + A_2(h^3 + 3h^2k + \text{etc.}) + \text{etc.}$, (2).

Again, since X, A, A_1, A_2, A_3, etc., are functions of x, if we change x into $x + k$ they become $X + Ak + A_1k^2 + A_2k^3 + \text{etc.}$, $A + A'k + A''k^2 + \text{etc.}$, $A_1 + A'_1k + A''_1k^2 + \text{etc.}$, $A_2 + A'_2k + A''_2k^2 + \text{etc.}$, and so on; in which $X + Ak +$

etc., is clearly deduced from the right member of (1) by changing h into k, and A' is the first derived function of A, A'_1 that of A_1, and so on; hence, if we change x in (1) into $x + k$, it is clear that X' will be changed into X'', since it is the value of X which corresponds to $x + h + k$; consequently, substituting the values of X, A, A_1, etc., resulting from changing x into $x + k$, we shall have $X'' = X + Ak + A_1k^2 + \text{etc.} + (A + A'k + \text{etc.})h + (A_1 + A'_1k + \text{etc.})h^2 + (A_2 + A'_2k + \text{etc.})h^3 + \text{etc.} = X + A(h + k) + A_1(h^2 + k^2) + A_2(h^3 + k^3) + \text{etc.} + (A'h + A'_1h^2 + A'_2h^3 + A'_3h^4 + \text{etc.})k + (A''h + A''_1h^2 + A''_2h^3 + \text{etc.})k^2 + (A'''h + A'''_1h^2 + \text{etc.})k^3 + \text{etc.}$, (3).

Because the first members of (2) and (3) are equal, it is clear that their second members must also be equal, and it is evident that they must be identical, so as to leave h and k arbitrary; and it is clear that they will be identical, if we put the coefficient of the simple power of k in the second member of (2) equal to that of k in the second member of (3); consequently, we must have $A + 2A_1h + 3A_2h^2 + 4A_3h^3 + \text{etc.} = A + A'h + A'_1h^2 + A'_2h^3 + \text{etc.}$, (4).

Since h must be arbitrary, it is clear that the coefficient of any power of h in the first member of (4) must equal the coefficient of the same power of h in the second member; consequently, we must have the equations $A = A$, $2A_1 = A'$, $3A_2 = A'_1$, $4A_3 = A'_2$, $5A_4 = A'_3$, and so on (5).

Because A, the coefficient of the simple power of h in the right member of (1), is called the first derived function of X, we may express it by X_1; and because A' is the first derived function of A, we may call it the second derived function of X, and denote it by X_2; and in like manner we may call the first derived function of A' the third derived function of X, and represent it by X_3; and so on.

Hence, because $A' = X_2$, the second of (5) becomes $2A_1 = X_2$, which gives $A_1 = \frac{X_2}{2} = \frac{X_2}{1 . 2}$; and thence we have $A'_1 = \frac{X_3}{1 . 2}$, and the third of (5) is reduced to $3A_2 = \frac{X_3}{1 . 2}$, or we get $A_2 = \frac{X_3}{1 . 2 . 3}$; and in a similar way we get from the fourth of (5) $A_3 = \frac{X_4}{1 . 2 . 3 . 4}$; and so on. Hence, by substituting the values of A, A_1, A_2, A_3, etc., in the right member of (1), it will be reduced to $X' = X + X_1 \,.\, h + \frac{X_2}{1\,.\,2} \,.\, h^2 + \frac{X_3}{1\,.\,2\,.\,3} \,.\, h^3 + \frac{X_4}{1\,.\,2\,.\,3\,.\,4} \,.\, h^4 +$ etc. (a), whose law of continuation is manifest; (a), which is usually called *Taylor's theorem*, is the development which we proposed to find. It is clear that X_1 can be obtained from X by changing x into $x + h$, and putting the coefficient of the simple power of h in the expansion equal to X_1; and if we change x into $x + h$ in X_1, the coefficient of the simple power of h in the expansion will clearly equal X_2; and in a similar way, if we change x in X_2 into $x + h$, the coefficient of the simple power of h in the expansion will equal X_3; and so on.

To show the use of (a), take the following

EXAMPLES.

Ex. 1.—To find the expansion of x^4 when x is changed into $x + h$, or to develop $(x + h)^4$ according to the ascending powers of h.

Here X is represented by x^4, and allowing $x^4 + 4x^3h$ to be the first two terms of the expansion of $(x + h)^4$, we shall have $X_1 = 4x^3$, and allowing the first two terms of the expansion of $(x + h)^3$ to be $x^3 + 3x^2h$, we shall have $4(x + h)^3 = 4x^3 + 12x^2h +$; consequently, $12x^2$, the coefficient of the simple power of h, will equal X_2; and from $12(x + h)^2 = 12x^2 + 24xh +$ we get $X_3 = 24x$; and from $24(x + h) = 24x + 24h$ we get $X_4 = 24$; and since X_4 does not contain x, it is clear that X_5, X_6, etc., are (each) equal to 0.

Hence, putting $(x + h)^4$, x^4, $4x^3$, $12x^2$, $24x$, and 24, severally, for X', X, X_1, X_2, X_3, X_4, in (a), we shall get $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$, as required.

Ex. 2.—To develop $3x^3 - 5x^2 + 2x - 7$, when x is changed into $x + h$.

Since $9x^2 - 10x + 2$, $18x - 10$, 18, 0, 0, etc., are easily seen to be the first, second, third, etc., derived functions, we get (from (a)) $3(x + h)^3 - 5(x + h)^2 + 2(x + h) - 7 = 3x^3 - 5x^2 + 2x - 7 + (9x^2 - 10x + 2)h + (9x - 5)h^2 + 3h^3$, as required.

Ex. 3.—To develop $x^4 - 2x^3$, when x is changed into $x + h$.

Here the first, second, etc., derived functions are $4x^3 - 6x^2$, $12x^2 - 12x$, $24x - 12$, 24, 0, 0, etc.; consequently, we get $(x + h)^4 - 2(x + h)^3 = x^4 - 2x^3 + (4x^3 - 6x^2)h + (6x^2 - 6x)h^2 + (4x - 2)h^3 + h^4$, as required.

Ex. 4.—Supposing the first derived function of x^n to be nx^{n-1} for any value of n, it is required to develop $(x + h)^n$ according to the ascending powers of h.

Here the first, second, etc., derived functions are nx^{n-1}, $n(n-1)x^{n-2}$, $n(n-1)(n-2)x^{n-3}$, $n(n-1)(n-2)(n-3)x^{n-4}$, and so on; consequently, from (a) we get $(x + h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{1\,.\,2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}h^3 +$ etc., as required.

2. According to what has heretofore been shown, we may apply (a) to expand any function of $x + h$, according to the ascending powers of h, when particular values are assigned to x and h, provided X, X_1, X_2, etc., do not any of them become infinite; should any of these quantities, as X_3, become infinite, then the expansion will be correct no further than to the term which contains X_3.

If for x, in (a), we put the particular value 0, while h is arbitrary; then if we represent the values of X, X_1, X_2, etc., which correspond to $x = 0$ by (X), (X_1), (X_2), etc., (a) will become $X' = (X) + (X_1)\,.\,h + \frac{(X_2)}{1.2}\,.\,h^2 +$ etc.; or since we may clearly (here) use x for h, we shall get $X' = (X) + (X_1)\,.\,x + \frac{(X_2)}{1\,.\,2}\,.\,x^2 + \frac{(X_3)}{1\,.\,2\,.\,3}\,.\,x^3 +$ etc., (b), which is called *MacLaurin's Theorem;* where it will be noticed that X' is a function of x and known quantities, and that (X) is the value of X' which corresponds to $x = 0$; (X_1), (X_2), etc., being the values of the first, second, etc., derived functions which correspond to $x = 0$.

MacLaurin's Theorem is often very useful in the expansion of functions. Thus, if we wish to expand $(b + x)^m$ according to the ascending powers of x, we have, as in Taylor's Theorem, $(b + x + h)^m = (b + x)^m + m(b + x)^{m-1}h +$

$\frac{m(m-1)}{1\,.\,2}(b+x)^{m-2}h^2+$ etc.; consequently, if in this we put $x=0$, and change h into x, it will become $(b+x)^m=b^m+mb^{m-1}x+\frac{m(m-1)}{1\,.\,2}b^{m-2}x^2+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}b^{m-3}x^3+$ etc., as required.

It may be added, that if any function of x, when expanded by MacLaurin's Theorem, contains infinite terms, it will clearly show that the Theorem is not applicable to the expansion.

3. If we subtract X from the two members of (a), we shall get $X'-X=X_1\,.\,h+\frac{X_2}{1.2}\,.\,h^2+\frac{X_3}{1.2.3}\,.\,h^3+$ etc., (d), for the variation of X, arising from changing x into $x+h$ in X, and then subtracting X from the result.

If we represent $X'-X$ by writing the Greek letter Δ (delta) before or to the left of X, we shall have $X'-X=\Delta X$; and in a similar manner, if we represent $x+h$ by x', we may express $h=x'-x$ by Δx; consequently, if we use ΔX and Δx for $X'-X$ and h, (d) may be written in the form $\Delta X=X_1\,.\,\Delta x+\frac{X_2}{1.2}\,.\,(\Delta x)^2+\frac{X_3}{1.2.3}\,.\,(\Delta x)^3+$ etc., (e); which is (sometimes) called an equation of finite differences, because ΔX and Δx are supposed to have definite values, and Δ is said to be the *characteristic of finite differences.*

Again, if we represent the first term of the right member of (d) by writing d before or to the left of X, we shall have the equation $dX=X_1\,.\,h$, (f), which is called the *differential* of X, since it is only a part of the difference in the right member of (d).

If, for uniformity of notation, we represent h in (f) by dx, and then divide both members of the equation by $h=dx$, we shall get $\frac{dX}{dx}=X_1$, (g); in which $\frac{dX}{dx}$ is called the *first differential coefficient,* since dx multiplied by it gives dX, the differential of X; and it is clear, from (g), that the value of the first differential coefficient is expressed by the first derived function of X.

If we divide the two members of (d) by h, it will be reduced to $\frac{X'-X}{h} = X_1 + \frac{X_2}{1.2}h + \frac{X_3}{1.2.3}.h^2 + \text{etc.}, \quad (h)$; and if in this we put $h = 0$, the right member of the equation will be reduced to X_1, the first derived function of X, which is the same as the value of $\frac{dX}{dx}$ in (g); while the first member of (h) is reduced to the form $\frac{0}{0}$, since the factor which is common to the numerator and denominator is put equal to 0.

Hence, *if we actually divide the variation or difference* $X' - X$ *of* X *by* h, *the corresponding variation or difference of the independent variable* x, *and put* $h = 0$ *in the quotient, the result will be the first derived function of* X, *or it will be the value of the first differential coefficient* $\frac{dX}{dx}$.

Thus, we get the first differential coefficient of x^2 by changing x into $x + h$, which gives $(x + h)^2 = x^2 + 2xh + h^2$ or $(x + h)^2 - x^2 = 2xh + h^2$; and by indicating the division by h, in the first member of this equation, and performing it in the second member, we get $\frac{(x + h)^2 - x^2}{h} = 2x + h$; consequently, putting $h = 0$ in the quotient, we have $2x$ for the first derived function, or $\frac{dx^2}{dx} = 2x$. In a similar way from $3x^3$ we get $\frac{d3x^3}{dx} = \frac{3dx^3}{dx} = 9x^2$.

Because X_2 is the first derived function of X_1, it follows, from what has been done, that we shall have $X_2 = \frac{dX_1}{dx}$, or, since $X_1 = \frac{dX}{dx}$, we shall have $X_2 = \frac{d\left(\frac{dX}{dx}\right)}{dx}$. Because the quotient $\frac{dX}{dx}$ is actually independent of dx, we may clearly express its differential coefficient by $\frac{d^2X}{(dx)^2}$, and call it the second differential coefficient of X; and in the same way we may express the third differential coefficient by $\frac{d^3X}{(dx)^3}$, and so on. Hence, we shall have $X_2 = \frac{d^2X}{(dx)^2}$, $X_3 = \frac{d^3X}{(dx)^3}$, and so on; consequently, by substituting the differential coefficients for X_1, X_2, X_3, etc., in (a), it will be reduced to $X' = X + \frac{dX}{dx} . h + \frac{d^2X}{(dx)^2} \cdot \frac{h^2}{1 . 2} + \frac{d^3X}{(dx)^3} \cdot \frac{h^3}{1 . 2 . 3}$ + etc., (a').

Because X' is supposed to be the same function of $x + h$ that X is of x, if we represent X by $f(x) = a$ function of x, we must express X' by $f(x + h) = a$ function of $x + h$. Hence, putting $f(x)$ and $f(x + h)$ for X and X', (a') becomes $f(x + h) = f(x) + \frac{df(x)}{dx} . h + \frac{d^2f(x)}{(dx)^2} \cdot \frac{h^2}{1 . 2}$ + etc., (a''); which agrees with the form in which Taylor's Theorem is usually given.

(**50.**) Still supposing X to be a function of x, and that X, X'_1, X'_2, X'_3, ... $X'n$ are the values of X, which correspond to x, $x + h$, $x + 2h$, $x + 3h$, ... $x + nh$, severally; we propose to find $X'n$.

1. We shall use ΔX, $\Delta X'_1$, $\Delta X'_2$, etc., to represent the variations of X, X'_1, X'_2, etc., which result from changing x into $x + h$; and in a similar way we shall express the variations of ΔX, $\Delta X'_1$, $\Delta X'_2$, etc. (which result from changing x into $x + h$), by Δ^2X, $\Delta^2X'_1$, $\Delta^2X'_2$, etc., and the variations of these will be expressed by Δ^3X, $\Delta^3X'_1$, $\Delta^3X'_2$, etc., and so on.

Hence, if we change x in X into $x + h$, we shall get $X + \Delta X = X'_1$, since $X + \Delta X$ and X'_1 are different expressions for the value of X, which corresponds to $x + h$.

And if in $X + \Delta X = X'_1$ we change x into $x + h$, we shall get $X + 2\Delta X + \Delta^2X = X'_1 + \Delta X'_1 = X'_2$, since X is changed into $X + \Delta X$ and ΔX into $\Delta X + \Delta^2X$, and that X'_2 corresponds to $x + 2h$.

Similarly, from $X + 2\Delta X + \Delta^2X = X'_2$, we shall get $X + 3\Delta X + 3\Delta^2X + \Delta^3X = X'_2 + \Delta X'_2 = X'_3$, and so on to any extent.

Hence, it is clear that we shall have $X'n = X + n\Delta X + \frac{n(n-1)}{1 . 2}\Delta^2X + \frac{n(n-1)(n-2)}{1 . 2 . 3}\Delta^3X + \ldots . + n\Delta^{n-1}X + \Delta^nX$, ($a$); for it is plain, from the law of the coefficients in the successive changed values of X, that the coefficients of

ΔX, $\Delta^2 X$, $\Delta^3 X$, etc., must be the same as the coefficients of the second, third, fourth, etc., terms of a binomial when raised to the n^{th} power; while the numbers to be applied to Δ in the successive terms are the same as the indices of the second term of the binomial in the expansion.

It is clear that in order to get X'_n from (*a*), n, X, ΔX, $\Delta^2 X$, etc., must be known.

2. Supposing n, X, X'_1, $X'_2 \ldots . X'_n$ to be given, we will now show how to find ΔX, $\Delta^2 X$, etc., which *are called the first terms of the first, second, etc., orders of differences.*

From the equations $X + \Delta X = X'_1$, $X'_1 + \Delta X'_1 = X'_2$, $X'_2 + \Delta X'_2 = X'_3$, and so on, we easily get $\Delta X = X'_1 - X$ $\Delta X'_1 = X'_2 - X'_1$, $\Delta X'_2 = X'_3 - X'_2$, and so on.

And from $\Delta^2 X = \Delta X'_1 - \Delta X$, $\Delta^2 X'_1 = \Delta X'_2 - \Delta X'_1$, and so on, we get $\Delta^2 X = X'_2 - X'_1 - (X'_1 - X) = X'_2 - 2X'_1 + X$ $\Delta^2 X'_1 = X'_3 - 2X'_2 + X'_1$, $\Delta^2 X'_2 = X'_4 - 2X'_3 + X'_2$, and so on; similarly, we get $\Delta^3 X = X'_3 - 3X'_2 + 3X'_1 - X$, $\Delta^3 X'_1 = X'_4 - 3X'_3 + 3X'_2 - X_1$, etc.; the law of continuation of the successive differences being evident.

Hence, it is clear that the first difference of the n^{th} order will be expressed by $\Delta^n X = X'_n - nX'_{n-1} + \frac{n(n-1)}{1\,.\,2}X'_{n-2} - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}X'_{n-3} + \text{etc.}$, (*b*); observing that the coefficients are the same and have the same signs as in the corresponding terms of $(X' - 1)^n$ when expanded according to the descending powers of X', while the numbers n, $n-1$, $n-2$, etc., joined to X' (below) are the same as the exponents of X' in the corresponding terms of the expansion; noticing that the last term is X if n is an even number, and $-X$ when n is an odd number. If we write the right member of (*b*) in a contrary order, it will evidently become $\Delta^n X = \pm \left[X - nX'_1 + \frac{n(n-1)}{1\,.\,2}X'_2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}X'_3 + \text{etc.}\right]$, (*c*); observing that $+$ must be used for $\pm$ when n is an even number, and that $-$ must be used for $\pm$ when n is an odd number; and it is clear that the coefficients and their signs are the same as those which correspond to them in the expansion of $(-1 + X')^n$, according to the ascending powers of X', while the numbers 1, 2, 3, 4, etc., which are joined to X (below) are the same as the exponents of X′ in the corresponding terms of the expansion.

3. To show the use of the formulæ (*a*), (*b*), and (*c*), take the following

EXAMPLES.

Ex. 1.—To find the $(n + 1)^{th}$ term of the series 1, 3, 6, 10, 15, 21, etc.

Here, X, X'_1, X'_2, X'_3, etc., are severally represented by 1, 3, 6, 10, etc.; consequently, we have $\triangle X = X'_1 - X = 3 - 1 = 2$, $\triangle X_1' = X'_2 - X'_1 = 6 - 3 = 3$, $\triangle X'_2 = X'_3 - X'_2 = 10 - 6 = 4$, and so on, for the first order of differences; and $\triangle^2 X = X'_2 - X'_1 - (X'_1 - X) = X'_2 - 2X'_1 + X = 3 - 2 = 1$, $\triangle^2 X'_1 = X'_3 - X'_2 - (X'_2 - X'_1) = 4 - 3 = 1$, and so on, are the differences of the second order; and the differences of the third, fourth, etc., orders are 0.

Hence, putting 1, 2, 1, 0, 0, etc., for X, $\triangle X$, $\triangle^2 X$, $\triangle^3 X$, $\triangle^4 X$, etc., severally, (*a*) becomes $X'_n = 1 + 2n + \frac{n(n-1)}{1\,.\,2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)\,.\,(n+2)}{1\,.\,2}$ = the term whose number, reckoned from the first term 1 (of the series), is n, or it is the $(n + 1)^{th}$ term of the series; and because it expresses the terms of the series in a general form, it is said to be its *general term;* thus, if for n we put 0, 1, 2, 3, 4, etc., we get $\left(\text{from } \frac{(n+2)\,.\,(n+1)}{1\,.\,2}\right)$, 1, 3, 6, 10, 15, 21, etc., the terms of the series.

Ex. 2.—To find the sum of n terms of the preceding series.

Here, we form the new series 0, 1, (1 + 3), (1 + 3 + 6), (1 + 3 + 6 + 10), (1 + 3 + 6 + 10 + 15), etc., whose differences of the first order are 1, 3, 6, 10, 15, etc., which are the terms of the given series; and 2, 3, 4, etc., are the differences of the second order; also, 1, 1, 1, 1, etc., are the differences of the third order; and the differences of the following orders equal 0. Hence, putting 0, 1, 2. 1, 0, 0, etc., for X, $\triangle X$, $\triangle^2 X$, $\triangle^3 X$, etc., in (*a*), we have $X'_n = 0 + n + n(n - 1) + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} = \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$ = the sought sum.

Thus, if we put 5 for n, we get $1 + 3 + 6 + 10 + 15 =$

$\frac{5 \times 6 \times 7}{1 \times 2 \times 3} = 35$; and $n = 6$ gives $1 + 3 + 6 + 10 + 15 + 21 = \frac{6 \times 7 \times 8}{1 \times 2 \times 3} = 56$.

Hence, *to find the sum of n terms of any series, we must in* (a) *put* 0 *for* X, *the first term of the series for* ΔX, *the first term of the first order of differences of the series for* $\Delta^2 X$, *the first term of the second order of differences of the series for* $\Delta^3 X$, *and so on.*

Thus, to find the sum of n terms of the series of cubes 1, 8, 27, 64, 125, 216, etc., we have 7, 12, and 6 for the first terms of its first, second, and third order, of differences, the differences of higher orders being 0; consequently, since 1 is the first term of the series, we get (from (a),) $X'n = 0 + n + \frac{n(n-1)}{1 \,.\, 2} \times 7 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} \times 12 + \frac{n(n-1)(n-2)(n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \times 6 = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4}$ = the sum of n terms, as required.

Ex. 3.—Given, 1.690196, 1.698970, 1.707570, 1.716003, the logarithms of 49, 50, 51, and 52, to find the logarithm of 50.75.

Here we find 0.008774, – 0.000174, and 0.000007, for the first terms of the first, second, and third orders of differences; and subtracting 49 from 50.75, we get 1.75 for n.

Hence, from (a) we get $X'_n = \log.\ 50.75 = \log.\ 49 + 1.75 \times 0.008774 - \frac{1.75 \times (1.75 - 1)}{1 \,.\, 2} \times 0.000174$ (when we omit the remaining term on account of its minuteness), or $\log.\ 50.75 = \log.\ 49 + 0.0153545 - 0.0001143 = 1.7054362$, as required; which to six decimal places agrees with the logarithm of 50.75, as found directly from the tables of logarithms.

Remark.—When any intermediate term of a series is found (as above), the term thus found *is said to be interpolated.*

Ex. 4.—From the same logarithms as in the preceding question, it is required by (b) or (c) to find the logarithm of 53.

Here, by taking the first of the fourth order of differences, we get $\Delta^4X = X'_4 - 4X'_3 + 6X'_2 - 4X'_1 + X$, which, by rejecting Δ^4X (on the supposition that it is of no importance in the investigation), becomes $X'_4 - 4X'_3 + 6X'_2 - 4X_1 + X = 0$; consequently, we easily get $X'_4 = 4(X'_1 + X'_3) - (6X'_2 + X)$. Hence, putting the logarithms 49, 50, 51, and 52 for X, X'_1, X'_2, X'_3, severally, we get $X'_4 = 1.724276 =$ the logarithm of 53, which agrees with the logarithm as found from the tables.

Ex. 5.—Given the series 2, 12, 20, 30, etc., to find the term which is wanting between 2 and 12.

Putting the third order of differences equal to 0, and proceeding as in the last question, the sought term will be found to be 6.

Ex. 6.—To find the first term of the third order of differences of the series of cubes 1, 8, 27, 64, etc.

From $\Delta^3X = -(X - 3X'_1 + 3X'_2 - X'_3)$, the sought difference will be found to equal 6.

Ex. 7.—Required the first term of the fourth order of differences of the series 1, 6, 20, 50, 105, 196, etc.

From $\Delta^4X = X - 4X'_1 + 6X'_2 - 4X'_3 + X'_4$, the sought difference will be found to equal 2.

4. *Remarks.*—1. If we have the equation $S = m + ax + bx^2 + cx^3 + dx^4 +$ etc., to infinity, or its equivalent, $S - m = ax + bx^2 + cx^3 + dx^4 +$ etc., (1); then by multiplying the members of the equation by $1 - x$, we get $(S - m)(1 - x) = ax + (b - a)x^2 + (c - b)x^3 + (d - c)x^4 +$ etc.; or, by subtracting ax from the two members, we have $(S - m) \times (1 - x) - ax = \Delta a.x^2 + \Delta b.x^3 + \Delta c.x^4 +$ etc., (2), since $b - a$, $c - b$, $d - c$, etc., are clearly the first differences of a, b, c, etc.

If we multiply the members of (2) by $1 - x$, and proceed in the same way as before, we shall get $[(S - m)(1 - x) - ax](1 - x) - \Delta a.x^2 = (\Delta b - \Delta a)x^3 + (\Delta c - \Delta b)x^4 +$ etc. $= \Delta^2a.x^3 + \Delta^2b.x^4 +$ etc., (3), since $\Delta b - \Delta a$, $\Delta c - \Delta b$, etc., are clearly the first differences of Δa, Δb, etc., or the second differences of a, b, etc.; and in like manner we may find equations which involve the third, fourth, etc., differences of a, b, etc.

If we have the equation $S' - m' = a'x + b'x^3 + c'x^5 +$ etc., (4); then, multiplying its members by $1 - x^2$, and proceeding as before, we shall get $(S' - m')(1 - x^2) - a'x = \Delta a'.x^3 + \Delta b'.x^5 +$ etc., (5; and thence $[(S' - m')(1 - x^2) - a'x](1 - x^2) - \Delta a'.x^3 = \Delta^2 a'.x^5 +$ etc., (6), and so on.

If the differences of any order which result from (1) or (4) equal 0, we can evidently find S or S′, which are called the generating functions of the series; also, if the differences do not equal 0, yet if they are so small that they may be rejected, we can find the approximate value of S or S′.

To illustrate what has been done, we shall take the following

EXAMPLES.

Ex. 1.—To find the generating function of $S = 1 + 2x + 3x^2 + 4x^3 + 5x^4 +$ etc.

Comparing the equation to (1), we get $m = 1$, $a = 2$, $b = 3$, $c = 4$, etc., $\Delta a = b - a = 3 - 2 = 1$, $\Delta b = c - b = 4 - 3 = 1$, the first differences being each equal to 1; consequently, the second differences will be 0. Hence, (3) will become $[(S - 1)(1 - x) - 2x](1 - x) - x^2 = (S - 1)(1 - x)^2 - 2x + x^2 = S(1 - x)^2 - 1 = 0$, which gives $S = \frac{1}{(1 - x)^2}$, as required.

Ex. 2.—To find the generating function of $S = 1 - 2x + 2x^2 - 2x^3 +$ etc.

Putting $x = -y$, the series becomes $S = 1 + 2y + 2y^2 + 2y^3 +$ etc., whose first differences being 0, we shall get $(S - 1)(1 - y) - 2y = S(1 - y) - (1 + y) = 0$, which gives $S = \frac{1 + y}{1 - y}$, or since $y = -x$, we get $S = \frac{1 - x}{1 + x}$, as required.

Ex. 3.—To find the generating function of $S = 1 + x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + 36x^6 +$ etc.

Here the differences of the third order being 0, we get $(S - 1)(1 - x)^3 - (x + 2x^2)(1 - x) = 2x^3$; consequently, we shall have $S = \frac{1 - 2x + 4x^2 - x^3}{(1 - x)^3}$, as required.

Ex. 4.—To find the generating function of $S' = 1 + 3x + 5x^3 + 7x^5 + 9x^7 +$ etc.

Because the second differences are 0, if in (6) we put $m' = 1$, $a' = 3$, $\triangle a' = 5 - 3 = 2$, we shall get $[(S' - 1)(1 - x^2) - 3x](1 - x^2) - 2x^3 = S'(1 - x^2)^2 - 1 + 2x^2 - x^4 - 3x + x^3 = 0$, which gives $S' = \dfrac{1 + 3x - 2x^2 - x^3 + x^4}{(1 - x^2)^2}$, as required.

Ex. 5.—To find the generating function of $S = 1 + x + 5x^2 + 13x^3 + 41x^4 + 121x^5 +$ etc.

Here our method is not applicable, because there is no order of the differences of the coefficients which equals 0. Nevertheless, the coefficients are found to be connected with each other by a very simple law; for if we multiply the coefficient of x by 3, and that of x^2 by 2, their sum gives $3 + 10 = 13 =$ the coefficient of x^3; and if we multiply the coefficient of x^2 by 3, and that of x^3 by 2, their sum gives $15 + 26 = 41$, the coefficient of x^4, and so on. Hence, the given series is clearly a recurring series, such that $(3x^2, 2x)$ is the scale of relation of each of its terms (after the first two terms) to the two preceding terms.

And it is clear, from what was formerly shown of such a series, that the given series must be the expansion of a fraction of the form $\dfrac{1 + Ax}{1 - 2x - 3x^2}$, or that we must have $\dfrac{1 + Ax}{1 - 2x - 3x^2} = 1 + x + 5x^2 + 13x^3 + 41x^4 + 121x^5 +$ etc., in which A is to be found so to satisfy the equation. To determine A, we develop the fraction, and thence get $1 + (A + 2)x +$ etc. $= 1 + x +$ etc.; consequently, because the series must be identical, we must have $A + 2 = 1$, or $A = -1$, and of course $\dfrac{1 - x}{1 - 2x - 3x^2}$ is the generating function of the given series.

Hence, it is clear that the method to be used, in order to find the generating function of a series, must be varied according to the nature of the case.

2. If the generating function of a series equals the sum of its terms, or differs insensibly from it, *it is called the sum of the series;* otherwise, *the series is called the development or expansion of the generating function.*

The generating function of a series can sometimes be found by adding its terms as they are given, or by changing them into equivalent forms.

EXAMPLES.

Ex. 1.—To find the sum of n terms of the series $S = \frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \ldots\ldots + \frac{1}{n(n+1)}$.

Here, by adding the terms as they are given, we have $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} = \frac{4}{6} = \frac{2}{3}$, $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} = \frac{9}{12} = \frac{3}{4}$, $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \frac{1}{4\,.\,5} = \frac{16}{20} = \frac{4}{5}$, and so on; consequently, it is clear that we shall have $S = \frac{n}{n+1}$ = the generating function, as required.

Again, by changing the terms of the series into equivalent forms, we shall get $S = \frac{1}{1\,.\,2} + \frac{1}{2\,.\,3}$ + etc. $= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots\ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$, as appears by erasing the terms which destroy each other.

It may be added, that the series can be obtained from the development of the generating function.

Thus, since $\frac{n}{n+1} = \frac{n}{2n - (n-1)}$, we shall get $\frac{n}{n+1} = \frac{1}{1\,.\,2} + \frac{n-1}{2(n+1)}$; and since $\frac{n-1}{2(n+1)} = \frac{n-1}{2[(3n-3)-(2n-4)]} = \frac{1}{2\,.\,3} + \frac{n-2}{3(n+1)}$, we get $\frac{n}{n+1} = \frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{n-2}{3(n+1)}$; also, because $\frac{n-2}{3(n+1)} = \frac{n-2}{3[(4n-8)-(3n-9)]} = \frac{1}{3\,.\,4} + \frac{n-3}{4(n+1)}$, we get $\frac{n}{n+1} = \frac{1}{1\,.\,2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \frac{n-3}{4(n+1)}$; and we may proceed in the same way indefinitely.

Hence it is clear, if n is infinite, that we must take 1 for the generating function, and that the sum of the terms of the series will differ insensibly from 1.

Ex. 2.—To find the sum of n terms of the series $S = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots . \frac{1}{n(n+1)(n+2)}$.

Changing the terms into equivalent expressions, we get $S = \left(\frac{1}{1.2} - \frac{1}{2.3}\right) \div 2 + \left(\frac{1}{2.3} - \frac{1}{3.4}\right) \div 2 + \ldots . + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) \div 2 = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$, as required.

Ex. 3.—To find the sum of n terms of the series $S = 1.2 + 2.3 + 3.4 + \ldots . + n(n+1)$.

Here we have $S = (1.2.3 - 0.1.2) \div 3 + (2.3.4 - 1.2.3) \div 3 + (3.4.5 - 2.3.4) \div 3 + \ldots . + [n(n+1)(n+2) - (n-1)n(n+1)] \div 3 = \frac{n(n+1)(n+2)}{3}$.

3. We will now give the substance of the preceding remark in a more general form.

To the end in view, we shall resume the series $S = \frac{1}{1.2} + \frac{1}{2.3} + \ldots . + \frac{1}{n(n+1)}$, and we shall represent the sum of $n+1$ terms of the same series by $S' = \frac{1}{1.2} + \frac{1}{2.3} + \ldots . + \frac{1}{(n+1)(n+2)}$; then, by subtraction, we get $S' - S = \frac{1}{(n+1)(n+2)}$, or $S' - S = -\left(\frac{1}{n+2} - \frac{1}{n+1}\right)$, (1).

Because the first and second members of (1) are of like forms, and that S and $\frac{1}{n+1}$ are changed into S' and $\frac{1}{n+2}$, when n is changed into $n+1$, it is clear that (1) can be satisfied by assuming $S = C - \frac{1}{n+1}$, and $S' = C - \frac{1}{n+2}$, in which C, called *the arbitrary constant*, is independent of n. To determine C, we put 0 for n; then, since $n = 0$ gives

$S = 0$, the equation $S = C - \frac{1}{n+1}$ is reduced to $0 = C - 1$ or $C = 1$; consequently, putting 1 for C, we get $S = 1 - \frac{1}{n+1}$, which is the same expression for the sum of n terms that we have previously found.

It is evident that $S' - S$ and $\frac{1}{n+2} - \frac{1}{n+1}$ are the differences of S and $\frac{1}{n+1}$, which result from changing n into $n + 1$; consequently, (1) is equivalent to the equation $\triangle S = - \triangle \frac{1}{n+1}$, (2).

If we denote the reverse of the method of taking the difference of a quantity by writing $\int$ before or to the left of the quantity, then, from (2), we shall get $\int \triangle S = \int - \triangle \frac{1}{n+1}$, or $S = C - \frac{1}{n+1}$; because the operation indicated by $\triangle$ is destroyed by the reverse operation denoted by $\int$, and that C, called the arbitrary constant, is added to the right member of the equation to correct the result, or for generality, since the difference of $C - \frac{1}{n+1}$ is the same as that of $-\frac{1}{n+1}$, because C does not contain n.

It is evident that this process which is called *integration* is substantially the same as that previously given (though it is, perhaps, less simple), and that C can be found in the same way as before.

It clearly follows from what we have done, that any term of a series is the difference of the sum of all the preceding terms taken with their proper signs, and that in order to find the sum of the preceding terms from the difference, the difference must be put in the form of an exact difference, after the manner of the right member of (1) or (2); where it may be noticed that the finding the sum of the terms from the difference is called *the finding the integral*, or *integration*.

To illustrate what has been done, take the following

EXAMPLES.

Ex. 1.—To find the sum of n terms of the series $S = (a+b)+(a+2b)+(a+3b)+\ldots\ldots+(a+(n-1)b)$.

Here we have $a+nb$ for the difference of the sum of n terms of the series; and since $(n+1)^2-n^2=2n+1$, we have $n=\frac{(n+1)^2-n^2-1}{2}=\frac{\Delta n^2-1}{2}$; consequently, putting this value for n, we have $a+nb=a+\frac{b\Delta n^2-b}{2}=\frac{2a-b+b\Delta n^2}{2}$; and since $n+1-n=1=\Delta n$, the expression is reduced to $\frac{(2a-b)\Delta n+b\Delta n^2}{2}$; consequently, by taking the integral, we shall get $\int(a+nb)=C+\frac{(2a-b)n+bn^2}{2}$, or the sum of n terms of the series, is expressed by $S=C+\frac{2a+(n-1)b}{2}\times n$.

To determine C, we put $n=0$, which gives $S=0$, and thence we have $0=C$; consequently, the sum of n terms of the series is expressed by $S=\frac{a+a+(n-1)b}{2}\times n$.

Ex. 2.—To find the sum of n terms of the series $S=1+4+9+16+25+\ldots\ldots+n^2$.

Here we have $\Delta S=(n+1)^2=n^2+2n+1=\frac{3n^2+6n+3}{3}=\frac{3n^2+3n+1}{3}+\frac{6n+3+1}{6}$, and (since $3n^2+3n+1=(n+1)^3-n^3=\Delta n^3$, $6n+3=3(n+1)^2-3n^2=3\Delta n^2$, and $(n+1)-n=\Delta n$) we shall have $\Delta S=\frac{\Delta n^3}{3}+\frac{3\Delta n^2+\Delta n}{6}=\frac{2\Delta n^3+3\Delta n^2+\Delta n}{6}$; consequently, by taking the integral, we get $S=\frac{2n^3+3n^2+n}{6}=\frac{n(n+1)(2n+1)}{6}$, noticing that it is not necessary to use any constant, since it will equal 0.

SECTION X.

POWERS AND INVOLUTION.

(**1.**) According to 9, of Sec. I., the products obtained by the successive multiplications of any number or quantity (*called the root*) by itself, are called *powers of the root.*

Thus, if A is taken for the root, it is said to be the first power of A; $A \times A = A^2$ is called the second power or square of A; $A \times A \times A = A^2 \times A = A^3$ is called the third power or cube of A; $A \times A \times A \times A = A^3 \times A = A^4$ is called the fourth power of A; and generally, $A \times A \times A \times \ldots . \times A$ to n factors $= A^n$ is called the n^{th} power of A, and n, which denotes the number of times that A enters as a factor into A^n, is called the *index* or *exponent* of the power.

(**2.**) The process used in obtaining any power of the root *is called Involution;* and when the power is formed, the root is said to have been *involved or raised to the power.*

(**3.**) Because any power is formed or derived from the root by multiplication, it is evident that we shall have the following rule for raising any root to any proposed power.

RULE.

1. *By the rules of Multiplication, multiply the root by itself, and the product by the root again, and so on, until the number of multiplications is one less than the number of units in the index of the power; then the product thus obtained will be the required power.*

2. *By the rule of signs in multiplication, when the root is positive, all its powers will be positive; but if the root is negative, all the even powers, or those whose index is an even number, will be positive, and all the uneven powers, or those whose index is an uneven number, will be negative, or will have the same sign as the root.*

3. *If the root is a fraction, it follows from the rule for the multiplication of fractions, that the power will be found*

by raising the numerator and denominator to the proposed power; also, if the root is of a mixed form, or partly integral and partly fractional, it may be reduced to the form of an improper fraction, and then the power may be found as in the case of a fraction.

4. *If the root consists of factors, the power may be found by raising the factors to the power, and taking their product for the sought power.*

5. *Because powers of the same root may be multiplied by adding their indices for the index of their product, it is clear if the root has no index expressed, that by giving it or its factors the index of the power for an index, the power will be expressed as required; also, if the root or its factors have indices, then the index of the root or the indices of its factors, when multiplied by the index of the power, will express the power, as required.*

6. *If we take different powers of the root, such that the sum of their indices equals the index of the power to which the root is to be raised, then the product of the powers will equal the required power of the root.*

Remark.—It is clear from 2 of the rule that it is impossible for an even power of a number or quantity to equal a negative number or quantity; thus, $x^2 = -2$, $x^4 = -a^4$, $y^6 = -b^4$ are impossible equations, since there are no real values of x and y which can satisfy them; consequently, x and y are said to be impossible or imaginary.

To illustrate the rule, take the following

EXAMPLES.

1. To find the second, third, fourth, and fifth powers of a and $-b$.

Here, $a \times a = a^2$, $a \times a \times a = a^2 \times a = a^3$, $a \times a \times a \times a = a^2 \times a^2 = a^3 \times a = a^4$, $a \times a \times a \times a \times a = a^3 \times a^2 = a^5$ are the required powers of a; and $-b \times -b = b^2$, $-b \times -b \times -b = b^2 \times -b = -b^3$, $-b \times -b \times -b \times -b = b^2 \times b^2 = -b^3 \times -b = b^4$, $-b \times -b \times -b \times -b \times -b = b^2 \times b^2 \times -b = -b^3 \times b^2 = -b^5$, are the required powers of $-b$.

2. To find the square of $3a^2b^3$ and the cube of $7abc^2$

Here we have $3a^2b^3 \times 3a^2b^3 = 3^2a^4b^6 = 9a^4b^6$, and $(7abc^2)^3 = 7^3a^3b^3c^6 = 343a^3b^3c^6$, as required.

3. To find the fourth power of $\frac{2ab^{\frac{1}{2}}c}{3ef^{\frac{1}{4}}}$ and the square of $-\frac{3c^5d^6}{4a^2b^3}$.

Here, $\left(\frac{2ab^{\frac{1}{2}}c}{3ef^{\frac{1}{4}}}\right)^4 = \frac{2^4a^4b^2c^4}{3^4e^4f} = \frac{16a^4b^2c^4}{81e^4f}$, and $\left(-\frac{3c^5d^6}{4a^2b^3}\right)^2 = \left(\frac{3c^5d^6}{4a^2b^3}\right)^2$ $= \frac{3^2c^{10}d^{12}}{4^2a^4b^6} = \frac{9c^{10}d^{12}}{16a^4b^6}$, are the required powers.

4. To find the square of $a - \frac{2b}{3c}$, and the cube of $-\left(4\frac{2}{3}\right)$.

Here we have $\left(a - \frac{2b}{3c}\right)^2 = \left(\frac{3ac - 2b}{3c}\right)^2 = \frac{(3ac - 2b)^2}{(3c)^2} = \frac{(3ac - 2b)^2}{9c^2} = \frac{9a^2c^2 - 12abc + 4b^2}{9c^2}$, and $\left(-4\frac{2}{3}\right)^3 = \left(-\frac{14}{3}\right)^3 = -\frac{2744}{27}$, for the sought answers.

5. Given, $(a + b)^2 = a^2 + 2ab + b^2$ and $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, to find $(a + b)^5$, or the fifth power of $a + b$.

Here, $(a + b)^3 \times (a + b)^2 = (a + b)^5$ gives $(a + b)^5 = (a^3 + 3a^2b + 3ab^2 + b^3) \times (a^2 + 2ab + b^2) = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, as required.

(4.) Because the raising of compound or polynomial expressions to powers by common multiplication is often long and very operose, we will now proceed to show how the process may generally be greatly abridged. To the end in view, we shall commence by giving the

INVESTIGATION OF THE BINOMIAL THEOREM.

1. Supposing $b + h$ to stand for the binomial, and n for any positive whole number, then from (C), in the remarks under Ex. 8 (p. 48), in Multiplication, and from (A) in the remarks under Ex. 6 (p. 70), in Division, we get the equations, $(b+h)^n = b^n + [(b + h)^{n-1} + (b + h)^{n-2}b + (b + h)^{n-3}b^2 + \ldots + (b + h)b^{n-2} + b^{n-1}] \times h = b^n + nb^{n-1}h +$ terms depending on h^2, h^3, etc., (1); and $(b+h)^{-n} = b^{-n} + [b^{-1}(b+h)^{-n} + b^{-2}(b+h)^{-(n-1)} +$

$b^{-3}(b+h)^{-(n-2)} + b^{-4}(b+h)^{-(n-3)} + \dots\dots + b^{-n}(b+h)^{-1}] \times -h = b^{-n} - nb^{-(n+1)}h +$ terms depending on h^2, h^3, etc., (2).

Because $n-1$, $n-2$, $n-3$, etc., are positive whole numbers, it follows from the expansion in (1), that we shall have $(b+h)^{n-1} = b^{n-1} + (n-1)b^{n-2}h +$ etc., $(b+h)^{n-2}b = [b^{n-2} + (n-2)b^{n-3}h +$ etc.$] \times b = b^{n-1} + (n-2)b^{n-2}h +$ etc., $(b+h)^{n-3}b^2 = b^{n-1} + (n-3)b^{n-2}h +$ etc., and so on; consequently, from (1), we shall get $[(n-1) + (n-2) + (n-3) + (n-4) + \dots\dots + 1$, making $n-1$ terms$]\ b^{n-2}h^2 = \frac{n(n-1)}{1\,.\,2}b^{n-2}h^2$, as appears from the Appendix to Multiplication and Division; consequently, we shall have $(b+h)^n = b^n + nb^{n-1}h + \frac{n(n-1)}{1\,.\,2}b^{n-2}h^2 +$ etc. If we proceed with (2) in the same way as before, we shall get $(b+h)^{-n} = b^{-n} - nb^{-(n+1)}h + \frac{n(n+1)}{1\,.\,2}b^{-(n+2)}h^2 -$ etc.; which can also clearly be obtained from the expansion of (1), by changing the sign of n.

Hence, we shall have $(b+h)^{n-1} = b^{n-1} + (n-1)b^{n-2}h + \frac{(n-1)(n-2)}{1\,.\,2}b^{n-3}h^2 +$ etc., $(b+h)^{n-2}b = b^{n-1} + (n-2)b^{n-2}h + \frac{(n-2)(n-3)}{1\,.\,2}b^{n-3}h^2 +$ etc., $(b+h)^{n-3}b^2 = b^{n-1} + (n-3)b^{n-2}h + \frac{(n-3)(n-4)}{1\,.\,2}b^{n-3}h^2 +$ etc., and so on; consequently, from (1) we shall get $\left[\frac{(n-1)(n-2)}{1\,.\,2} + \frac{(n-2)(n-3)}{1\,.\,2} + \frac{(n-3)(n-4)}{1\,.\,2} + \dots\dots + \frac{2\,.\,1}{1\,.\,2}\right] \times b^{n-3}h^3 =$ (by the Appendix before cited) $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} \times b^{n-3}h^3$, which is the fourth term of the expansion of $(b+h)^n$; and proceeding in like manner with (2), we get $-\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}b^{-(n+3)}h^3$ for the fourth term of the expansion of $(b+h)^{-n}$. Proceeding in like manner from the fourth terms of the expansions to get the fifth terms, and so on, it is clear that we shall have $(b+h)^n = b^n + nb^{n-1}h + \frac{n(n-1)}{1\,.\,2}b^{n-2}h^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} \times$

$b^{n-3}h^3 + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}b^{n-4}h^4$ + etc., to $(n+1)$ terms, (1′); and $(b+h)^{-n} = b^{-n} - nb^{-(n+1)}h + \frac{n(n+1)}{1\,.\,2}b^{-(n+2)}h^2 - \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}b^{-(n+3)}h^3$ + etc., to an unlimited number of terms, (2′); which are the required expansions of (1) and (2).

2. We will now investigate the theorem, when the exponent of the power is a positive or negative vulgar fraction.

Putting $a = b + h$ or $a - b = h$, and using $\frac{n}{m}$ to represent any vulgar fraction, it is clear that we shall have $\frac{a^{\frac{n}{m}} - b^{\frac{n}{m}}}{a - b} =$

$$\frac{a^{\frac{n}{m}} - b^{\frac{n}{m}}}{a^{\frac{m}{m}} - b^{\frac{m}{m}}} = \frac{a^{\frac{n-1}{m}} + a^{\frac{n-2}{m}}b^{\frac{1}{m}} + a^{\frac{n-3}{m}}b^{\frac{2}{m}} + \dots + b^{\frac{n-1}{m}}, \text{ to } n \text{ terms.}}{a^{\frac{m-1}{m}} + a^{\frac{m-2}{m}}b^{\frac{1}{m}} + a^{\frac{m-3}{m}}b^{\frac{2}{m}} + \dots + b^{\frac{m-1}{m}}, \text{ to } m \text{ terms,}}$$

as is evident by dividing the numerator and denominator by their common factor $a^{\frac{1}{m}} - b^{\frac{1}{m}}$.

If the members of the equation are multiplied by $a - b = h$, and $b^{\frac{n}{m}}$ added to the resulting products, and $(b + h)$ is put for a in the result, we shall get $(b + h)^{\frac{n}{m}} = b^{\frac{n}{m}} +$

$$\frac{(b+h)^{\frac{n-1}{m}} + (b+h)^{\frac{n-2}{m}}b^{\frac{1}{m}} + (b+h)^{\frac{n-3}{m}}b^{\frac{2}{m}} + \dots + b^{\frac{n-1}{m}}, \text{ to } n \text{ terms,}}{(b+h)^{\frac{m-1}{m}} + (b+h)^{\frac{m-2}{m}}b^{\frac{1}{m}} + (b+h)^{\frac{m-3}{m}}b^{\frac{2}{m}} + \dots + b^{\frac{m-1}{m}}, \text{ to } m \text{ terms,}} \times h, \text{ (3).}$$

If in (3) we put 1 for m, and then change n into $\frac{n}{m}$, it will become $(b + h)^{\frac{n}{m}} = b^{\frac{n}{m}} + [(b + h)^{\frac{n}{m}-1} + (b + h)^{\frac{n}{m}-2}b + (b + h)^{\frac{n}{m}-3}b^2$ + etc., to an unlimited number of terms] $\times h$, (3′); which is clearly the same result that we shall get from the division of the numerator of the fraction in (3), by its denominator.

Again, if we divide $a - b = a^{\frac{m}{m}} - b^{\frac{m}{m}}$ by $a^{\frac{1}{m}} - b^{\frac{1}{m}}$, and $a^{\frac{-n}{m}} - b^{\frac{-n}{m}}$ by $-b^{\frac{1}{m}} + a^{\frac{1}{m}}$, we shall get $\frac{a^{\frac{-n}{m}} - b^{\frac{-n}{m}}}{a - b} =$

$$- \frac{a^{\frac{-n}{m}}b^{-\frac{1}{m}} + a^{\frac{-n+1}{m}}b^{-\frac{2}{m}} + a^{\frac{-n+2}{m}}b^{-\frac{3}{m}} + \dots + a^{-\frac{1}{m}}b^{-\frac{n}{m}}, \text{ to } n \text{ terms}}{a^{\frac{m-1}{m}} + a^{\frac{m-2}{m}}b^{\frac{1}{m}} + a^{\frac{m-3}{m}}b^{\frac{2}{m}} + \dots + b^{\frac{m-1}{m}}, \text{ to } m \text{ terms}}, \text{ (4).}$$

If we put h for $a-b$, and $b+h$ for a, the equation is easily changed to

$$(b+h)^{-\frac{n}{m}} = b^{-\frac{n}{m}} + \frac{(b+h)^{-\frac{n}{m}}b^{-\frac{1}{m}} + (b+h)^{\frac{-n+1}{m}}b^{\frac{-2}{m}} + (b+h)^{\frac{-n+2}{m}}b^{\frac{-3}{m}} + \dots\dots (b+h)^{-\frac{1}{m}}b^{-\frac{n}{m}}}{(b+h)^{\frac{m-1}{m}} + (b+h)^{\frac{m-2}{m}}b^{\frac{1}{m}} + (b+h)^{\frac{m-3}{m}}b^{\frac{2}{m}} + \dots\dots + b^{\frac{m-1}{m}}} \times -h, \ (5).$$

If we put 1 for m, and change n into $\frac{n}{m}$, (5) will become $(b+h)^{-\frac{n}{m}} = b^{-\frac{n}{m}} + [(b+h)^{-\frac{n}{m}}b^{-1} + (b+h)^{-\frac{n}{m}+1}b^{-2} + (b+h)^{-\frac{n}{m}+2}b^{-3} + \text{etc.}] \times -h$, (5′), which is clearly the same as to convert the fraction in (5) into a series.

Because $b^{\frac{n}{m}}$ is the first term of the expansion of $(b+h)^{\frac{n}{m}}$ in (3), it is clear that $b^{\frac{n-1}{m}}$ is the first term of the expansion of $(b+h)^{\frac{n-1}{m}}$, and that each term in the numerator of the fraction in (3) gives $b^{\frac{n-1}{m}}$ for the first term of its expansion, so that all the terms give $nb^{\frac{n-1}{m}}$; and in a similar way, the terms of the denominator give $mb^{\frac{m-1}{m}}$; consequently, (3) is reducible to $(b+h)^{\frac{n}{m}} = b^{\frac{n}{m}} + \dfrac{nb^{\frac{n-1}{m}} + \text{terms in } h}{mb^{\frac{m-1}{m}} + \text{terms in } h} \times h = b^{\frac{n}{m}} + \frac{n}{m}b^{\frac{n}{m}-1}h +$ terms depending on h^2, h^3, etc., (3″), as is evident from the division of the numerator of the fraction by its denominator; and similarly, (5) gives $(b+h)^{-\frac{n}{m}} = b^{-\frac{n}{m}} - \frac{n}{m}b^{-(\frac{n}{m}+1)}h +$ terms in h^2, h^3, etc., (5″), which can at once be obtained from (3″), by changing the sign of $\frac{n}{m}$.

Again, because the first terms of the expansions of $(b+h)^{\frac{n}{m}-1}$, $(b+h)^{\frac{n}{m}-2}b$, $(b+h)^{\frac{n}{m}-3}b^2$, etc., are $b^{\frac{n}{m}-1}$, $b^{\frac{n}{m}-1}$, $b^{\frac{n}{m}-1}$, etc., we get from (3′) $(b+h)^{\frac{n}{m}} = b^{\frac{n}{m}} + (b^{\frac{n}{m}-1} + b^{\frac{n}{m}-1} + b^{\frac{n}{m}-1} + \text{etc.}) \times h +$ etc., so that the coefficient of h is expressed in an indeterminate form, when its true value is $\frac{n}{m}b^{\frac{n}{m}-1}$; consequently, the conversion of (3) into (3′) converts

$\frac{n}{m}$ into $\frac{n}{n-(n-m)} = 1 + \frac{n-m}{n-m-(n-2m)} = 1 + 1 + \frac{n-2m}{(n-2m)-(n-3m)} = 1 + 1 + 1 + \frac{n-3m}{n-3m-(n-4m)}$, and so on, indefinitely; and it is clear that similar observations are applicable to (5) and (5′).

Because $\left(\frac{n}{m}-1\right)b^{\frac{n}{m}-2}h$, $\left(\frac{n}{m}-2\right)b^{\frac{n}{m}-2}h$, $\left(\frac{n}{m}-3\right)b^{\frac{n}{m}-2}h$, etc., are the second terms of the expansions of $(b+h)^{\frac{n}{m}-1}$, $(b+h)^{\frac{n}{m}-2}b$, $(b+h)^{\frac{n}{m}-3}b^2$, etc., in (3′), it follows that we shall have $\left[\left(\frac{n}{m}-1\right)+\left(\frac{n}{m}-2\right)+\left(\frac{n}{m}-3\right)+ \text{ etc.}\right]b^{\frac{n}{m}-2}h^2 =$ (since, by the Appendix to Multiplication and Division, $\left(\frac{n}{m}-1\right)+\left(\frac{n}{m}-2\right)+$ etc. is the expansion of $\frac{\frac{n}{m}\left(\frac{n}{m}-1\right)}{1\,.\,2}$) $\frac{\frac{n}{m}\left(\frac{n}{m}-1\right)}{1\,.\,2}b^{\frac{n}{m}-2}h^2$, for the third term of the expansion of $(b+h)^{\frac{n}{m}}$; and by changing the sign of $\frac{n}{m}$, it will become the third term of the expansion of $(b+h)^{-\frac{n}{m}}$.

Proceeding in a similar way, we readily find $\frac{\frac{n}{m}\left(\frac{n}{m}-1\right)}{1\,.\,2}\,.\,\frac{\left(\frac{n}{m}-2\right)}{3}b^{\frac{n}{m}-3}h^3$ for the fourth term of the expansion of $(b+h)^{\frac{n}{m}}$; and by changing the sign of $\frac{n}{m}$, it will become the fourth term of the expansion of $(b+h)^{-\frac{n}{m}}$; and it is clear that the law of the terms of the expansions is the same as when the exponent of the power of the binomial is a positive or negative whole number.

3. If any positive or negative irrational number is represented by p, the expansion of $(b+h)^p$ will be of the same general form as the expansions of $(b+h)^n$, $(b+h)^{-n}$.

For let n' denote any very small rational number, which has the same sign as p, then taking the series n', $2n'$, $3n'$,

$4n'$, etc., we shall clearly find two successive numbers which may be expressed by qn' and $qn' + n'$, such that p shall be numerically greater than qn', and less than $qn' + n'$, so that p will have qn' and $qn' + n'$ for its limits. Hence, $(b + h)^p$ must clearly have $(b + h)^{qn'}$ and $(b + h)^{qn'+n'}$ for its limits; consequently, since n' may be taken so small that either limit shall differ insensibly from the other, it follows that $(b + h)^p$ may be considered as coinciding with either of them; hence, because $(b + h)^{qn'}$ and $(b + h)^{qn'+n'}$ can be expanded into series of the same form as $(b + h)^n$ or $(b + h)^{-n}$, it is clear that $(b + h)^p$ can also be expanded in a similar way, when p is put for n or $-n$.

4. If n stands for any real positive or negative number, it follows from what has been done that we shall have $(b + h)^n = b^n + nb^{n-1}h + \frac{n(n-1)}{1\ .\ 2}b^{n-2}h^2 +$ etc, in which b and h are clearly arbitrary.

If for h we put $-h$, we shall get $(b - h)^n = b^n - nb^{n-1}h + \frac{n(n-1)}{1\ .\ 2}b^{n-2}h^2 -$ etc., since the odd powers of h are negative, and the even powers positive.

Hence, $(b \pm h)^n = b^n \pm nb^{n-1}h + \frac{n(n-1)}{1\ .\ 2}b^{n-2}h^2 \pm \frac{n(n-1)}{1\ .\ 2}\ .\ \frac{(n-2)}{3} \times b^{n-3}h^3 + \frac{n(n-1)(n-2)(n-3)}{1\ .\ 2\ .\ 3\ .\ 4}b^{n-4}h^4 \pm$ etc., (a), which is such that if the number of any term is represented by $m + 1$, the term itself will be expressed by $\pm \frac{n(n-1)(n-2) \times \ldots . \times (n-m+1)}{1\ .\ 2\quad 3\ \times \ldots . \times\quad m}b^{n-m}h^m$, (b), in which $+$ must be used for $\pm$ when m is an even number, and $-$ must be used when m is an odd number; thus, if $m = 2$ the term becomes $\frac{n(n-1)}{1\ .\ 2}b^{n-2}h^2$, and if $m = 3$ it becomes $\pm \frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}b^{n-3}h^3$; consequently, all the terms of (a) can be deduced from (b), *which is called the general term* of (a).

If in $b \pm h$ we call b the first or leading term, and $\pm h$ the second or following term, then it is clear that if we multiply any term of (a) by the index of the leading term in it, diminish the index by unity, multiply the result by the following term (or $\pm h$), and divide by the index of the following term in the product, the result will equal the next successive term of (a); consequently, all the terms of (a) can clearly be deduced from b^n, taken as the first term of the expansion in (a); and it may be added, that the method here given for finding the successive terms of (a) is substantially the same as what is called *Newton's Binomial Theorem.*

EXAMPLES.

1. To find the third power or cube of $a \pm b$.

Putting a for b, and b for h, and 3 for n, in (a), we get $(a \pm b)^3 = a^3 \pm 3a^{3-1}b + \frac{3\times(3-1)}{1\,.\,2}a^{3-2}b^2 \pm \frac{3\times(3-1)\times(3-2)}{1\,.\,2\,.\,3}a^{3-3}b^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$; noticing that $a^{3-3} = a^0 = 1$, and that all the following terms of the series are reduced to 0, because they have $3 - 3 = 0$ for a factor.

2. To expand $(a + b)^6$ and $(x - y)^5$.

Here, we have $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, and $(x - y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

3. To expand $(3a + 4b)^3$, $(5a - 2b)^4$.

Here, putting $3a$ for b, and $4b$ for $\pm h$, in (a), and using 3 for n, we get $(3a + 4b)^3 = (3a)^3 + 3 \times (3a)^2 \times 4b + 3 \times (3a) \times (4b)^2 + (4b)^3 = 27a^3 + 108a^2b + 144ab^2 + 64b^3$; also using $5a$ for b, $-2b$ for $\pm h$, and 4 for n, in (a), we have $(5a - 2b)^4 = (5a)^4 - 4 \times (5a)^3 \times 2b + 6 \times (5a)^2 \times (2b)^2 - 4 \times 5a \times (2b)^3 + (2b)^4 = 625a^4 - 1000a^3b + 600a^2b^2 - 160ab^3 + 16b^4$.

Remarks.—If the coefficients in the first of these expansions are added, their sum will be found to equal $343 = 7^3$; and the coefficients in the second expansion, when added according to their signs, will be found to equal $81 = 3^4$; results which are clearly as they ought to be, as is evident by putting 1 for a and 1 for b.

4. To expand $(a \pm b)^{-1}$ and $(x + y)^{-2}$.

Here, using a for b, $\pm b$ for $\pm h$, and -1 for n, in (a), we get $(a \pm b)^{-1} = a^{-1} \mp a^{-2}b + a^{-3}b^2 \mp a^{-4}b^3 + \text{etc.} = \frac{1}{a} \mp \frac{b}{a^2} +$

$\frac{b^2}{a^3} \mp \frac{b^3}{a^4}$ + etc., which is clearly the same result that we shall get from the division of 1 by $a \pm b$, as it evidently ought to be; and in a similar way we get $(x + y)^{-2} = x^{-2} - 2x^{-3}y + 3x^{-4}y^2 - 4x^{-5}y^3 + \text{etc.} = \frac{1}{x^2} - \frac{2y}{x^3} + \frac{3y^2}{x^4} - \frac{4y^3}{x^5} + \text{etc.}$, which is also the same result that the division of 1 by $(x + y)^2 = x^2 + 2xy + y^2$ will give.

(5.) We will now apply (a) to the expansion of any power of a polynomial expression.

Let $a + bx + cx^2 + dx^3 + \text{etc.}$, represent a polynomial, a being its first term, b the coefficient of x in its second term, c that of x^2 in its third term, and so on, according to the order of the letters in the alphabet; then the polynomial may clearly be supposed to be derived from a by changing a into $a + bx$, b into $b + cx$, c into $c + dx$, and so on.

If we put a for b and $bx + cx^2 + \text{etc.}$, for $\pm h$ in (a), and represent the index of the power by n, we shall have $(a + bx + cx^2 + \text{etc.})^n = a^n + na^{n-1}(bx + cx^2 + \text{etc.}) + \frac{n(n-1)}{1 \,.\, 2}a^{n-2}(bx + cx^2 + \text{etc.})^2 + \text{etc.} = a^n + na^{n-1}x(b + cx + \text{etc.}) + \frac{n(n-1)}{1 \,.\, 2}a^{n-2}x^2(b + cx + \text{etc.})^2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}a^{n-3}x^3(b + cx + \text{etc.})^3 + \frac{n(n-1)(n-2)(n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \times a^{n-4}x^4(b + cx + \text{etc.})^4 +$ etc., (c); and it is clear that the powers of $b + cx + \text{etc.}$, in (c) admit of similar expansions, and so on.

If the terms in (c) are arranged according to the ascending powers of x, and we express the coefficients of the powers of x by using the vertical bar, we shall get

$$(a + bx + cx^2 + \text{etc.})^n = a^n + na^{n-1}bx + \left.\begin{array}{r} \frac{n(n-1)}{1 \,.\, 2}a^{n-2}b^2 \\ + na^{n-1}c \end{array}\right| x^2 + \left.\begin{array}{r} \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}a^{n-3}b^3 \\ + n(n-1)a^{n-2}bc \\ + na^{n-1}d \end{array}\right| x^3 + \left.\begin{array}{r} \frac{n(n-1)(n-2)(n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}a^{n-4}b^4 \\ \frac{n(n-1)(n-2)}{1 \,.\, 2}a^{n-3}b^2c \\ \frac{n(n-1)}{1 \,.\, 2}a^{n-2}(c^2 + 2bd) \\ + na^{n-1}e \end{array}\right| x^4 +$$

etc., (d), for the required expansion.

If the terms of the polynomial are arranged according to the descending powers of x, or if the coefficients are detached, or 1 is put for x (provided any deficient terms are supplied by terms whose coefficients equal 0), it is clear that the expansion may be obtained as in (d); observing that the coefficients of the powers of x, or the terms of the given polynomial, are supposed to be represented by the letters a, b, c, d, etc., taken alphabetically.

If we have any coefficient in the expansion in (d), and suppose the letters of each of its terms to be written in alphabetical order, it is clear, from (c), that we can find the next successive coefficient by the following

RULE.

1. *When the last letter of any term of the given coefficient is not the last coefficient in the given polynomial.*

Multiply the term by the index of the last letter in the term, and subtract unity from the same exponent (of the letter) in the product, and multiply the result by the next (successive) letter of the alphabet; then the product will be a term of the required coefficient.

2. *When the last two letters of any term of the given coefficient are successive letters of the alphabet.*

Find the term of the sought coefficient, which results from the last letter of the term, as in 1; *then multiply the given term by the index of the last letter but one in the term, subtract one from the index of the same letter, and add one to the index of the following or last letter in the term; then the result, when divided by the increased index of the last letter (of the term), will be a (new) term of the sought coefficient.*

3. *All the terms which can be obtained by* (1,) *and* (2,) *from the given coefficient, when duly ordered and connected by their proper signs, will constitute the required coefficient, because the other letters which may enter the term of the given coefficient must (according to* (c)*) be considered as not giving any terms in the required coefficient.*

4. *In the case of* (1), *when the last letter is not raised to any power, it will be sufficient to change it into the next*

successive letter in the alphabet, in order to get the sought term.

5. *If the last letter in any term of the given coefficient is the last coefficient in the given polynomial, it clearly can not give any term in the required coefficient, since the successive letters in the polynomial equal* 0.

Hence, we perceive how we may proceed in order to deduce the expansion in (d) from its first term, a^n.

Remark.—Our rule is substantially the same as that given by Arbogast, at p. 25 of his work entitled "Du Calcul. Des Derivations."

EXAMPLES.

Ex. 1.—To expand $(1 + 2x - 3x^2)^3$, according to the ascending powers of x.

Using a, b, and c for 1, 2, and -3, we get $(a + bx + cx^2)^3$

$$= a^3 + 3a^2bx + \left.\begin{matrix}3ab^2\\3a^2c\end{matrix}\right| x^2 + \left.\begin{matrix}b^3\\6abc\end{matrix}\right| x^3 + \left.\begin{matrix}3b^2c\\3ac^2\end{matrix}\right| x^4 + 3bc^2x^5 + c^3x^6.$$

For $3a^2b$, the coefficient of x, has been obtained from a^3 regarded as the coefficient of $x^0 = 1$, by (1) of the rule; the term $3ab^2$ of the coefficient of x^2 has been obtained from $3a^2b$ by multiplying by 2 the index of a and subtracting 1 from the index, and multiplying the result by b, and dividing the product by 2, the increased index of b, according to (2) of the rule, and $3a^2c$, the remaining term of the coefficient of x^2 has been found from $3a^2b$ by changing b into c, according to (4); also, b^3, the first term of the coefficient of x^3, has been found from $3ab^2$ of the coefficient of x^2, by changing a into b, and dividing by 3 the increased index of b agreeably to (2).; and the remaining term, $6abc$, of the coefficient of x^3 has been obtained from $3ab^2$ by (1); noticing that the term $3a^2c$ of the coefficient of x^2 can not give any term in the coefficient of x^3, since the letters a and c are not successive letters of the alphabet, and that c is the last coefficient in $a + bx + cx^2$, agreeably to (3) and (4), and so on.

Restoring the values of the letters, we shall have $(1 + 2x - 3x^2)^3 = 1 + 6x + 3x^2 - 28x^3 - 9x^4 + 54x^5 - 27x^6$, which is clearly correct, since the sum of the coefficients in $1 + 2x -$

$3x^2$ is 0, and that the sum of the coefficients in the expansion is also 0.

Ex. 2.—To expand $(2x^2 + 3x - 3)^4$ according to the descending powers of x.

Here, using a, b, and c for 2, 3, and -3, we have

$$(ax^2 + bx + c)^4 = a^4x^8 + 4a^3bx^7 + \left.\begin{array}{c} 6a^2b^2 \\ 4a^3c \end{array}\right| x^6 + \left.\begin{array}{c} 4ab^3 \\ 12a^2bc \end{array}\right| x^5 + \left.\begin{array}{c} b^4 \\ 12ab^2c \\ 6a^2c^2 \end{array}\right| x^4 + \left.\begin{array}{c} 4b^3c \\ 12abc^2 \end{array}\right| x^3 + \left.\begin{array}{c} 6b^2c^2 \\ 4ac^3 \end{array}\right| x^2 + 4bc^3x + c^4,$$

as required.

Restoring the values of the letters, we get $(2x^2 + 3x - 3)^4 = 16x^8 + 96x^7 + 120x^6 - 216x^5 - 351x^4 + 324x^3 + 270x^2 - 324x + 81$, which is clearly correct, since the sum of the coefficients (in the expansion) is 16, which is the fourth power of 2, the sum of the coefficients in $2x^2 + 3x - 3$.

Ex. 3.—To expand $(a + b + c + d)^2$.

Here we have

$$(a + b + c + d)^2 = a^2 + 2ab + \begin{array}{c} b^2 \\ 2ac \end{array} + \begin{array}{c} 2bc \\ 2ad \end{array} + \begin{array}{c} c^2 \\ 2bd \end{array} + 2cd + d^2.$$

Ex. 4.—To expand $(ax^m + bx^{m-1} + cx^{m-2} + dx^{m-3} + \text{etc.})^n$, according to the descending powers of x.

Because a^nx^{mn} is clearly the first term of the expansion, we easily get

$$(ax^m + bx^{m-1} + cx^{m-2} + \text{etc.})^n = a^nx^{mn} + na^{n-1}bx^{mn-1} + \left.\begin{array}{r} \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 \\ na^{n-1}c \end{array}\right| x^{mn-2} + \left.\begin{array}{r} \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 \\ n(n-1)\,a^{n-2}\,bc \\ na^{n-1}d \end{array}\right| \times$$

$$x^{mn-3} + \left.\begin{array}{c} \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}b^4 \\ \frac{n(n-1)(n-2)}{1\,.\,2}a^{n-3}b^2c \\ \frac{n(n-1)}{1\,.\,2}a^{n-2}(c^2 + 2bd) \\ na^{n-1}e \end{array}\right| x^{mn-4} + \text{etc.}$$

Remark.—If we put $m = 8$, $a = 81 = 3^4$, $b = -216$, $c = 0$, $d = 336$, $e = -56$, $f = -224$, $g = 0$, $h = 64$, $i = 16$, $k = 0$, $l = 0$, and so on; then, if we put $n = \frac{1}{4}$, our formula becomes $(3^4x^8 - 216x^7 + 0\,.\,x^6 + 336x^5 - 56x^4 - 224x^3 + 0\,.\,x^2 + 64x + 16)^{\frac{1}{4}} = \pm\,(3x^2 - 2x - 2) =$ the fourth root of the first member of the equation, since the index $\frac{1}{4}$ clearly indicates that its fourth root is to be taken; and since the root is of an even degree, it may have either of the signs + or −, which is indicated by writing the ambiguous sign ± before the root. Thus, since ax^m is represented by 3^4x^8, a^nx^{mn}, in the general expansion, is represented by $3^{4\times\frac{1}{4}} \times x^{8\times\frac{1}{4}} = 3x^2$, the first term of the root; and the term $na^{n-1}bx^{mn-1}$, in the general expansion, becomes $\frac{1}{4}(3^4)^{\frac{1}{4}-1} \times (-216)x^{8\times\frac{1}{4}-1} = -2x$; and because $c = 0$, the third term of the general expansion becomes $\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 \times x^{mn-2} = \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{1\,.\,2} \times \frac{(3^4)^{\frac{1}{4}}}{3^8} \times (216)^2 = -\frac{3}{32} \times \frac{3}{81^2} \times 216^2 = -2$, the remaining term of the root; and it is easy to perceive that the root is exact.

SECTION XI.

EVOLUTION.

(1.) Evolution, or the extraction of roots, is the reverse of Involution, and consists in finding the root of any number or quantity regarded as a power of the root.

Thus, since A × A × A × .... × A to n factors $= A^n$, the n^{th} power of A, it follows that A is the n^{th} root of A^n; consequently, in order to get the n^{th} root of a number or quantity, we must resolve (or separate) it into n equal factors, and take one of the factors for the root.

(2.) According to 10 of Sec. I., $\sqrt[n]{A^n}$ is used to signify the n^{th} root of A^n, $\sqrt{}$ being called the surd sign, and n written above it is called its index or exponent, since it indicates what root of the number or quantity under the surd is to be extracted.

Because $\sqrt[n]{A^n} = A$, it follows that the extraction of the n^{th}

root of A^n is equivalent to freeing A of its index n, or consists in reducing the index to unity; consequently, since $A^{\frac{n}{n}} = A^1 = A$, we may also express the n^{th} root of A^n by $A^{\frac{n}{n}}$, or by $(A^n)^{\frac{1}{n}}$, supposing the meaning to be, that the index of A is to be multiplied by $\frac{1}{n}$; hence, $\sqrt[n]{A^n}$ and $(A^n)^{\frac{1}{n}}$ are equivalent expressions, and so are $\sqrt{(ab)}$ and $(ab)^{\frac{1}{2}}$, since ab is to be considered and treated as being an exact square; and in like manner $\sqrt[n]{(a+b-c)^m}$ and $[(a+b-c)^m]^{\frac{1}{n}}$ are equivalent expressions, which signify the n^{th} root of the m^{th} power of $a+b-c$ or the m^{th} power of the n^{th} root of $a+b-c$, as is clear by putting $a+b-c=x^n$ or $x=(a+b-c)^{\frac{1}{n}}=$ the n^{th} root of $a+b-c$.

(3.) If we represent the n^{th} root of A^n by x, we shall have $x^n = A^n$ or $x^n - A^n = 0$, (1), and according to what has heretofore been shown (regarding x as the independent variable), we shall have $nx^{n-1} = 0$ or $x^{n-1} = 0$, (2), for the first derived equation of (1); where it may be noticed that (1) is called a *binomial equation*, since its first member consists of only two terms.

Since the first member of (2) is the $(n-1)^{th}$ power of x, it clearly has $n-1$ roots, whose common value is 0; consequently, (1) can not have more than one positive nor more than one negative root; for if it can, then 0, a root of (2), must lie between every successive two of them, which is impossible.

Hence, and from what has heretofore been shown, if n is an odd number, it results that (1) can have only one real root, whose sign is contrary to that of $-A^n$, or the same as that of A^n; consequently (1) must have $n-1$ impossible or imaginary roots.

Similarly, if n is an even number and A^n positive, (1) has only two real roots, one of which is positive and the other negative; consequently, the remaining $n-2$ roots must be imaginary; also, n still being an even number, if A^n is negative; then, since (1) can not have an even number of positive or negative roots, it follows that its roots must all be imaginary.

(4.) If $n = pq$, (1) becomes $x^{pq} - A^{pq} = 0$ or $x^{pq} = A^{pq}$, (3); which, by extracting the q^{th} root, gives $x^p = A^p$, and thence by extracting the p^{th} root we get $x = A$; and if we first extract the p^{th} root, and then the q^{th} root of the result, we shall still have $x = A$; the same result that we shall get from the extraction of the pq^{th} root of the members of (3).

Hence, when the index of the root is composite or consists of factors, we may clearly extract the root denoted by any one of the factors, and then taking any one of the remaining factors as an index, we may extract the root of the root found, and so on until all the factors have been used; then the final root will be the same that we shall get from the extraction of the root denoted by the given index.

Thus, for $\sqrt[4]{16}$ we may write $\sqrt{\sqrt{16}} = 2 =$ the fourth root of 16; and for $\sqrt[6]{64}$ we may write $\sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2$ or $\sqrt{\sqrt[3]{64}} = \sqrt{4} = 2 =$ the sixth root of 64.

(5.) It is clear that $A^{\frac{1}{n}}$, which expresses the n^{th} root of A, may be written in the form $A^{\frac{1}{n}} = A^{\frac{1}{n+p}} \times A^{\frac{p}{(n+p)^2}} \times A^{\frac{p^2}{(n+p)^3}} \times$ etc., (1); for by adding the indices of A, in the right member of the equation, their sum becomes $\frac{1}{n+p} + \frac{p}{(n+p)^2} + \frac{p^2}{(n+p)^3} + \text{etc.} = \frac{1}{n+p-p} = \frac{1}{n}$, so that the members of (1) become identical.

If $n = 3$ and $p = 1$, (1) becomes $A^{\frac{1}{3}} = A^{\frac{1}{4}} \times A^{\frac{1}{16}} \times A^{\frac{1}{64}}$, $\times$ etc., consequently the extraction of the cube root can be reduced to that of fourth roots, and of course to that of square roots, and their product continued indefinitely; thus, if $A = 2$, we get (by using eight factors in the preceding product) $2^{\frac{1}{3}} = 1.2599 =$ the cube root of 2, correct to four decimals.

It is evident that p, in (1), may be positive or negative, provided that the indices of A are sensibly less than unity.

It is clear, from what has been done, that the extraction of the n^{th} root may be reduced to the extraction of square roots and their products.

(6.) Supposing r^n to be an exact power of r (r being rational, at least in one of its letters), and that r is not the product of rational factors, then if for r we put $r + y$, we shall get $(r + y)^n = r^n + nr^{n-1}y$ + etc.; consequently, if we find the greatest common divisor of r^n and nr^{n-1} (the coefficient of y, which is the first derived function of r^n), we shall get r^{n-1}, and then if we divide r^n by r^{n-1}, we shall get r, the root of r^n.

EXAMPLES.

Ex. 1.—To find the square roots of $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$.

Putting $a + y$ for a in these expressions, we get $2(a + b)$ and $2(a - b)$ for the coefficients of y; consequently, we shall get $a + b$ and $a - b$ for the square roots, which may clearly be taken with the sign + or −.

Ex. 2.—To find the square root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

Putting $a + y$ for a, we get $4a^3 - 6a^2x + 6ax^2 - 2x^3$ for the coefficient of y; also, putting $x + y$ for x, we get $4x^3 - 6ax^2 + 6a^2x - 2a^3$. Because the sought root is clearly the greatest common divisor of these coefficients, we add half the second taken in a reverse order to the first, and thence get $4a^3 - 6a^2x + 6ax^2 - 2x^3 - a^3 + 3a^2x - 3ax^2 + 2x^3 = 3a^3 - 3a^2x + 3ax^2 = 3a(a^2 - ax + x^2)$; consequently, $\pm(a^2 - ax + x^2)$ equals the required root.

Ex. 3.—To extract the cube root of $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.

Putting $a + y$ for a, we get $6(a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1)$ for the coefficient of y; consequently, dividing the given expression by $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$, we get $a - 1$ for the quotient, which is clearly the sixth root of the given expression, and of course $(a - 1)^2 = a^2 - 2a + 1$ must be the cube root, as required.

Ex. 4.—To find the square root of $x^2z^2 - 2x^2z + x^2 - 2xz^2 + 4xz - 2x + z^2 - 2z + 1$.

Putting $x + y$ for x, we get $2xz^2 - 4xz + 2x - 2z^2 + 4z - 2 = 2(xz^2 - 2xz + x - z^2 + 2z - 1)$ for the coefficient of y; consequently, dividing the given expression by $xz^2 - 2xz + x - z^2 + 2z - 1$, we get $x - 1$ for one of the factors of the root; and putting $z + y$ for z, by proceeding as before, we get $z-1$ for the other factor of the root, so that the sought root equals $(x-1)(z-1)=xz-x-z+1$.

Hence, when the root r of r^n is composed of independent factors, we perceive how we must proceed in order to find them.

Remark.—It is clear that the preceding method of finding the roots of polynomials is essentially the same as that of finding the equal roots of equations.

(7.) From what has been done, we deduce the following rule for finding any root of a monomial quantity.

RULE.

1. *Resolve or separate the quantity into as many equal factors as there are units in the index or number which denotes the root to be extracted, then one of the equal factors will be the required root. If the monomial has a coefficient, we may extract its root, as in Arithmetic; then the root thus found, being prefixed as a coefficient to the product of the letters in the quantity, after their indices have been severally divided by the index of the root, will be the root, as required, which is clearly the same as to take the product of the roots of all the factors of the quantity for its root.*

2. *If the monomial is a fraction, the root of the numerator, when divided by that of the denominator, will be the root, as required.*

3. *If the index of the root is an odd number, the sign of the root must be the same as that of the given monomial; but if the index of the root is an even number, the root must have the ambiguous sign ± prefixed to it, to show that it may be either positive or negative.*

If the monomial is negative, and the index of the root an even number, then the root will clearly be an imaginary quantity.

EXAMPLES.

1. To extract the square root of $9a^2$.

Because $9a^2 = 3a \times 3a = -3a \times -3a$, the root equals $\pm 3a$.

2. To find the cube root of $64a^3b^6$, and the fifth root of $-243a^5b^{10}c^{15}$.

Because $64a^3b^6 = 4ab^2 \times 4ab^2 \times 4ab^2$, and $-243a^5b^{10}c^{15} = -3ab^2c^3 \times -3ab^2c^3 \times -3ab^2c^3 \times -3ab^2c^3 \times -3ab^2c^3$, the roots are $4ab^2$ and $-3ab^2c^3$.

Otherwise, since $\sqrt[3]{64a^3b^6} = \sqrt[3]{4^3a^3b^6} = 4^{\frac{3}{3}}a^{\frac{3}{3}}b^{\frac{6}{3}} = 4ab^2$, and $\sqrt[5]{-243a^5b^{10}c^{15}} = -\sqrt[5]{3^5a^5b^{10}c^{15}} = -3^{\frac{5}{5}}a^{\frac{5}{5}}b^{\frac{10}{5}}c^{\frac{15}{5}} = -3ab^2c^3$, we have $4ab^2$ and $-3ab^2c^3$ for the roots, as before.

3. To find the square root of $\frac{9a^2b^4}{16c^4d^2}$, and the cube root of $-\frac{27a^6b^9c^{12}}{125d^{15}e^{18}}$.

Here we have $\pm\sqrt{\frac{9a^2b^4}{16c^4d^2}} = \pm\frac{\sqrt{9a^2b^4}}{\sqrt{16c^4d^2}} = \pm\frac{3ab^2}{4c^2d}$, and $\sqrt[3]{-\frac{27a^6b^9c^{12}}{125d^{15}e^{18}}} = -\frac{\sqrt[3]{27a^6b^9c^{12}}}{\sqrt[3]{125d^{15}e^{18}}} = -\frac{3a^2b^3c^4}{5d^5e^6}$.

4. To find the square root of $4a^3b^5$, and the cube root of $81a^4b^7$.

Here $\pm\sqrt{4a^3b^5} = \pm\sqrt{4a^2b^4 \times ab} = \pm\sqrt{4a^2b^4} \times \sqrt{ab} = \pm 2ab^2\sqrt{ab}$, and $\sqrt[3]{81a^4b^7} = \sqrt[3]{27a^3b^6 \times 3ab} = \sqrt[3]{27a^3b^6} \times \sqrt[3]{3ab} = 3ab^2\sqrt[3]{3ab}$.

Remarks.—These roots are in part freed from the signs of the roots, and in part affected by them; consequently, because the roots can not be wholly freed from the signs of the roots (or from the radical signs), they are called *surds or irrational quantities.*

Also, because $2ab^2\sqrt{ab} = 2ab^2 \times \sqrt{ab}$, and $3ab^2\sqrt[3]{3ab} = 3ab^2 \times \sqrt[3]{3ab}$, $2ab^2$, and $3ab^2$, which are called the rational factors of the roots, are sometimes called the *coefficients* of $\sqrt{ab}$ and $\sqrt[3]{3ab}$, which are their irrational or surd factors.

5. To find the square root of $-9a^2b^4$, and the sixth root of $-64a^{12}b^{24}$.

Here $\pm\sqrt{-9a^2b^4} = \pm\sqrt{9a^2b^4} \times \sqrt{-1} = \pm 3ab^2\sqrt{-1}$, and $\pm\sqrt[6]{-64a^{12}b^{24}} = \pm\sqrt{\sqrt[3]{-64a^{12}b^{24}}} = \pm\sqrt{-4a^4b^8} = \pm\sqrt{4a^4b^8} \times \sqrt{-1} = \pm 2a^2b^4\sqrt{-1}$.

Remark.—These roots are *impossible or imaginary*, because $\sqrt{-1}$ is *impossible in real numbers;* and $3ab^2$, $2a^2b^4$ are called the *coefficients* of $\sqrt{-1}$, in the roots.

6. To find the square root of $a^2b^3c^5 = a^2b^2c^4 \times bc$.

Ans. $\pm abc^2\sqrt{bc}$.

7. To find the fourth root of $a^{4m}b^{8n}c^m$. *Ans.* $\pm a^m b^{2n} c^{\frac{m}{4}}$.

8. To find the cube root of $2ab^5c^8$. *Ans.* $bc^2\sqrt[3]{2ab^2c^2}$.

9. To find the fifth root of $-32a^{12}b^{\frac{15}{2}}c^{\frac{13}{4}}$.

Ans. $-2a^2a^{\frac{2}{5}}b^{\frac{3}{2}}c^{\frac{13}{20}}$.

10. To find the sixth root of $64a^{17}b^{24}c^{30}$. *Ans.* $\pm 2a^2b^4c^5a^{\frac{5}{6}}$.

11. To find the m^{th} root of $\frac{3^{mn}a^{pm}b^{2m}}{c^{qm}d^{em}}$. *Ans.* $\frac{3^n a^p b^2}{c^q d^e}$.

12. To find the fourth root of $\frac{16a^{-8}}{81b^{-12}}$.

Ans. $\pm \frac{2a^{-2}}{3b^{-3}} = \pm \frac{2b^3}{3a^2}$.

(8.) Before giving the usual rules for the extraction of the roots of compound or polynomial quantities, we will premise the following

PROPOSITION.

Any expression of the form $1 + \frac{b}{a}$, in which a and b have like or unlike signs, may be resolved into factors of a similar form.

For if n stands for any positive whole number, it is clear that we shall have $1 + \frac{b}{a} = \frac{na+b}{na} \times \frac{na+2b}{na+b} \times \frac{na+3b}{na+2b} \times \ldots\ldots \times \frac{na+nb}{na+(n-1)b} = \left(1 + \frac{b}{na}\right) \times \left(1 + \frac{b}{na+b}\right) \times \left(1 + \frac{b}{na+2b}\right) \times \ldots\ldots \times \left(1 + \frac{b}{na+(n-1)b}\right)$ to n factors, (1), as required

Hence, having reduced any number or quantity to the form $A^n + B$, such that A^n is an exact n^{th} power of A, we shall have $A^n + B = A^n\left(1 + \frac{B}{A^n}\right) = A^n \times \left(1 + \frac{B}{nA^n}\right) \times$

$\left(1+\frac{B}{nA^n+B}\right)\times\left(1+\frac{B}{nA^n+2B}\right)\times\ldots\ldots\times\left(1+\frac{B}{nA^n+(n-1)B}\right)$ to n factors, (2).

If B is very small in comparison to A^n, then by omitting the terms which contain B in the denominators of the factors of the right member of (2), we shall approximately have $A^n + B = A^n \times \left(1 + \frac{B}{nA^n}\right)^n$, or, extracting the n^{th} root, we get $\sqrt[n]{A^n + B} = A \times \left(1 + \frac{B}{nA^n}\right) = A + \frac{B}{nA^{n-1}}$, (3).

It is clear from (2) that $\sqrt[n]{A^n + B}$, or the n^{th} root of $A^n +$ B, will be less than $A + \frac{B}{nA^{n-1}}$ and greater than $A + \frac{AB}{nA^n + (n-1)B}$.

To perceive the use of what has been done, take the following

EXAMPLES.

Ex. 1.—To resolve $2 = 1 + 1 = 1 + \frac{1}{1}$ into two factors, and thence find its square root.

Putting $a = 1$, $b = 1$, and $n = 2$, (1) gives $2 = 1 + \frac{1}{1} = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{2} \times \frac{1}{3}$; consequently, $1 + \frac{1}{2} = 1.5$ and $1 + \frac{1}{3} = 1.33\dot{3}.$ are the required factors.

Because the square root of 2 is greater than $1.33\dot{3}.$ and less than 1.5, it is easy to perceive that the first part of it must be 1.41; indeed, if we represent the root by $1 + x$, we shall have $(1 + x)^2 = 2 = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{2} \times \frac{1}{3} = 1 + 0.5 + 0.333\dot{3}. + \frac{1}{6} = 1.83\dot{3}. + \frac{1}{6}$, or since $(1 + x)^2 = 1 + 2x + x^2$, we shall have $1 + 2x + x^2 = 1.83\dot{3}. + 0.16\dot{6}.$, or $2x + x^2 = 0.83\dot{3}. + 0.16\dot{6}.$, which, by rejecting x^2 and $0.16\dot{6}.$ is reduced to $2x = 0.83\dot{3}.$, which gives $x = 0.41 +$, and of course we have $1 + x = 1.41$ for the approximate value of the root, as above.

Since $2 = \frac{(1.41)^2 \times 2}{(1.41)^2} = (1.41)^2 \times \frac{2}{1.9881} = (1.41)^2 \times 1.0059856$, we get, by resolving 1.0059856 into two factors as before, 1.0029928 and 1.0029838 for the factors; consequently (as before) we have $1.0029928 \times 1.0029838 = 1.0029928 + 0.0029838 + = 1.0059766 +$, and $1 + \frac{.0059766}{2} = 1.0029883$ is the approximate square root of 1.0059856. Hence, we shall have $\sqrt{2} = \sqrt{(1.41)^2} \times \sqrt{1.0059856} = 1.41 \times 1.0029883 = 1.4142135 +$ for the square root of 2; which is correct in all its figures.

Ex. 2.—To find the square root of 3.

Since $3 = 1 + \frac{2}{1}$, if we put $a = 1$, $b = 2$, and $n = 2$, (1) gives $3 = (1 + 1)\left(1 + \frac{1}{2}\right)$, and in a similar way $1 + 1 = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)$; consequently, $3 = \left(1 + \frac{1}{2}\right)^2 \times \left(1 + \frac{1}{3}\right)$. Because $1 + \frac{1}{3} = \left(1 + \frac{1}{6}\right)\left(1 + \frac{1}{7}\right) = 1 + \left(\frac{1}{6} + \frac{1}{7}\right) + \frac{1}{6} \times \frac{1}{7}$, we get $1 + \left(\frac{1}{6} + \frac{1}{7}\right) \div 2 = 1.1547$ for the approximate square root of $1 + \frac{1}{3}$; consequently, from $3 = \left(1 + \frac{1}{2}\right)^2\left(1 + \frac{1}{3}\right)$, we shall get $\sqrt{3} = 1.5 \times 1.1547 = 1.73205 +$ for the approximate square root of 3; which is correct in all its figures.

Ex. 3.—To find the square root of 17.3056.

Since 16 is the greatest integral square in 17, we shall put the number in the form $17.3056 = \frac{16 \times 17.3056}{16} = 4^2 \times 1.0816$; consequently, resolving 1.0816 into two factors by (1), we shall have $1.0816 = 1.0408 \times 1.0392 = 1.0408 + 0.0392 + = 1.08 +$; consequently, we shall have $1 + \frac{.08}{2} = 1.04$ for the square root of 1.0816.

Hence, we shall have $\sqrt{17.3056} = \sqrt{4^2} \times \sqrt{1.0816} = 4 \times 1.04 = 4.16$ for the square root of the given number, the root being exact.

Ex. 4.—To resolve 11 into five factors, and thence obtain its fifth root.

Because $11 = 1 + \frac{10}{1}$, if we put $a = 1$, $b = 10$, and $n = 5$,

(1) gives $11 = (1+2)\left(1+\frac{2}{3}\right)\left(1+\frac{2}{5}\right)\left(1+\frac{2}{7}\right)\left(1+\frac{2}{9}\right)$, as required; consequently, the fifth root of 11 lies between the limits 3 and $1 + \frac{2}{9} = 1.22\dot{2}$.

If we multiply the factors in the preceding value of 11 together, we shall have $11 = 1 + \left(2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}\right)$ + the sum of the products of every two of the terms $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}$. + the sum of the products of every three of them, etc. Now if we represent the 5th root of 11 by $1 + x$, we shall have $11 = (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, which is clearly of a form analogous to the preceding value of 11; consequently, if we equate the corresponding terms, we shall get $5x = 2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} = 2 + 0.666\dot{6}. + 0.4 + 0.2857 + 0.222\dot{2}. = 3.5745$, or $x = \frac{3.5745}{5} = 0.7149$.

Again, if we take the sum of the products of every two of the terms $2, \frac{2}{3}, \frac{2}{5}$, etc., we shall have $10x^2 = 2 \times \left(\frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}\right) + \frac{2}{3} \times \left(\frac{2}{5} + \frac{2}{7} + \frac{2}{9}\right) + \frac{2}{5} \times \left(\frac{2}{7} + \frac{2}{9}\right) + \frac{2}{7} \times \frac{2}{9} = 3.1490 + 0.6052 + 0.20316 + 0.0634 = 4.0207$; which clearly shows that the preceding value of x is a little too great, and that the true value of x lies between 0.6 and 0.7, so that 1.6 are clearly the first figures of the root. Since $11 = (1.6)^5 \times \frac{11}{(1.6)^5} = (1.6)^5 \times (1.049041 +)$, we have to extract the root of 1.049041 +, which, by our method, will be found to be 1.00962 +.

Hence, we shall have $\sqrt[5]{11} = 1.6 \times (1.00962 +) = 1.61539 +$ for the fifth root of 11, correctly found to five decimal places.

Ex. 5.—To extract the cube root of 48228544.

Since $48228544 = 1000000 \times \frac{48228544}{1000000} = (100)^3 \times 48.228544$, it is evident that the extraction of the root is reduced to that of 48.228544.

Also, because 27 is the greatest integral cube in 48, and that $48.228544 = 27 \times \frac{48.228544}{27} = 3^3 \times 1.786242 +$, it is clear that the extraction of the root is reduced to that of $1.786 +$.

To extract the root of $1.786 +$, we resolve it into the factors $1.262 +$, $1.207 +$, and $1.171 +$; consequently, we shall have $1 + \frac{0.640}{3}$ for the root.

Hence, we get $48228544 = (100)^3 \times 3^3 \times \left(1 + \frac{0.640}{3}\right)^3 = (100)^3 \times (3.64)^3 = (364)^3$, which gives $\sqrt[3]{48228544} = 364$, the root being exact.

Remark.—It is clear that the reduction of 48228544 to 48.228544 is equivalent to the separation of the number from the right into periods of three figures each, according to the common rule for the extraction of the cube root given in Arithmetic.

Ex. 6.—To find the n^{th} root of $1 + \frac{b}{a}$.

By taking the n^{th} root of the members of (1), we shall have $\left(1 + \frac{b}{a}\right)^{\frac{1}{n}} = \left(1 + \frac{b}{na}\right)^{\frac{1}{n}} \times \left(1 + \frac{b}{na + b}\right)^{\frac{1}{n}} \times \left(1 + \frac{b}{na + 2b}\right)^{\frac{1}{n}} \times \ldots\ldots \times \left(1 + \frac{b}{na + (n-1)b}\right)^{\frac{1}{n}}$, which expresses the n^{th} root, as required.

Because (by (1)) $1 + \frac{b}{na} = \left(1 + \frac{b}{n^2a}\right) \times \left(1 + \frac{b}{n^2a + b}\right) \times \left(1 + \frac{b}{n^2a + 2b}\right) \times \ldots\ldots \times \left(1 + \frac{b}{n^2a + (n-1)b}\right)$, if b is so small in comparison to na that for $n^2a + b$, $n^2a + 2b$, etc., we may write n^2a, n^2a, etc.; then we shall have $1 + \frac{b}{na} = \left(1 + \frac{b}{n^2a}\right)^n$ or $\left(1 + \frac{b}{na}\right)^{\frac{1}{n}} = 1 + \frac{b}{n^2a}$ very nearly. And in like manner we shall have $\left(1 + \frac{b}{na + b}\right)^{\frac{1}{n}} = 1 + \frac{b}{n(na + b)}$ very nearly, and $\left(1+\frac{b}{na+2b}\right)^{\frac{1}{n}}=1+\frac{b}{n(na+2b)}$ very nearly, and so on.

Hence, by substituting the approximate values of $\left(1+\frac{b}{na}\right)^{\frac{1}{n}}$, $\left(1+\frac{b}{na+b}\right)^{\frac{1}{n}}$, we shall get $\left(1+\frac{b}{a}\right)^{\frac{1}{n}}=\left(1+\frac{b}{n^2a}\right) \times \left(1+\frac{b}{n(na+b)}\right) \times \left(1+\frac{b}{n(na+2b)}\right) \times \dots\dots \times \left(1+\frac{b}{n(na+(n-1)b)}\right)$ very nearly.

If we take the product of the factors in the right member of the preceding equation, we shall clearly get $\left(1+\frac{b}{a}\right)^{\frac{1}{n}}=1+\frac{1}{n}\times\left[\frac{b}{na}+\frac{b}{na+b}+\frac{b}{na+2b}+\dots\dots+\frac{b}{na+(n-1)b}\right]$, (2), very nearly; for it is clear that the products $\frac{b}{n^2a}\times\frac{b}{n(na+b)}$, $\frac{b}{n(na+b)}\times\frac{b}{n(na+2b)}$, etc., and $\frac{b}{n^2a}\times\frac{b}{n(na+b)}\times\frac{b}{n(na+2b)}$, etc., and so on, may be rejected on account of their minuteness in comparison to the terms which are retained in (2).

If we convert $\frac{b}{na+b}$, $\frac{b}{na+2b}$, ... $\frac{b}{na+(n-1)b}$ into series arranged according to the ascending powers of b, we shall have $\frac{b}{na+b}=\frac{b}{na}-\frac{b^2}{n^2a^2}+\frac{b^3}{n^3a^3}-\frac{b^4}{n^4a^4}+$ etc., $\frac{b}{na+2b}=\frac{b}{na}-\frac{2b^2}{n^2a^2}+\frac{2^2b^3}{n^3a^3}-\frac{2^3b^4}{n^4a^4}+$ etc., $\frac{b}{na+3b}=\frac{b}{na}-\frac{3b^2}{n^2a^2}+\frac{3^2b^3}{n^3a^3}-\frac{3^3b^4}{n^4a^4}+$ etc., ... $\frac{b}{na+(n-1)b}=\frac{b}{na}-\frac{(n-1)b^2}{n^2a^2}+\frac{(n-1)^2b^3}{n^3a^3}-\frac{(n-1)^3b^4}{n^4a^4}+$ etc.

Hence, by substituting the preceding values, we get $\frac{b}{na}+\frac{b}{na+b}+\frac{b}{na+2b}+\dots\dots+\frac{b}{na+(n-1)b}=\frac{b}{na}+\frac{b}{na}+$

$$\frac{b}{na} + \ldots . n \text{ terms} - \frac{b^2}{n^2a^2}\,[1 + 2 + 3 + 4 + \ldots . + n - 1] + \frac{b^3}{n^3a^3}\,[1^2 + 2^2 + 3^2 + \ldots . + (n-1)^2] - \frac{b^4}{n^4a^4}\,[1^3 + 2^3 + 3^3 + \ldots . + (n-1)^3] + \frac{b^5}{n^5a^5}\,[1^4 + 2^4 + 3^4 + \ldots . + (n-1)^4] - \text{etc.} = \frac{nb}{na} - \frac{b^2}{n^2a^2} \times \frac{n(n-1)}{2} + \frac{b^3}{n^3a^3} \times \frac{n(1-n)(1-2n)}{2 \times 3} - \frac{b^4}{n^4a^4} \times \frac{n^2(n-1)^2}{4} + \text{etc.} = \frac{b}{a} + \frac{\left(\frac{1}{n} - 1\right)}{2}\frac{b^2}{a^2} + \frac{\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{2 \,.\, 3}\frac{b^3}{a^3} - \frac{\left(\frac{1}{n} - 1\right)^2}{4}\frac{b^4}{a^4} + \text{etc.}$$

Substituting the value of $\frac{b}{na} + \frac{b}{na+b} + \text{etc.}$, in (2), we shall get $\left(1 + \frac{b}{a}\right)^{\frac{1}{n}} = 1 + \frac{1}{n}\frac{b}{a} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1 \,.\, 2}\frac{b^2}{a^2} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{1 \,.\, 2 \,.\, 3}\frac{b^3}{a^3} - \frac{\left(\frac{1}{n} - 1\right)^2}{4n}\frac{b^4}{a^4} + \text{etc.}$, (3), very nearly.

If we expand $\left(1 + \frac{b}{a}\right)^{\frac{1}{n}}$ by the Binomial Theorem, we shall get $\left(1 + \frac{b}{a}\right)^{\frac{1}{n}} = 1 + \frac{1}{n}\frac{b}{a} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1 \,.\, 2}\frac{b^2}{a^2} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{1 \,.\, 2 \,.\, 3}\frac{b^3}{a^3} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\left(\frac{1}{n} - 3\right)}{1 \,.\, 2 \,.\, 3 \,.\, 4}\frac{b^4}{a^4} + \text{etc.}$, (4), whose first four terms agree with those of (3).

Since n is supposed to be a whole number not less than 2, and that in (2) b is regarded as being small in comparison to na, it is clear $\left(\text{since } \frac{b}{na} \text{ is generally much smaller than } \frac{b}{a}\right)$ that when $\frac{b}{a}$ is very small, it will be much more exact to find

$\left(1 + \frac{b}{a}\right)^{\frac{1}{n}}$ from (2) than to take the first four terms of (4) for its value.

Again, if we suppose n to be a very great number, it is plain that for $\frac{1}{n} - 1$, $\frac{1}{n} - 2$, $\frac{1}{n} - 3$, etc., we may, without sensible error, write -1, -2, -3, etc.; consequently, when n is very great, (3) or (4) will be reduced to $\left(1 + \frac{b}{a}\right)^{\frac{1}{n}} = 1 + \frac{1}{n}\left(\frac{b}{a} - \frac{b^2}{2a^2} + \frac{b^3}{3a^3} - \frac{b^4}{4a^4} + \text{etc.}\right)$ very nearly, (5), which shows that if n is infinitely great, (2) will be as exact as (4) continued to infinity.

By putting $h = \frac{b}{a} - \frac{b^2}{2a^2} + \frac{b^3}{3a^3} - \frac{b^4}{4a^4} + \text{etc.}$, (6), (5) will become $\left(1 + \frac{b}{a}\right)^{\frac{1}{n}} = 1 + \frac{h}{n}$, (7), supposing n to be unlimitedly great.

If we raise the members of (7) to the mn^{th} power, it will become $\left(1 + \frac{b}{a}\right)^{m} = \left(1 + \frac{h}{n}\right)^{mn}$, or expanding the right member of the equation as in (4), we shall get $\left(1 + \frac{b}{a}\right)^{m} = 1 + mh + \frac{m(mn-1)}{1\,.\,2}\frac{h^2}{n} + \frac{m(mn-1)(mn-2)}{1\,.\,2\,.\,3}\frac{h^3}{n^2} +$, etc.; or since we must clearly write mn for $mn - 1$, mn for $mn - 2$, etc., so that the second member may (like the first member) be independent of n, we shall have $\left(1 + \frac{b}{a}\right)^{m} = 1 + mh + \frac{m^2h^2}{1\,.\,2} + \frac{m^3h^3}{1\,.\,2\,.\,3} + \frac{m^4h^4}{1\,.\,2\,.\,3\,.\,4} + \text{etc.}$, (8), which is called the *Exponential Theorem*, in which any real positive or negative number may evidently be used for m, according to the nature of the case.

In order to apply (8) to the extraction of a root of $1 + \frac{b}{a}$, it will be necessary to find h, which is called the Hyperbolic or Napierian Logarithm of $\left(1 + \frac{b}{a}\right)$ from (6), and to substitute it for h.

Thus, to extract the 31st root of 5, we get from the tables of hyperbolic logarithms $h = 1.60943791$, and putting $\frac{1}{31}$ for m, wo shall have $mh = \frac{1.60943791}{31} = 0.05191735$.

Hence, from (8) we get $5^{\frac{1}{31}} = 1 + 0.05191735 + 0.00134770 + 0.00002332 + 0.00000029 = 1.0532886 +$ for the root, which is correct to seven decimal places.

Because we may proceed in a similar way to extract any root of a number, it follows that the extraction of the roots of numbers is reduced to the calculation of their hyperbolic logarithms.

Also, since the common logarithm of a number multiplied by 2.30258, 50929, 94045, 68402 (the hyperbolic logarithm of 10) equals the hyperbolic logarithm of the number, it results that the extraction of any root of a number is reduced to the calculation of the common logarithm of the number.

(9.) We will now proceed to show how to extract the n^{th} root of a polynomial, regarded as being an exact n^{th} power.

Supposing the polynomial to be arranged according to the descending or ascending powers of one of its letters, and that A^n denotes the n^{th} power of the sum of any number of terms of the root supposed to have been found, then if we subtract A^n from the polynomial, and denote the remainder by B, it is clear that the polynomial will be reduced to the form $A^n + B$, (1), which agrees with the form given in (2), at page 289, whose approximate n^{th} root was there shown to be expressed by $A + \frac{B}{nA^{n-1}}$, (2).

Now, since the first or left-hand term of the arranged polynomial must clearly be (or be regarded as being) an exact n^{th} power, we may put it for A^n, and placing its n^{th} root in the form of a quotient to the polynomial written as a dividend, by subtracting the n^{th} power of the quotient or root found from the polynomial, the remainder will be expressed by B; then, dividing the first or left-hand term of B by nA^{n-1}, we shall get the second term of the root, which must be put for the second term of the quotient.

If we now denote the sum of the terms of the root found (taken with their proper signs) by A, and subtract A^n from the (arranged) polynomial, the first term of the remainder, divided by the same divisor as before, will give the third term of the root.

If we now represent the sum of the three terms of the root found by A, and proceed as before (by using the same divisor), we shall get the fourth term of the root, and so on, to any extent that may be required.

Hence, the n^{th} root of a polynomial may be found by the following

RULE.

1. *Arrange the terms of the polynomial according to the descending or ascending powers of one of its letters, and extract the n^{th} root of its first or left-hand term, and place it in the quotient for the first term of the root, and subtract the n^{th} power of the first term of the root from the polynomial, and bring down the first term of the remainder (to the place of the remainder) for a dividend.*

2. *Raise the first term of the root to the power whose exponent is $n-1$, and multiply the power by n, for the* constant divisor; *then divide the dividend by the constant divisor, and put the result in the quotient for the second term of the root, and subtract the n^{th} power of the root now found from the polynomial, and bring down the first term of the remainder for a second dividend.*

3. *Divide the second dividend by the constant divisor for the third term of the root; then proceed in the same way as before, with the root now found and the constant divisor, to find the fourth term of the root, and so on, to any extent required.*

EXAMPLES.

1. To extract the cube root of $x^6+6x^5+96x-64-40x^3$.

Arranging the terms according to the ascending powers of x, since $n=3$, and $n-1=2$, we shall get

$$-64+96x-40x^3+6x^5+x^6 \,\underline{|-4+2x+x^2}= \text{the root.}$$

$$\underline{-64}$$

$$\text{const. div. } \overline{=48 \mid 96x} = \text{first dividend.}$$

$$\overline{\underline{-64+96x-48x^2+8x^3}}$$

$$48 \mid 48x^2 = \text{the second dividend.}$$

$$\overline{-64+96x-40x^3+6x^5+x^6}$$

Here, because $-64 = -(4)^3$, we have -4 for the first term of the root, and $3(-4)^2 = 48$ is the constant divisor.

Subtracting -64 from the polynomial, and bringing down the first term of the remainder, we get $96x$ for the dividend, which, divided by the constant divisor, gives $2x$ for the second term of the root. Then, subtracting $(-4 + 2x)^3 = -64 + 96x - 48x^2 + 8x^3$ from the polynomial, we get $48x^2$ for a second dividend, which, divided by the constant divisor, gives x^2 for the third term of the root.

And since $(-4 + 2x + x^2)^3 = -64 + 96x - 40x^3 + 6x^5 + x^6$, when subtracted from the polynomial, gives 0 for the remainder, it follows that the root is exact.

Remark.—If we take the first derived function of the given expression, and reject the factor 6, which is common to its terms, we shall get $x^5 + 5x^4 - 20x^2 + 16$; then $x^4 + 4x^3 - 4x^2 - 16x + 16$ will be the greatest common divisor of this and the given expression.

Hence, if we divide the given expression by the greatest common divisor, we shall find $x^2 + 2x - 4 = -4 + 2x + x^2$ for the cube root of the given expression, as we clearly ought to do, according to what has heretofore been shown.

2. To find the square root of $4x^4 - 4ax^3 - 3a^2x^2 + 2a^3x + a^4$.

If we extract the root of the polynomial under its given form, we shall get $2x^2 - ax - a^2$ for the root; but if we write the terms in a reverse order, and then extract the root, it will be found to be $a^2 + ax - 2x^2$; consequently, the expression admits of two square roots, which may be expressed by $\pm(2x^2 - ax - a^2)$, by using $+$ for $\pm$ for one of the roots, and using $-$ for $\pm$ for the other root.

3. To find the fourth root of $81x^8 - 216x^7 + 336x^5 - 56x^4 - 224x^3 + 64x + 16$. *Ans.* $\pm(3x^2 - 2x - 2)$.

4. To find the square root of $a^2 - 2ab - 2ac + b^2 + 2bc + c^2$. *Ans.* $\pm(a - b - c)$.

5. To find the cube root of $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$. *Ans.* $x^2 - 2ax + a^2$.

6. To find the fourth root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$. *Ans.* $\pm(2a - 3x)$.

Remark.—The answer may be found by extracting the square root of the square root of the given expression, as is clear, since 4, the index of the required root, equals 2×2.

7. To find the fifth root of $-1 + 10x - 40x^2 + 80x^3 - 80x^4 + 32x^5$. *Ans.* $-1 + 2x$.

8. To find the cube root of $a^3 - 6a + 12a^{-1} - 8a^{-3}$.

Ans. $a - \frac{2}{a}$.

9. To extract the square root of $a^{2m} - 6a^{2m-p}x^p + 4a^m x^m + 9a^{2m-2p}x^{2p} - 12a^{m-p}x^{m+p} + 4x^{2m}$. *Ans.* $a^m - 3a^{m-p}x^p + 2x^m$.

10. To extract the fourth root of $\frac{3x^4+12x^3+18x^2+12x+3}{3x^4-12x^3+18x^2-12x+3}$.

Reducing the expression to its lowest terms, and dividing the root of its numerator by that of its denominator, the root will equal $\frac{1+x}{1-x}$.

11. To extract the square root of $1 + \frac{4ab}{a^2 - 2ab + b^2}$.

Reducing the expression to the form of an improper fraction, and extracting the root of its numerator and denominator, the answer will be $\frac{a+b}{a-b}$.

12. To extract the square root of $4 - \frac{4a}{2a-x} + \frac{a^2}{4a^2-4ax+x^2}$.

Ans. $\frac{3a-2x}{2a-x}$.

13. To extract the cube root of $a^3 - \frac{3a^2c}{b+d} + \frac{3ac^2}{(b+d)^2} - \frac{c^3}{(b+d)^3}$. *Ans.* $a - \frac{c}{b+d} = \frac{a(b+d)-c}{b+d}$.

Remarks.—1st. If we have to extract the n^{th} root of an expression of the form $a^n \pm b$, whose root is not exact, then by putting $\pm b = a^n x$, or $x = \pm \frac{b}{a^n}$, we shall get $a^n \pm b = a^n + a^n x = a^n(1 + x)$, or $\sqrt[n]{a^n \pm b} = a\sqrt[n]{1+x} = a(1+x)^{\frac{1}{n}}$; consequently, the extraction of the root is reduced to the extraction of the n^{th} root of $1 + x$. Thus, to find the square root of 38, since $38 = 36 + 2 = 36\left(1 + \frac{2}{36}\right) = 6^2\left(1 + \frac{1}{18}\right)$, we shall have $\sqrt{38} = 6\left(1 + \frac{1}{18}\right)^{\frac{1}{2}}$; also, to get the cube root of 24, we have $24 = 27 - 3 = 3^3\left(1 - \frac{1}{9}\right)$, or $\sqrt[3]{24} = 3\left(1 - \frac{1}{9}\right)^{\frac{1}{3}}$.

2d. We will now apply our rule to the extraction of the n^{th} root of $1 + x$, supposing the terms to be arranged according to the ascending powers of x.

Thus, since 1 is the n^{th} root of 1, we shall have 1 for the first term of the n^{th} root of $1 + x$, and n will evidently be the constant divisor. Then, dividing x by n, we shall have $\frac{x}{n}$ for the second term of the root, and of course $1 + \frac{x}{n}$ are the first two terms of the root.

Raising $1 + \frac{x}{n}$ to the n^{th} power by the binomial theorem, we shall have $\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{n(n-1)}{1\ .\ 2}\frac{x^2}{n^2} + \text{etc.} = 1 + x + \frac{\left(1 - \frac{1}{n}\right)}{1\ .\ 2}x^2 + \text{etc.}$, which, subtracted from $1 + x$, gives $\frac{\frac{1}{n} - 1}{1\ .\ 2}x^2$ for the first term of the remainder, which, divided by n, gives $\frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1\ .\ 2}x^2$ for the third term of the root; consequently, $1 + \frac{x}{n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1\ .\ 2}x^2$ are the first three terms of the root.

To find the next term of the root, we may treat $1 + \frac{x}{n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1\ .\ 2}x^2$ as being a binomial whose first term is 1, and second term $\frac{x}{n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1\ .\ 2}x^2$; then, since it will clearly suffice to notice only those terms which contain x^3, we shall have

$\left(1+\frac{x}{n}+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\,.\,2}x^2\right)^n=\frac{n(n-1)}{1\,.\,2}\times\frac{2x}{n}\times\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\,.\,2}x^3+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\frac{x^3}{n^3}=\frac{n(n-1)}{1\,.\,2}(1-n)\frac{x^3}{n^3}+\frac{n(n-1)(n-2)x^3}{1\,.\,2\,.\,3\;\;n^3}$ $=\frac{\left(1-\frac{1}{n}\right)\left(\frac{1}{n}-2\right)}{2\,.\,3}x^3$; consequently, subtracting this from $1+x$, and dividing the remainder (in x^3) by n, we shall get $\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1\,.\,2\,.\,3}x^3$ for the fourth term of the root.

Hence, we shall clearly have $1+\frac{1}{n}x+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\,.\,2}x^2+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\,.\,2}\times\frac{\left(\frac{1}{n}-2\right)}{3}x^3$ + etc. for the n^{th} root of $1+x$, which is the same that the expansion of $(1+x)^{\frac{1}{n}}$, according to the ascending powers of x by the binomial theorem, will give; consequently, since the root of $1+x$ can more readily be found by the binomial theorem than by our rule, it ought to be preferred in practice.

3d. To perceive the use of the binomial theorem in the extraction of the roots of expressions of the form $1+x$, take the following

EXAMPLES.

Ex. 1.—To extract the square root of 38.

Since $\sqrt{38}=6\left(1+\frac{1}{18}\right)^{\frac{1}{2}}$, if in $(1+x)^{\frac{1}{n}}=1+\frac{1}{n}x+\frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\,.\,2}x^2$ + etc., we put 2 for n, and $\frac{1}{18}$ for x, we shall get $\sqrt{38}=6\left(1+\frac{1}{2}\times\frac{1}{18}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\,.\,2}\times\frac{1}{18^2}+\text{etc.}\right)=6(1+0.02777\dot{7}.-0.00038580+0.00001071-0.00000037+\text{etc.})=6.1644138$ +, which is correct to seven decimal places.

Ex. 2.—To extract the cube root of $124 = 125 - 1 = 125 \times \left(1 - \frac{1}{125}\right)$.

Here, by using 3 for n, and $-\frac{1}{125}$ for x, we shall get

$\sqrt[3]{124} = 5\left(1 - \frac{1}{125}\right)^{\frac{1}{3}} = 5(1 - 0.002666\dot{6}. - 0.00000711 -$ etc.) $= 4.9866313$, which is correct to seven decimal places.

Thus, we have in this question found the value of $\sqrt[3]{124}$ more readily than that of $\sqrt{38}$ in the preceding question, because $\frac{1}{125}$ is much smaller than $\frac{1}{18}$ in comparison to 1 or unity, so that the terms of the series diminish much more rapidly in this than in the preceding question.

Ex. 3. To extract the fourth root of 3.

Since $3 = 1 + 2 = \left(1 + \frac{1}{3}\right) \cdot \left(1 + \frac{1}{4}\right) \cdot \left(1 + \frac{1}{5}\right) \cdot \left(1 + \frac{1}{6}\right) \cdot \left(1 + \frac{1}{7}\right) \cdot \left(1 + \frac{1}{8}\right)$, by extracting the fourth root we have

$$\sqrt[4]{3} = \left(1 + \frac{1}{3}\right)^{\frac{1}{4}} \cdot \left(1 + \frac{1}{4}\right)^{\frac{1}{4}} \cdot \left(1 + \frac{1}{5}\right)^{\frac{1}{4}} \cdot \left(1 + \frac{1}{6}\right)^{\frac{1}{4}} \cdot \left(1 + \frac{1}{7}\right)^{\frac{1}{4}} \cdot \left(1 + \frac{1}{8}\right)^{\frac{1}{4}}.$$

Hence, extracting the roots of the binomials in the right member of the equation by the binomial theorem to six decimal places, and then taking their products, we get $\sqrt[4]{3} = 1.31607$ correct to five decimal places.

Remarks.—1st. We have given the preceding question for the purpose of showing how the binomial theorem may be applied to the extraction of any root of a number. For since any whole number can clearly be resolved into binomial factors whose first terms are 1, and second terms are very small in comparison to 1, it is evident that, by extracting the roots of the factors, and then taking their product, we shall get the root of the proposed number.

Thus, since $20 = 16 + 4 = 16\left(1 + \frac{1}{4}\right) = (1 + 1)^4\left(1 + \frac{1}{4}\right) =$

$\left(1+\frac{1}{3}\right)^4 \cdot \left(1+\frac{1}{4}\right)^5 \cdot \left(1+\frac{1}{5}\right)^4$, we shall get $\sqrt[n]{20} = \left(1+\frac{1}{3}\right)^{\frac{4}{n}} \cdot \left(1+\frac{1}{4}\right)^{\frac{5}{n}} \cdot \left(1+\frac{1}{5}\right)^{\frac{4}{n}} = \left(\left(1+\frac{1}{3}\right)^{\frac{1}{n}}\right)^4 \cdot \left(\left(1+\frac{1}{4}\right)^{\frac{1}{n}}\right)^5 \cdot \left(\left(1+\frac{1}{5}\right)^{\frac{1}{n}}\right)^4$ for an expression for the n^{th} root of 20; consequently, if we put particular numbers for n, by finding the values of $\left(1+\frac{1}{3}\right)^{\frac{1}{n}}$, $\left(1+\frac{1}{4}\right)^{\frac{1}{n}}$, and $\left(1+\frac{1}{5}\right)^{\frac{1}{n}}$ by the binomial theorem, we shall readily get the roots of 20, which correspond to the particular values of n.

2d. It may not be improper to give in this place the following method of approximating to the root of a number of the form $1 + \frac{b}{a}$, in which $\frac{b}{a}$ is very small in comparison to 1.

Thus, supposing n to denote the root to be extracted, we shall, by the binomial theorem, have $\sqrt[n]{1 + \frac{b}{a}} = 1 + \frac{b}{na} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1 \,.\, 2}\frac{b^2}{a^2} + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1 \,.\, 2 \,.\, 3}\frac{b^3}{a^3} +$ etc., (1).

If we put $A = na + \frac{n-1}{2}b + \frac{1-n^2}{3 \,.\, 4}\frac{b^2}{na} + \frac{n(n^2-1)}{2 \,.\, 3 \,.\, 4}\frac{b^3}{n^2a^2}$ + etc., (2), which is of the form $A = na + A_1 b + A_2\frac{b^2}{na} + A_3\frac{b^3}{n^2a^2} + A_4\frac{b^4}{n^3a^3}$ + etc., (3), such that the coefficient of $\frac{b^m}{n^{m-1}a^{m-1}}$ is given by the equation $A_m + A_{m-1}\left(\frac{1-n}{2}\right) + A_{m-2}\left(\frac{(1-n)(1-2n)}{2 \,.\, 3}\right) + A_{m-3}\left(\frac{(1-n)(1-2n)(1-3n)}{2 \,.\, 3 \,.\, 4}\right) + \ldots + A_1\left(\frac{(1-n)(1-2n)(1-3n) \times \ldots \times (1-(m-1)n)}{2 \,.\, 3 \,.\, 4 \times \ldots \times m}\right) + \frac{(1-n)(1-2n)(1-3n) \times \ldots \times (1-mn)}{2 \,.\, 3 \,.\, 4 \times \ldots \times (m+1)} = 0$, (4); then (1) will be reduced to $\sqrt[n]{1+\frac{b}{a}} = 1 + \frac{b}{A}$, (5), since b divided by the value of A will make the right member of (5) identical with that of (1).

It is clear that we may put 1 for a in (2) and (5), provided $\frac{b}{a}$ is put for b.

Because n is supposed to be a whole number not less than 2, if a and b have like signs, and the terms of (2) decrease in a high ratio from left to right, it is clear that if we take an odd number of the terms of (2) for A, the root as found from (5) will be too great, while an even number of the terms of (2) being taken for A will make the root too small.

It may be noticed if $1 + \frac{b}{a}$ exceeds 1 by a very small number, that by putting $\frac{1}{n}$ for n, in (2) and (5), we shall clearly get an approximate value of the n^{th} power of $1 + \frac{b}{a}$.

To perceive the use of what has been done, take the following

EXAMPLES.

1. To extract the square root of $2 = 1 + 1 = 1 + \frac{1}{1}$.

Putting $a = 1$, $b = 1$, and $n = 2$, the first three terms of (2) give $A = 2 + \frac{1}{2} - \frac{1}{8} = \frac{19}{8}$, and the first four terms give $A = 2 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = \frac{39}{16}$; consequently, the square root of 2 has $1 + \frac{8}{19} = 1.42 +$ and $1 + \frac{16}{39} = 1.41 +$ for limits, and of course 1.41 must be the first three figures of the root.

Since $2 = (1.41)^2 \times \frac{2}{(1.41)^2} = (1.41)^2 \times 1.00598561 +$, we get $\sqrt{2} = 1.41\sqrt{1.00598561 +} = 1.41 \times 1.00298833 + = 1.4142135 +$, correct to seven decimals; noticing that $\sqrt{1.00598561 +} = 1.00298833 +$ has been found by putting $a = 1$, $b = 0.00598561 +$, and $n = 2$, in (2), etc.

2. To extract the thirteenth root of 1.02.

Putting $a = 1$, $b = 0.02$, and $n = 13$, (2) gives $A = 13 + 0.12 - 0.0004307 +$ etc.; consequently, the thirteenth root of 1.02 has $1 + \frac{2}{1312} = 1.0015243 +$ and $1 + \frac{2}{1311.95693 +}$

$= 1.0015244 +$ for limits, and of course we have $1.0015243 +$ for the root, correct to seven decimals.

3. To find the first five figures of the seventh root of 1.5.
Ans. 1.0596.

4. To find the square of $1.1 = 1 + \frac{1}{10}$.

Putting $a = 1$, $b = \frac{1}{10}$, and $n = \frac{1}{2} = 0.5$, (2) gives $A = \frac{1}{2} - \frac{1}{40} + \frac{1}{800} - \frac{1}{16000} +$; consequently, the square lies between the limits $1 + \frac{800}{3810} = 1.2097 +$ and 1.210001, from which it is clear that we must have $(1.1)^2 = 1.21$.

5. To find the twentieth power of 1.02 to four decimal places.

Putting $a = 1$, $b = 0.02$, and $n = \frac{1}{20}$, (2) gives $A = \frac{1}{20} - \frac{19}{2000} + \frac{133}{200000} - \frac{133}{20000000} +$ etc.; consequently, using the sums of three and then of four terms of this series, we get $A = \frac{1}{20} - \frac{19}{2000} + \frac{133}{200000} = \frac{8233}{200000}$ and $A = \frac{1}{20} - \frac{19}{2000} + \frac{133}{200000} - \frac{133}{20000000} = \frac{823167}{20000000}$; consequently, $1 + \frac{4000}{8233} = 1.48584 +$ and $1 + \frac{400000}{823167} = 1.48592 +$ being limits of the power, 1.4858 must be the required figures. It is evident that by carrying the series (2) sufficiently far we may find the power to any required degree of exactness.

3d. It is clear that we can apply our rule, or the Binomial Theorem, to the extraction of the roots of series; but they can clearly be more readily found by the rule given for finding the powers of binomials.

To understand the applications of the rule to the extraction of the roots of series, take the following

EXAMPLES.

1. To extract the square root of the series $1 + 2x + 3x^2 + 4x^3 + 5x^4 +$ etc.

According to the rule, we shall have $(a + bx + cx^2 +$

$$dx^3 + ex^4 + \text{etc.})^n = a^n + na^{n-1}bx + \left.\begin{array}{r}\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2\\ na^{n-1}c\end{array}\right| x^2 +$$

$$\left.\begin{array}{r}\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3\\ n(n-1)a^{n-2}bc\\ na^{n-1}d\end{array}\right| x^3 + \left.\begin{array}{r}\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}b^4\\ \frac{n(n-1)}{1\,.\,2}(c^2 + 2bd)a^{n-2}\\ na^{n-1}e\end{array}\right| x^4$$

etc., (1).

It is clear from (1), that if we put $\frac{1}{2}$ for n, and 1, 2, 3, 4, etc., severally for a, b, c, d, etc., we shall get $(1 + 2x + 3x^2 + \text{etc.})^{\frac{1}{2}} = \pm(1 + x + x^2 + x^3 + \text{etc.})$, for the root.

2. To extract the cube root of $8x^3 + 36x^4 + 102x^5 + 231x^6 + \text{etc.} = x^3(8 + 36x + 102x^2 + \text{etc.})$.

Putting $\frac{1}{3}$ for n, and 8, 36, etc., for a, b, c, etc., in (8), we get $x(8 + 36x + 102x^2 + \text{etc.})^{\frac{1}{3}} = 2x + 3x^2 + 4x^3 + \text{etc.}$, for the root.

3. To extract the fourth root of $1 + 2x + 3x^2 + \text{etc.}$, or the square root of $1 + x + x^2 + \text{etc.}$

Ans. $\pm\left(1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128} + \text{etc.}\right)$.

4. To extract the square root of $x^2 - 2x^4 + 3x^6 - 4x^8 + \text{etc.} = x^2(1 - 2x^2 + 3x^4 - \text{etc.})$.

Ans. $\pm(x - x^3 + x^5 - x^7 + x^9 - \text{etc.})$.

(**10.**) We will now proceed to show how to obtain the rules which are commonly used in the extraction of square and cube roots.

1st. To extract the square root of a polynomial.

Suppose the polynomial, whose root is to be extracted, is arranged according to the powers of one of its letters, and that A stands for the sum of any number of terms of the root considered as known; then it is clear that the polynomial will be reduced to the form $A^2 + B$.

By our general rule for the extraction of roots, if we divide the first term of the remainder B (obtained by subtracting A^2 from the arranged polynomial) by the first term of 2A, we shall get a new term of the root, which we shall represent by b; then, according to the rule, subtracting $(A+b)^2 = A^2 + 2Ab + b^2$ from $A^2 + B$, we shall get $A^2 + B - (A + b)^2 = B - (2Ab + b^2) = B - (2A + b)b$ for the remainder; from this remainder we must, as before, get another term of the root, and so on, to any required extent.

Since the remainder $B - (2A + b)b$ is of the form of a remainder in common division, B being the dividend, $2A+b$ the divisor, and b the quotient, it is customary to call the divisor 2A (which we have used in finding b) the *incomplete divisor;* since $2A + b$ is the *complete* or *true* divisor.

Hence *we can extract the square root of a polynomial by the following*

RULE.

1. *Arrange the polynomial according to the powers of one of its letters, and put the square root of its first term in the quotient for the first term of the root, and subtract the square of the root found from the polynomial, and its first term will be destroyed; then the remaining terms of the polynomial will constitute* the first remainder.

2. *Take twice the first term of the root for the first incomplete divisor, and divide the first term of the first remainder by it, and put the result in the quotient for the second term of the root, and annex the second term of the root to the incomplete divisor, and the result will be the complete divisor. Multiply the complete divisor by the second term of the root, and subtract the product from the first remainder, and the result will constitute* the second remainder.

3. *Take twice the root now found for the incomplete divisor, and divide the first term of the second remainder by its first term, and the result put in the quotient will be the third term of the root; then complete the divisor as before, and proceed to find* the third remainder; *and so on to any required extent.*

EXAMPLES.

1. To extract the square root of $x^4 + 3x^2 + 1 - 2x^3 - 2x$.

Arranging the polynomial according to the descending powers of x, etc., we shall have

$$\begin{array}{r|l}
 & x^4 - 2x^3 + 3x^2 - 2x + 1 \;\big|\; \underline{x^2 - x + 1} = \text{the root.} \\
 & \underline{x^4} \\
2x^2 - x & -2x^3 + x^2 \\ \hline
2x^2 - 2x + 1 & 2x^2 - 2x + 1 \\
 & 2x^2 - 2x + 1
\end{array}$$

Explanation.—Here, since $x^4 = (x^2)^2$, we get x^2 for the first term of the root, whose square, x^4, subtracted from the polynomial, gives $-2x^3 + 3x^2 - 2x + 1$ for the first remainder, then $2x^2$ is the first incomplete divisor, and dividing $-2x^3$, the first term of the first remainder, by it, we get $-x$ for the second term of the root, and annexing $-x$ to $2x^2$, we have $2x^2 - x$ for the first complete divisor.

Multiplying $2x^2 - x$ by $-x$ we get the product $-2x^3 + x^2$, which subtracted from the first remainder gives $2x^2 - 2x + 1$ for the second remainder. Taking twice the root now found, we have $2x^2 - 2x$ for the second incomplete divisor; and dividing $2x^2$, the first term of the second remainder, by $2x^2$, the first term of the second incomplete divisor, we get 1 for the third term of the root, and annexing 1 to the second incomplete divisor, we get $2x^2 - 2x + 1$ for the second complete divisor.

Multiplying the second complete divisor by 1, the third term of the root, we get $2x^2 - 2x + 1$ for the product, which, subtracted from the second remainder, gives 0 for the remainder; consequently, $x^2 - x + 1$ is the exact root of the given polynomial; indeed, we have $(x^2 - x + 1)^2 = x^4 - 2x^3 + 3x^2 - 2x + 1$.

2. To extract the square root of $9x^4 + 12x^3 - 26x^2 - 20x + 25$. *Ans.* $3x^2 + 2x - 5$.

If we arrange the polynomial according to the ascending powers of x, and extract the root, it will be found to be $5 - 2x - 3x^2$; which is clearly as it ought to be, since any even root may be taken with either of the signs + or —.

3. To extract the square root of $a^4 - 6a^3x + 13a^2x^2 - 12ax^3 + 4x^4$. *Ans.* $a^2 - 3ax + 2x^2$.

4. To extract the square root of $\frac{4}{9} - \frac{4}{3}x + \frac{4}{3}x^2 - \frac{x^3}{2} + \frac{x^4}{16}$.

Ans. $\frac{2}{3} - x + \frac{x^2}{4}$.

5. To extract the square root of $c^2x^6 - 2bcx^5 + b^2x^4 + 2acx^3 - 2abx^2 + a^2$. *Ans.* $cx^3 - bx^2 + a$.

6. To extract the square root of $x^{\frac{2}{3}} - 2bx^{\frac{8}{15}} + b^2x^{\frac{2}{5}} + 2cx^{\frac{1}{3}} - 2bcx^{\frac{1}{5}} + c^2$. *Ans.* $x^{\frac{1}{3}} - bx^{\frac{1}{5}} + c$.

2d. To extract the cube root of a polynomial.

Arranging the polynomial according to the powers of one of its letters, it appears (as in the case of square roots) that it may be reduced to the form $A^3 + B$; supposing A to stand for the sum of any number of terms of the root considered as having been found, and B as the remainder resulting from the subtraction of A^3 from the polynomial.

According to our general rule for the extraction of roots, if we divide the first term of B by the first term of $3A^2$, and represent the quotient by b, then b will be a new term of the root; then subtracting $(A + b)^3 = A^3 + 3A^2b + 3Ab^2 + b^3$ from $A^3 + B$, we shall get $A^3 + B - (A + b)^3 = B - (3A^2b + 3Ab^2 + b^3) = B - (3A^2 + 3Ab + b^2)b$ for the remainder; and from this remainder we must, as before, get another term of the root, and so on to any required extent.

Because the remainder $B - (3A^2 + 3Ab + b^2)b$ is of the form of a remainder in common division, B being the dividend, $3A^2 + 3Ab + b^2$ the divisor, and b the quotient, since $3A^2$, the divisor used to get b, is but a part of the divisor, it is called the *incomplete divisor*.

Hence, we can extract the cube root of a polynomial by the following

RULE.

1. *Arrange the polynomial according to the powers of one of its letters, and put the cube root of its first term in the quotient for the first term of the root; then subtracting the cube of the root found from the polynomial, its first term will be destroyed, and its remaining terms will constitute* the first remainder.

2. *Take three times the square of the root found for the first incomplete divisor, and divide the first term of the first remainder by it, and put the result in the quotient for the second term of the root. Complete the divisor by adding to it three times the last term of the root multiplied by the pre-*

ceding part of it for the first correction, *and then add the square of the last term of the root, for* the second correction, *to the sum, and the divisor will be completed.*

Multiply the complete divisor by the last term of the root, and subtract the product from the first remainder; then the result will constitute the second remainder.

3. *To the preceding complete divisor add once the first of the preceding corrections and twice the second, and the sum will equal three times the square of the root now found, which will be the second incomplete divisor; then divide the first term of the second remainder by the first term of the second incomplete divisor, and put the result in the quotient for the third term of the root. Complete the divisor in the same way as before, and thence get* the third remainder; *and then proceed as before to find three times the square of the root now found, or the third incomplete divisor, and so on to any required extent.*

EXAMPLES.

1. To extract the cube root of $x^6 + 9x^4 - 3x^5 - 12x - 13x^3 + 8 + 18x^2$.

$$\begin{array}{r|l|l}
 & x^6-3x^5+9x^4-13x^3+18x^2-12x+8 & \underline{x^2-x+2} = \\
 & x^6 & [\text{the root.} \\
3x^4-3x^3+\overline{x^2} & -3x^5+3x^4-x^3 & \\
-3x^3+2x^2 & & \\
\hline
3x^4-6x^3+3x^2 & & \\
6x^2-6x+4 & 6x^4-12x^3+18x^2-12x+8 = \text{the 2 rem.} & \\
\hline
3x^4-6x^3+9x^2-6x+4 & 6x^4-12x^3+18^2-12x+8 & \\
\end{array}$$

Explanation.—Since $x^6 = (x^2)^3$, x^2 must be the first term of the root, and $(x^2)^3 = x^6$ subtracted from the polynomial gives $-3x^5$ for the first term of the first remainder; and $3(x^2)^2 = 3x^4$ is the first incomplete divisor, which is contained in $-3x^5$, $-x$ times; consequently, $-x$ is the second term of the root. The first correction of the incomplete divisor is $x^2 \times -3x = -3x^3$, and $(-x)^2 = x^2$ is the second correction, so that $3x^4-3x^3+x^2$ is the first complete divisor; which, multiplied by $-x$, and the product subtracted from the first remainder, gives $6x^4$ for the first term of the second remainder. By adding $-3x^3$ and $2x^2$ to the first complete divisor, we get $3x^4 - 6x^3 + 3x^2 = 3(x^2 - x)^2$ for the second incomplete divisor; whose first term, $3x^4$, being contained in $6x^4$ two times, it follows that 2 must be the third term of the root. The first correction of the second incomplete divisor is $(x^2 - x) \times 3 \times 2 = 6x^2 - 6x$, and $(2)^2 = 4$ is the second correction; consequently, $3x^4 - 6x^3 + 9x^2 - 6x + 4$ is the second complete divisor. Because the product of the second complete divisor and 2, when subtracted from the second remainder, gives 0 for the remainder, it follows that the root is exact.

2. To extract the cube root of $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$. *Ans.* $a + b + c$.

3. To extract the cube root of $27 + 54x^2 + 36x^4 + 8x^6$. *Ans.* $3 + 2x^2$.

4. To extract the cube root of $-\frac{1}{27} - \frac{1}{4}x^2 + \frac{1}{6}x + \frac{x^3}{8}$. *Ans.* $\frac{1}{2}x - \frac{1}{3}$.

5. To extract the cube root of $a^3b^3 - 3a^{\frac{5}{2}}b^3 + 3a^2b^3 - a^{\frac{3}{2}}b^3$. *Ans.* $ab - a^{\frac{1}{2}}b$.

6. To extract the cube root of $x^{6n} - 9x^{5n} + 42x^{4n} - 117x^{3n} + 210x^{2n} - 225x^n + 125$. *Ans.* $x^{2n} - 3x^n + 5$.

7. To find the cube root of $8x^6 + 36x^5 + 138x^4 + 279x^3 + 483x^2 + 441x + 343$. *Ans.* $2x^2 + 3x + 7$.

Final Remarks.—The beautiful method of Horner, as exhibited in the scheme (E), given at p. 115, can be applied to the extraction of roots of all degrees, and that, whether the expressions whose roots are to be extracted are algebraic or arithmetical; observing, that an algebraic expression must be arranged according to the powers of one of its letters.

EXAMPLES.

Ex. 1.—To extract the square root of $4x^6 - 4x^4 + 12x^3 + x^2 - 6x + 9$.

Let y stand for the root, then we shall have $y^2 = 4x^6 - 4x^4 + 12x^3 + x^2 - 6x + 9$, or $y^2 + 0 . y - 4x^6 + 4x^4 - 12x^3 - x^2 + 6x - 9 = 0$.

Hence, detaching the coefficients of the powers of y, since the root of $4x^6$ is $2x^3$, we shall by the method get

$$\begin{array}{l l l|l}
1+0 & \multicolumn{2}{l|}{-4x^6+0+4x^4-12x^3-x^2+6x-9} & 2x^3+0-x+3=2x^3 \\
\underline{2x^3} & \multicolumn{2}{l|}{\underline{+4x^6 \quad -4x^4 \quad\quad +x^2}} & \underline{-x+3}=\text{the root.} \\
2x^3 & & -12x^3+6x-9 & \\
\underline{2x^3} & & \underline{+12x^3-6x+9} & \\
4x^3-\;x & & 0+\;0+0 & \\
\underline{\quad\;\;-\;x} & & & \\
4x^3-2x+3 & & &
\end{array}$$

Ex. 2.—To extract the cube root of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.

Representing the root by y, we easily get the equation $y^3+0.y^2+0.y-a^6+6a^5-15a^4+20a^3-15a^2+6a-1=0$. Since a^2 is the root of the first term, by detaching the coefficients, etc., we shall get

$$\begin{array}{l l l|l}
1+0+ & 0 & -a^6+6a^5-15a^4+20a^3-15a^2+6a-1 & a^2-2a+ \\
\;a^2 & a^4+ & \underline{a^6-6a^5+12a^4-\;8a^3} & 1=\text{the} \\
\underline{\;a^2} & \underline{2a^4} & \quad -3a^4+12a^3-15a^2+6a-1 & \underline{\text{root.}} \\
2a^2 & 3a^4 & \quad \underline{+3a^4-12a^3+15a^2-6a+1} & \\
\underline{\;a^2\quad\;\;} & \multicolumn{2}{l}{\underline{-6a^3+4a^2}\;\; 0\;+\;0\;+\;0\;+0+0} & \\
3a^2-2a & \multicolumn{2}{l}{3a^4-6a^3+4a^2} & \\
\underline{\quad\;-2a} & \multicolumn{2}{l}{\underline{\quad -6a^3+8a^2\quad}} & \\
3a^2-4a & \multicolumn{2}{l}{3a^4-12a^3+12a^2} & \\
\underline{\quad\;-2a\quad\;} & \multicolumn{2}{l}{\underline{\qquad\quad +\;3a^2-6a+1}} & \\
3a^2-6a+1 & \multicolumn{2}{l}{3a^4-12a^3+15a^2-6a+1} &
\end{array}$$

It is clear that we may proceed in a similar way to what we have done in these examples, in order to extract any algebraic root; and it is also easily seen how we may, by the same method, raise any algebraic expression to any proposed power.

We propose to show how to apply the same method to the extraction of arithmetical roots, in the remaining examples.

Ex. 3.—To extract the square root of 56169.

Because $56169 = 10000 \times \frac{56169}{10000} = (100)^2 \times 5.6169$, we shall have $\sqrt{56169} = 100\sqrt{5.6169}$; consequently, if we extract the root of 5.6169, and move the decimal point in the root two places to the right, we shall get the root of 56169. It may be noticed, since $\frac{56169}{10000} = \frac{56169}{100 \times 100} = \frac{561.69}{100} =$ 5.61.69, that we shall have $\sqrt{56169} = 100\sqrt{5.61.69}$, which is in accordance with the rule in Arithmetic for pointing off the number, preparatory to the extraction of its root.

Proceeding as in the preceding questions, to extract the root of 5.6169, we shall get

$$\begin{array}{rcl}
1 + 0 & - & 5.6169 \,\big|\, \underline{2.37} = \text{the root; consequently, 237 is the} \\
2 & + & \underline{4} \qquad\qquad\qquad \text{[root of 56169.} \\
\underline{2} & - & 1.6169 \\
4.3 & + & \underline{1.29} \\
\underline{.3} & - & 0.3269 \\
4.67 & + & \underline{0.3269} \\
 & & 0.0000
\end{array}$$

The analogy of this method of extracting the root to the common methods is so evident as scarcely to need any comment.

For 2 is the root of the greatest square in 5, 4 is the first incomplete divisor, and 1.61 the corresponding dividend; 4.3 is the first complete divisor, which, multiplied by .3, the second figure of the root, gives the product 1.29; and 4.6 is the second incomplete divisor, and 0.3269 the corresponding dividend, etc.

Ex. 3.—To extract the cube root of 41781923.

Here we have

$$\sqrt[3]{41781923} = 100\sqrt[3]{41.781923} = 100\sqrt[3]{41.781.923}.$$

To extract the cube root of 41.781.923, we have

$$\begin{array}{rrrl}
1 + 0 & + & 0 & -41.781.923 \,\big|\, \underline{3.47} = \text{the root; and 347 is the} \\
3 & + & 9 & \underline{+27} \qquad \text{[root of the given number.} \\
\underline{3} & & \underline{18} & -14.781.923 \\
6 & & 27 & \underline{+12.304} \\
\underline{3} & & \underline{3.76} & -\ \ 2.477.923 \\
9.4 & & 30.76 & \underline{+\ \ 2.477.923} \\
\underline{.4} & & \underline{3.92} & 0.000.000 \\
9.8 & & 34.68 & \\
\underline{.4} & & \underline{.7189} & \\
10.27 & & 35.3989 &
\end{array}$$

Here the first dividend is 14.781, and 27, 30.76 are its incomplete and complete divisors; also, 2.477.923 is the second dividend, its incomplete and complete divisors being 34.68 and 35.3989.

Ex. 4.—To extract the fifth root of 7, to seven decimal places.

Proceeding as in the preceding examples, we get

1 + 0	+ 0	+ 0	+ 0	− 7	1.4757731 + = the root.
1	1	1	1	1	
2	3	4	5		
3	6	10	10.9456	− 6.00000	
4	10	14.864	19.2080	4.37824	
5.4	12.16	20.656	21.22726501	− 1.62176	
5.8	14.48	27.440	23.34744405	1.4859085507	
6.2	16.96	28.846643	23.50681134 +		
6.6	19.60	30.288272	23.6667207 −	− 0.1358514493	
7.07	20.0949	31.765230	23.6891947 −	0.1175340567 +	
7.14	20.5947	31.873458875	23.7116794 −	− 0.0183173926 +	
7.21	21.0994	31.981871750		0.01658244 −	
7.28	21.6090	32.090468750			
7.355	21.645775	32.1057017 +		− 0.00173495 +	
7.360	21.682575	32.120937 +			
7.365	21.719400				
7.370	21.756250				
7.3757	21.761413 −				
	21.76657 +				

In this example, 6 is the first dividend, and 5, 10.9456 are its incomplete and complete divisors; 1.62176 is the second dividend, and 19.2080, 21.22726501 are its incomplete and complete divisors, and so on. Having found the first four decimals as in the preceding examples, we have found the remaining three by dividing the dividend + 0.00173495 + by its incomplete divisor 23.7116794 —.

SECTION XII.

SURDS, OR IRRATIONAL EXPRESSIONS.

(**1.**) When a root is indicated by the use of the surd sign or by a fractional index, and is such that the root can not be exactly extracted, *it is called a surd, or an irrational expression.*

Thus, $\sqrt{2}$, $\sqrt[3]{4}$, $8^{\frac{1}{5}}$, $\sqrt[4]{a^2}$, $\sqrt[n]{b}$, $3\sqrt[6]{8}$, and $\sqrt[3]{a^5b^2c}$ are surds, because the indicated roots can not be exactly extracted.

(**2.**) Any number or quantity which is not affected by the sign of any root, or which can be freed from the sign of any root by the exact extraction of the root, is said to be *rational.*

Thus, 3, 5, 7, a, bc, def, are rational; and so are $\sqrt{36}$, $\sqrt[3]{27}$, $\sqrt[4]{a^8}$, $\sqrt[5]{b^{10}}$, since their roots are ± 6, 3, $\pm a^2$, and b^2, rejecting imaginary roots.

(**3.**) If a monomial consists of two factors, such that one of them is rational, while the other is irrational, then the rational factor, when written before or to the left of the irrational factor, *is called its coefficient.* Thus, in $5\sqrt{2}$ we call 5 the coefficient of $\sqrt{2}$, and in $ab\sqrt[n]{cd}$, ab is called the coefficient of $\sqrt[n]{cd}$. If a monomial surd has no coefficient expressed, it is clear that the coefficient must be understood to be 1. Thus, $\sqrt[3]{5}$ is clearly the same as $1 \times \sqrt[3]{5}$, and $\sqrt[m]{ab} = 1 \times \sqrt[m]{ab}$.

(**4.**) Monomial surds, which do not differ from each other in any other respect than their coefficients and signs, are said to be *similar or alike*, while those whose surd factors are different are *dissimilar or unlike*. Thus, $5\sqrt[3]{4}$ and $-2\sqrt[3]{4}$ are similar surds, while $5\sqrt[3]{4}$ and $-2\sqrt[4]{4}$ are dissimilar surds.

(5.) We will now proceed to the consideration of surds, in the following cases.

CASE I.

To reduce a rational number or quantity to the form of a surd.

RULE.

Raise the number or quantity to such a power as is denoted by the index of the surd, and draw the surd, with its proper index, over the power; then the number or quantity will be expressed, as required.

EXAMPLES.

1. To reduce 2 to the form of the square root, or to find a number whose square root equals 2.

Here, since the index of the root is 2, we have $2^2 = 4$, and of course $\sqrt{2^2} = 2 = \sqrt{4}$, as required.

2. To put 3 under the form of the cube root, and 5 under the form of the fourth root. *Ans.* $\sqrt[3]{27}$ and $\sqrt[4]{625}$.

3. To put $7ab$ under the form of the square root, and $\frac{2a}{3b}$ under the form of the fifth root.

Ans. $\sqrt{49a^2b^2}$ and $\sqrt[5]{\frac{32a^5}{243b^5}}$.

4. To reduce a^3b^3 to the form of the cube root, and $-3a^2\sqrt{b}$ to the form of the square root. *Ans.* $\sqrt[3]{a^9b^9}$ and $-\sqrt{9a^4b}$.

5. To reduce $a-b$ to the form of the square root, and $a+b$ to the form of the cube root.

Ans. $\sqrt{a^2-2ab+b^2}$ and $\sqrt[3]{a^3+3a^2b+3ab^2+b^3}$.

6. To reduce $-\frac{a^{\frac{1}{5}}}{b^{\frac{1}{10}}}$ to the form of the fifth root, and $c^{\frac{2}{3}}d^{\frac{1}{4}}$ to the form of the twelfth root. *Ans.* $-\left(\frac{a}{b^{\frac{1}{2}}}\right)^{\frac{1}{5}}$ and $(c^8d^3)^{\frac{1}{12}}$.

Remark.—If we reduce the coefficient of any surd to the form of the surd, it is clear that it may be put under the surd sign.

EXAMPLES.

1. To put the coefficients in $3\sqrt{5}$ and $2\sqrt[3]{4}$ under the surd signs.

Since $3 = \sqrt{9}$, and $2 = \sqrt[3]{8}$, we shall have $3\sqrt{5} = \sqrt{9} \times \sqrt{5} = \sqrt{45}$ and $2\sqrt[3]{4} = \sqrt[3]{8} \times \sqrt[3]{4} = \sqrt[3]{32}$.

2. To put the coefficients in $\frac{2}{3}\sqrt{\frac{2}{3}}$ and $\frac{a}{b}\sqrt[3]{\frac{a}{b}}$ under the surds. *Ans.* $\sqrt{\frac{8}{27}}$ and $\sqrt[3]{\frac{a^4}{b^4}}$.

3. To put the coefficients in $\frac{b}{a}\sqrt[3]{\frac{a^4}{b^4}}$ and $\frac{c+d}{c-d} \times \sqrt[4]{\frac{c^2-2cd+d^2}{c^2+2cd+d^3}}$ under the surds. *Ans.* $\sqrt[3]{\frac{a}{b}}$ and $\sqrt[4]{\frac{c^2+2cd+d^2}{c^2-2cd+d^2}}$.

4. To put the coefficients in $-\frac{1}{3}\sqrt{9a}$ and $\frac{1}{4}\sqrt[3]{16}$ under the surds. *Ans.* $-\sqrt{a}$ and $\sqrt[3]{\frac{1}{4}}$.

5. To put the coefficients in $3a(c+d)^{\frac{1}{3}}$ and $\frac{1}{5a}(75a^3d^5)^{\frac{1}{2}}$ under the signs of the roots.

Ans. $(27a^3c + 27a^3d)^{\frac{1}{3}}$ and $(3ad^5)^{\frac{1}{2}}$.

CASE II.

To reduce surds having different indices to others which are equivalent, having a common index.

RULE.

1. *Take a common multiple (the least is to be preferred) of the indices of the surds for the common index; then divide the common index by the index of each of the surds, and the quotients multiplied by the indices of the numbers or quantities under the corresponding surds must be taken for the new indices of the numbers or quantities.*

2. *Put each number or quantity with its new index under the sign of the common root; then the surds will be reduced, as required.*

EXAMPLES.

1. To reduce $\sqrt{2}$ and $\sqrt[3]{5}$ to a common index.

Here 2 and 3 are the indices of the surds, whose least common multiple is 6; and $6 \div 2 = 3$, $6 \div 3 = 2$ are the new indices of 2 and 5 under the common surd; consequently, we have $\sqrt{2} = \sqrt[6]{2^3} = \sqrt[6]{8}$ and $\sqrt[3]{5} = \sqrt[6]{5^2} = \sqrt[6]{25}$, and the surds are clearly reduced, as required.

2. To reduce $\sqrt[4]{3}$, $\sqrt[5]{7}$, and the $\sqrt[6]{11}$ to a common index.

Since 60 is the least common multiple of 4, 5, and 6, the answer is $\sqrt[60]{3^{15}}$, $\sqrt[60]{7^{12}}$, $\sqrt[60]{11^{10}}$.

3. To reduce $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, $c^{\frac{1}{4}}$, and $d^{\frac{1}{5}}$ to a common index.

Here 60 is the least common multiple of 2, 3, 4, and 5, and $60 \div 2 = 30$, $60 \div 3 = 20$, $60 \div 4 = 15$, $60 \div 5 = 12$ are the new indices of a, b, c, and d; consequently, the answer is $(a^{30})^{\frac{1}{60}} = a^{\frac{30}{60}}$, $(b^{20})^{\frac{1}{60}} = b^{\frac{20}{60}}$, $(c^{15})^{\frac{1}{60}} = c^{\frac{15}{60}}$, $(d^{12})^{\frac{1}{60}} = d^{\frac{12}{60}}$.

4. Reduce $a^{\frac{2}{3}}$ and $b^{\frac{3}{4}}$ to a common index.

Ans. $(a^8)^{\frac{1}{12}}$ and $(b^9)^{\frac{1}{12}}$.

5. Reduce $\sqrt[3]{a^2}$, $\sqrt[4]{b^3}$, and $\sqrt[8]{c^5}$ to a common index.

Ans. $\sqrt[24]{a^{16}}$, $\sqrt[24]{b^{18}}$, and $\sqrt[24]{c^{15}}$.

6. Reduce $\sqrt[3]{(a+b)^2}$ and $\sqrt[5]{(c+d)^3}$ to a common index.

Ans. $\sqrt[15]{(a+b)^{10}}$ and $\sqrt[15]{(c+d)^9}$.

CASE III.

To reduce a surd to its most simple form.

RULE.

Resolve the number or quantity under the surd into two factors, such that one of them shall be the greatest exact power contained in the given surd; then extract its root, and it will be freed from the sign of the root. Multiply the coefficient of the given surd by the root thus found, and take the product for the coefficient of the remaining surd factor of the given surd.

EXAMPLES.

1. Reduce $3\sqrt{8}$ and $4\sqrt[3]{108}$ to their most simple forms.

Because $8 = 4 \times 2 = 2^2 \times 2$, and $108 = 27 \times 4 = 3^3 \times 4$, we have $3\sqrt{8} = 3\sqrt{2^2} \times \sqrt{2} = 3 \times 2 \times \sqrt{2} = 6\sqrt{2}$, and $4\sqrt[3]{108} = 4 \times \sqrt[3]{27} \times \sqrt[3]{4} = 4 \times 3 \times \sqrt[3]{4} = 12\sqrt[3]{4}$, and the surds have been reduced, as required.

2. Reduce $\sqrt{50a^3b^5}$ and $\sqrt[4]{243a^7c^9}$ to their most simple forms. *Ans.* $5ab^2\sqrt{2ab}$ and $3ac^2\sqrt[4]{3a^3c}$.

3. Reduce $\sqrt[3]{27a^4 + 54a^3b}$ and $\sqrt{\dfrac{48x^3}{27y5}}$ to their most simple forms. *Ans.* $3a\sqrt[3]{a + 2b}$ and $\sqrt{\dfrac{16x^2}{9y^6}} \times \sqrt{xy} = \dfrac{4x}{3y^3}\sqrt{xy}$.

4. Reduce $(625a^4b^3 + 1875a^5b^2)^{\frac{1}{4}}$ and $\left(\dfrac{81a^5}{25b^5}\right)^{\frac{1}{3}}$ to their most simple forms.

Ans. $5a[b^2(b + 3a)]^{\frac{1}{4}}$ and $\left(\dfrac{27a^3}{125b^6}\right)^{\frac{1}{3}} \times (15a^2b)^{\frac{1}{3}} = \dfrac{3a}{5b^2}\sqrt[3]{15a^2b}$.

5. Reduce $\dfrac{2}{3}\left(\dfrac{2}{3}\right)^{\frac{1}{2}}$ and $\dfrac{3}{4}\left(\dfrac{16}{9}\right)^{\frac{1}{3}}$ to their most simple forms.

Ans. $\dfrac{2}{9}\sqrt{6}$ and $\dfrac{1}{2}\sqrt[3]{6}$.

Remarks.—1. If the number or quantity under the surd can not be separated into two factors, such that one of them is an exact power of the kind denoted by the index of the surd, it is clear that the surd is irreducible.

Thus, $\sqrt{10} = \sqrt{5} \times \sqrt{2}$, $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, and $\sqrt[3]{21} = \sqrt[3]{7} \times \sqrt[3]{3}$, are irreducible surds.

2. If the number or quantity under the surd has the form of a fraction, then if the numerator and denominator are multiplied by such a number or quantity as will make the denominator of the resulting fraction an exact power of the kind denoted by the index of the surd, by taking the root of the denominator the given surd will clearly be reduced to the form of a fraction, whose numerator is under the surd while the denominator is freed from it.

Thus, $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2} = \frac{1}{2}\sqrt{6}$, and $\sqrt[3]{\frac{a}{b}} = \sqrt[3]{\frac{ab^2}{b^3}} = \frac{1}{b}\sqrt[3]{ab^2}$.

CASE IV.

Addition and subtraction of surds.

RULE.

1. *Reduce the surds to their most simple forms; then if the surds are similar, the surds will be added by taking the sum of the coefficients (according to their signs) for the coefficient of the common surd; and the surds will be subtracted by changing the signs of all the surds which are to be subtracted, and then proceeding as before.*

2. *If the surds are dissimilar, they must be added by uniting them according to their signs, + and —; and they must be subtracted, by changing the signs of the surds to be subtracted, and then proceeding as before.*

EXAMPLES.

1. To find the sum of $3\sqrt{12}$ and $5\sqrt{27}$.

Because $3\sqrt{12} = 3\sqrt{4} \times \sqrt{3} = 3 \times 2\sqrt{3} = 6\sqrt{3}$ and $5\sqrt{27} = 5\sqrt{9} \times \sqrt{3} = 15\sqrt{3}$, by adding the coefficients 6 and 15 we get 21 for the coefficient of the common surd ($\sqrt{3}$), and of course the required sum equals $21\sqrt{3}$.

2. To subtract $9\sqrt[3]{24}$ from $7\sqrt[3]{81}$.

Because $9\sqrt[3]{24} = 18\sqrt[3]{3}$ and $7\sqrt[3]{81} = 21\sqrt[3]{3}$, we have $7\sqrt[3]{81} - 9\sqrt[3]{24} = 21\sqrt[3]{3} - 18\sqrt[3]{3} = 3\sqrt[3]{3}$ for the difference.

3. To add $5\sqrt[3]{21}$ and $3\sqrt[4]{2}$, and to subtract $5\sqrt[3]{2}$ from $6\sqrt[3]{5}$.

Here we have $5\sqrt[3]{21} + 3\sqrt[4]{2}$ for the sum, and $6\sqrt[3]{5} - 5\sqrt[3]{2}$ for the difference.

4. To find the sum of $\sqrt{24}$, $\sqrt{54}$, and $-3\sqrt{6}$.

Ans. $2\sqrt{6} + 3\sqrt{6} - 3\sqrt{6} = 2\sqrt{6}$.

5. From the sum of $6\sqrt[3]{54a^5b^3}$ and $7bc\sqrt[3]{16a^2}$ subtract the sum of $5\sqrt[3]{2a^2}$ and $-2bc\sqrt[3]{128a^2}$.

Ans. $(18ab + 22bc - 5)\sqrt[3]{2a^2}$.

6. To add $3\sqrt{a^3 + 2a^2b}$ and $-\sqrt{4a^3 + 8a^2b}$.

Ans. $a\sqrt{a + 2b}$.

7. To find the sum of $4\sqrt{6}$, $\sqrt{\frac{8}{27}}$, and $2\sqrt{\frac{2}{3}}$.

Ans. $\frac{44}{9}\sqrt{6}$.

8. To subtract $\sqrt[3]{\frac{27a^7}{4b}}$ from $\sqrt[3]{\frac{125a^7}{4b}}$. *Ans.* $a^2\sqrt[3]{\frac{2a}{b}}$.

9. To add $6(a^5 + a^3b^2)^{\frac{1}{3}}$ and $7a(64a^2 + 64b^2)^{\frac{1}{3}}$.

Ans. $34a(a^2 + b^2)^{\frac{1}{3}}$.

10. To find the sum of $(a^{-m+5})^{\frac{1}{5}}$, $-3(a^{-m+10})^{\frac{1}{5}}$, and $4(a^{-m+15})^{\frac{1}{5}}$. *Ans.* $(a - 3a^2 + 4a^3)a^{\frac{-m}{5}} = \frac{a - 3a^2 + 4a^3}{\sqrt[5]{a^m}}$.

CASE V.

Multiplication and division of surds.

RULE.

By Case II. reduce the surds to equivalent ones having the same index, and then multiply or divide as required.

EXAMPLES.

1. To multiply $\sqrt{3}$ by $\sqrt[3]{2}$, and divide the product by $\sqrt[4]{5}$.

Here the common index is 12, and the surds are equivalent to $\sqrt[12]{3^6}$, $\sqrt[12]{2^4}$, and $\sqrt[12]{5^3}$; consequently, we shall have

$$\frac{\sqrt{3} \times \sqrt[3]{2}}{\sqrt[4]{5}} = \sqrt[12]{\frac{3^6 \times 2^4}{5^3}} = \sqrt[12]{\frac{11664}{125}}.$$

2. Find the product of $3a^{\frac{1}{2}}$, $4a^{\frac{1}{3}}$, and $5a^{\frac{2}{5}}$. *Ans.* $60a\sqrt[20]{a^7}$.

3. Divide $12a^{\frac{5}{2}}b^{\frac{3}{4}}$ by $3a^{\frac{1}{2}}b^{\frac{1}{3}}$. *Ans.* $4a^2b^{\frac{5}{12}}$.

4. Multiply $3\sqrt[3]{16}$ by $4\sqrt[3]{4}$. *Ans.* 48.

5. Divide $\sqrt{\frac{(a^2 + b^2)^2}{3m}}$ by $-\sqrt{\frac{9m^2}{a^2 + b^2}}$. *Ans.* $-\frac{a^2 + b^2}{3m}$.

6. Find the product of $7m\sqrt[6]{bc}$, $5n^2\sqrt[3]{ef}$, and $4p\sqrt[4]{e}$.

Ans. $140mn^2p\sqrt[24]{b^4c^4e^9f^8}$.

7. Multiply $8a^{\frac{1}{m}}$ by $4a^{\frac{1}{n}}$, and divide $6a^{\frac{1}{p}}$ by $3a^{\frac{1}{q}}$.

Ans. $32a^{\frac{m+n}{mn}}$ and $2a^{\frac{q-p}{pq}}$.

8. Find the product of $2^{\frac{1}{3}}$, $3^{\frac{1}{4}}$, and $4^{\frac{1}{5}}$. *Ans.* $\sqrt[60]{2^{44} \times 3^{15}}$.

9. Multiply $\sqrt{x} + \sqrt{y}$ by $\sqrt{x} - \sqrt{y}$. *Ans.* $x - y$.

10. Divide $(x^2 - y)^{\frac{1}{2}}$ by $(x - \sqrt{y})^{\frac{1}{2}}$. *Ans.* $(x + \sqrt{y})^{\frac{1}{2}}$.

11. Multiply $\sqrt[3]{x^4 + 2x^3y + x^2y^2}$ by $\sqrt[3]{x^2 + xy}$.

Ans. $x(x + y)$.

12. Divide $\sqrt[3]{x^3 + y^3}$ by $\sqrt[3]{x + y}$. *Ans.* $\sqrt[3]{x^2 - xy + y^2}$.

CASE VI.

To find powers and roots of surds.

RULE.

1. *To find a power of a surd, raise the surd to such a power as is denoted by the index of the power.*

2. *To find a root of a surd, extract such a root of the surd as is denoted by the index of the root to be extracted.*

EXAMPLES.

1. Find the square of $2\sqrt{3}$ and the cube of $5\sqrt[3]{9}$.

Here we have $(2\sqrt{3})^2 = 2\sqrt{3} \times 2\sqrt{3} = 2^2\sqrt{3^2} = 4 \times 3 = 12$, and $(5\sqrt[3]{9})^3 = 5^3\sqrt[3]{9^3} = 125 \times 9 = 1125$.

2. To extract the square root of $9\sqrt{2}$, and the cube root of $64\sqrt[3]{7}$.

Here we have $(9\sqrt{2})^{\frac{1}{2}} = 3\sqrt[4]{2}$, and $(64\sqrt[3]{7})^{\frac{1}{3}} = 4\sqrt[9]{7}$.

3. Find the cube of $ab^{\frac{1}{3}}c^{\frac{1}{6}}$. *Ans.* $a^3bc^{\frac{1}{2}}$.

4. Find the fourth root of $81a^{\frac{1}{2}}$. *Ans.* $3\sqrt[8]{a}$.

5. Find the fifth power of $3a^{\frac{1}{7}}b^{\frac{1}{9}}c^{\frac{1}{11}}$ and the fourth root of $256c^8d^{12}e^{\frac{1}{3}}$. *Ans.* $243a^{\frac{5}{7}}b^{\frac{5}{9}}c^{\frac{5}{11}}$ and $\pm 4c^2d^3e^{\frac{1}{12}}$.

6. Find the cube root of $\sqrt[5]{27a^3b^9e^{12}}$. *Ans.* $\sqrt[5]{3ab^3e^4}$.

7. Find the fourth power of $\frac{2}{3}\sqrt{\frac{2}{3}a}$ and the square root of $\frac{2}{5}\sqrt{\frac{2}{5}}$

Ans. $\frac{64}{729}a^2$ and $\left(\frac{8}{125}\right)^{\frac{1}{4}}$.

8. Find the square of $3 - \sqrt{2}$ and of $a + \sqrt{b}$.

Ans. $11 - 6\sqrt{2}$ and $a^2 + b + 2a\sqrt{b}$.

9. Find the square root of $x + 2\sqrt{xy} + y$ and the cube root of $a - 3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} - b$.

Ans. $\pm(\sqrt{x} + \sqrt{y})$ and $\sqrt[3]{a} - \sqrt[3]{b}$.

10. Find the square of $3\sqrt{a} - 4\sqrt{b}$ and the square root of $a + b + c + 2\sqrt{ab} - 2\sqrt{ac} - 2\sqrt{bc}$.

Ans. $9a - 24\sqrt{ab} + 16b$ and $\pm(\sqrt{a} + \sqrt{b} - \sqrt{c})$.

CASE VII.

To extract the square root of a binomial, supposing that one or both of its terms are under the sign of the square root.

Assume $x^2 = A + B$ and $y^2 = A - B$, then by addition and multiplication we have $x^2 + y^2 = A + B + A - B = 2A$ and $x^2y^2 = (A + B) \times (A - B) = A^2 - B^2$ or $xy = \sqrt{A^2 - B^2}$.

From $x^2 + y^2 = 2A$ and $2xy = 2\sqrt{A^2 - B^2}$, by addition and subtraction, we have $x^2 + 2xy + y^2 = (x + y)^2 = 2A + 2\sqrt{A^2 - B^2}$ or $\left(\frac{x + y}{2}\right)^2 = \frac{A}{2} + \sqrt{\frac{A^2 - B^2}{2}}$, and $\left(\frac{x - y}{2}\right)^2 = \frac{A}{2} - \sqrt{\frac{A^2 - B^2}{2^2}}$; consequently, by extracting the square roots and taking their sum and difference, we have $x = \sqrt{\frac{A + \sqrt{A^2 - B^2}}{2}} + \sqrt{\frac{A - \sqrt{A^2 - B^2}}{2}}$, (*a*), and $y = \sqrt{\frac{A + \sqrt{A^2 - B^2}}{2}} - \sqrt{\frac{A - \sqrt{A^2 - B^2}}{2}}$, (*b*).

The formulæ (*a*) and (*b*) will enable us to find the square roots of the binomials $A + B$ and $A - B$, whether A and B are or are not affected by the sign of any root.

EXAMPLES.

1. To extract the square root of $11 + 6\sqrt{2}$.

Putting 11 for A and $6\sqrt{2}$ for B, the binomial $A + B$ becomes $11 + 6\sqrt{2}$, as in the question; and we shall have $\sqrt{A^2 - B^2} = \sqrt{11^2 - (6\sqrt{2})^2} = \sqrt{121 - 72} = \sqrt{49} = 7$, and of course from ($a$) we have $\sqrt{11 + 6\sqrt{2}} = 3 + \sqrt{2}$, as required.

2. To extract the square root of $8 - 2\sqrt{15}$.

Putting $A = 8$ and $B = 2\sqrt{15}$, the binomial $A - B$ becomes $8 - 2\sqrt{15}$, as in the question; and we shall have $\sqrt{A^2 - B^2} = \sqrt{64 - 60} = 2$, and of course from ($b$) we get $\sqrt{8 - 2\sqrt{15}} = \sqrt{5} - \sqrt{3}$, as required.

3. To extract the square root of $5 + 12\sqrt{-1}$.

Here we have $A = 5$ and $B = 12\sqrt{-1}$, and $A^2 - B^2 = 25 - (12\sqrt{-1})^2 = 25 + 144 = 169$ or $\sqrt{A^2 - B^2} = 13$; consequently, from (a) we get $\sqrt{5 + 12\sqrt{-1}} = 3 + 2\sqrt{-1}$, as required.

4. To extract the square root of $\sqrt{7} \pm \sqrt{3}$.

Ans. $\sqrt{\dfrac{\sqrt{7} + 2}{2}} \pm \sqrt{\dfrac{\sqrt{7} - 2}{2}}$.

5. To extract the square root of $a + b + 2\sqrt{ab}$.

Ans. $\sqrt{a} + \sqrt{b}$.

6. To extract the square root of $2a^2 - x^2 + 2a\sqrt{a^2 - x^2}$.

Ans. $a + \sqrt{a^2 - x^2}$.

7. To extract the square root of $\pm\sqrt{-1} = 0 \pm \sqrt{-1}$.

Putting 0 for A and $\pm\sqrt{-1}$ for B, we get from (a) and (b) $\sqrt{\pm\sqrt{-1}} = \dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}$, as required.

8. Required, the square root of $3 - 4\sqrt{-1}$ and of $-7 - 24\sqrt{-1}$.

Ans. $2 - \sqrt{-1}$ and $3 - 4\sqrt{-1}$.

CASE VIII.

To extract any root of a binomial, whether its terms are rational or surd.

Assume $(x + y)^n = A + B$ and $(x - y)^n = A - B$, and put $(A + B)(A - B) = A^2 - B^2 = C^n$ or $C = \sqrt[n]{A^2 - B^2}$; consequently, we shall have $\frac{(x + y)^n + (x - y)^n}{2} = A$, or expanding the first member of the equation by the binomial theorem, we have $x^n + \frac{n(n-1)}{1\,.\,2}x^{n-2}y^2 + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}x^{n-4}y^4 + \text{etc.} = A$, and since $(x + y)^n \times (x - y)^n = (x^2 - y^2)^n = A^2 - B^2 = C^n$ gives $x^2 - y^2 = C$ or $y^2 = x^2 - C$, the preceding equation is easily reduced to $x^n + \frac{n(n-1)}{1\,.\,2}x^{n-2}(x^2 - C) + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}x^{n-4}(x^2 - C)^2 + \text{etc.} = A$.

Hence we have the equations $x^n + \frac{n(n-1)}{1\,.\,2}x^{n-2}(x^2 - C) + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}x^{n-4}(x^2 - C)^2 + \text{etc.} = A$, (a'), $y^2 = x^2 - C$, (b'), which being solved give x and y; consequently, we easily get $x + y = \sqrt[n]{A + B}$ or $x - y = \sqrt[n]{A - B}$, the n^{th} root of $A + B$ or $A - B$.

EXAMPLES.

1. To extract the square root of $7 + 4\sqrt{3}$.

Here, by putting 7 for A, and $4\sqrt{3}$ for B, and 2 for n, we shall have $\sqrt{A^2 - B^2} = \sqrt{7^2 - 16 \times 3} = 1 = C$; consequently, (a') reduces to $x^2 + x^2 - 1 = 2x^2 - 1 = 7$, or $x^2 = 4$, or $x = 2$, and from (b') $y^2 = x^2 - C = 4 - 1 = 3$, or $y = \sqrt{3}$. Hence, we have $\sqrt{7 + 4\sqrt{3}} = x + y = 2 + \sqrt{3}$, as required.

2. To extract the cube root of 24.

Put $24 = 48 - 24$, $A = 48$, and $B = 24$, and we get $\sqrt[3]{A^2 - B^2} = 12$; consequently, since $n = 3$, (a') reduces to $x^3 + 3x(x^2 - 12) = 48$, or $x^3 - 9x = 12$, which is very nearly satisfied by putting $x = 3.5223$. Putting the values of C and x in (b'), we get $y^2 = x^2 - C = (3.5223)^2 - 12 = 12.40659729 - 12 = 0.40659729$, or $y = \sqrt{0.40659729} = 0.6376$ very nearly. Hence, $\sqrt[3]{48 - 24} = \sqrt[3]{24} = x - y = 3.5223 - 0.6376 = 2.8847$, which is correct to three decimal places.

3. To extract the cube root of $99 + 70\sqrt{2}$.

Here $A = 99$, $B = 70\sqrt{2}$, and $\sqrt[3]{A^2 - B^2} = 1$; and since $n = 3$, (a') becomes $x^3 + 3x(x^2 - 1) = 4x^3 - 3x = 99$, which gives $x = 3$; consequently, from (b') $y^2 = x^2 - C = 9 - 1 = 8$, or $y = \sqrt{8} = 2\sqrt{2}$, and of course $\sqrt[3]{99 + 70\sqrt{2}} = x + y = 3 + 2\sqrt{2}$, as required.

4. To extract the cube root of $-10 - 9\sqrt{-3}$.

Here $A = -10$, $B = -9\sqrt{-3}$, and $\sqrt[3]{A^2 - B^2} = \sqrt[3]{100 - (9\sqrt{-3})^2} = \sqrt[3]{100 + 243} = \sqrt[3]{343} = 7 = C$; consequently, since $n = 3$, (a') becomes $x^3 + 3x(x^2 - 7) = 4x^3 - 21x = -10$, which gives $x = 2$, and $y^2 = x^2 - C = 4 - 7 = -3$ gives $y = \sqrt{-3}$. Hence, $\sqrt[3]{-10 - 9\sqrt{-3}} = x - y = 2 - \sqrt{-3}$ is the required answer.

5. Extract the fourth root of $1561 \pm 696\sqrt{5}$.

Ans. $3 \pm 2\sqrt{5}$.

6. Extract the fourth root of $-7 \pm 24\sqrt{-1}$.

Ans. $2 \pm \sqrt{-1}$.

7. Extract the cube root of $80 \pm 72\sqrt{-3}$.

Ans. $5 \pm \sqrt{-3}$.

CASE IX.

If we have an expression consisting of any number of terms, then if any number of its terms are surds, it is proposed to find the form of another expression (*called the multiplier*), such that the product of the given expression and the multiplier shall be rational, or free from surds.

RULE.

1. *When the given expression is a monomial surd, which has* 1 *or any rational number for a coefficient. Raise the surd to such a power as is denoted by the index* (*of the surd*), *and the surd sign will be removed; then divide the power by the surd, and the quotient will clearly be the multiplier.*

Thus, if $m\sqrt[n]{a}$ represents the given surd having m for its coefficient, we shall have $(\sqrt[n]{a})^n = a$, and $\dfrac{a}{\sqrt[n]{a}} = \dfrac{\sqrt[n]{a^n}}{\sqrt[n]{a}} = \sqrt[n]{\dfrac{a^n}{a}}$

$= \sqrt[n]{a^{n-1}} =$ the multiplier, and we shall have $\sqrt[n]{a} \times \sqrt[n]{a^{n-1}} = \sqrt[n]{a^n} = a$, which is rational.

2. *When the given expression consists of more than one term, being a binomial or a polynomial.*

If the terms under any of the surds are numbers, represent them by letters, using different letters for different numbers, and suppose the expression is represented by P. Putting P equal to 0, we shall get the equation $P = 0$, (1); then, by the common method of freeing the terms of an equation from any surds which may affect them, free the terms of (1) from surds, and then bring all the terms of the resulting equation into the first member of the equation, and let the equation thus obtained be denoted by $Q = 0$, (2). Since (1) and (2) exist together, it is clear that Q must be exactly divisible by P; consequently, if we divide Q by P, it is clear that the quotient will be the required multiplier, such that P, being multiplied by it, the product will be Q, a rational product.

When letters are used for numbers in P, restore the values of the letters in P, Q, and the multiplier; then they will be expressed as they ought to be in the final result.

EXAMPLES.

1. To find multipliers which shall free $3\sqrt{a^3}$ and $4\sqrt[3]{a^5}$ from surds. *Ans.* $\sqrt{a}$ and $\sqrt[3]{a}$.

2. To find a multiplier which shall free $3\sqrt{a} - \sqrt{b}$ from surds.

Assuming the equation $3\sqrt{a} - \sqrt{b} = 0$, or $3\sqrt{a} = \sqrt{b}$, then, by squaring, we have $9a = b$, or $9a - b = 0$; consequently, we have $\frac{9a - b}{3\sqrt{a} - \sqrt{b}} = 3\sqrt{a} + \sqrt{b} =$ the multiplier, and we shall have $(3\sqrt{a} - \sqrt{b}) \times (3\sqrt{a} + \sqrt{b}) = 9a - b$, which is rational.

3. To find multipliers which will make $5 - 2\sqrt{3}$, $4 + 5\sqrt{3}$, $2\sqrt{7} - 3\sqrt{5}$, and $4\sqrt{2} + \sqrt{3}$, rational.

Ans. $5 + 2\sqrt{3}$, $4 - 5\sqrt{3}$, $2\sqrt{7} + 3\sqrt{5}$, and $4\sqrt{2} - \sqrt{3}$.

4. To find a multiplier which will free $\sqrt{a} + \sqrt{b} + \sqrt{c}$ from surds.

Assuming the equation $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$, or $\sqrt{a} + \sqrt{b} = -\sqrt{c}$, we get $a + b + 2\sqrt{ab} = c$, or $2\sqrt{ab} = c - a - b$, which gives $4ab = (a + b - c)^2$, or $(a + b - c)^2 - 4ab = 0$; consequently, we have

$$\frac{(a+b-c)^2 - 4ab}{\sqrt{a} + \sqrt{b} + \sqrt{c}} = \frac{(a+b-c+2\sqrt{ab}) \times (a+b-c-2\sqrt{ab}}{\sqrt{a} + \sqrt{b} + \sqrt{c}}$$

$= (\sqrt{a} + \sqrt{b} - \sqrt{c}) \times (a + b - c - 2\sqrt{ab}) =$ the required multiplier.

5. To find a multiplier which will free $\sqrt{5} + \sqrt{3} - \sqrt{2}$ from surds.

Ans. $(\sqrt{5} + \sqrt{3} + \sqrt{2}) \times (5 + 3 - 2 - 2\sqrt{15}) = (\sqrt{5} + \sqrt{3} + \sqrt{2}) \times (6 - 2\sqrt{15}.)$

6. To find a multiplier that shall free $\sqrt[3]{a} - \sqrt[3]{b}$ from surds.

From the equation $\sqrt[3]{a} = \sqrt[3]{b}$ we have $a = b$, or $a - b = 0$; consequently, we have $\dfrac{a - b}{\sqrt[3]{a} - \sqrt[3]{b}} = \sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$ for the multiplier.

7. To find the multiplier which will free $\sqrt[3]{4} - \sqrt[3]{3}$ from surds.

Ans. $\sqrt[3]{16} + \sqrt[3]{12} + \sqrt[3]{9} = 2\sqrt[3]{2} + \sqrt[3]{12} + \sqrt[3]{9}$.

8. To find a multiplier that shall free $3\sqrt[3]{2} - 2\sqrt[3]{3} = \sqrt[3]{54} - \sqrt[3]{24}$ from surds. *Ans.* $\sqrt[3]{54^2} + \sqrt[3]{54 \times 24} + \sqrt[3]{24^2}$.

9. To free $\sqrt[3]{a} + \sqrt[3]{b}$ from surds.

Here we have $\sqrt[3]{a} = -\sqrt[3]{b}$ or $a + b = 0$; consequently, $\dfrac{a + b}{\sqrt[3]{a} + \sqrt[3]{b}} = \sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2} =$ the multiplier, which will free the given expression from surds.

10. Find the multiplier which will free $\sqrt[3]{3} + \sqrt[3]{2}$ from surds. *Ans.* $\sqrt[3]{9} - \sqrt[3]{6} + \sqrt[3]{4}$.

11. Find the multiplier which will free $2\sqrt[3]{3} + 3\sqrt[3]{2} = \sqrt[3]{24} + \sqrt[3]{54}$ from surds. *Ans.* $\sqrt[3]{24^2} - \sqrt[3]{24 \times 54} + \sqrt[3]{54^2}$.

12. To find a multiplier that will make $\sqrt[4]{a} + \sqrt[4]{b}$ rational.

Here we have $\sqrt[4]{a} + \sqrt[4]{b} = 0$, or $\sqrt[4]{a} = -\sqrt[4]{b}$, which gives $a = b$, or $a - b = 0$; consequently, $\frac{a-b}{\sqrt[4]{a} + \sqrt[4]{b}} = \sqrt[4]{a^3} - \sqrt[4]{a^2b} + \sqrt[4]{ab^2} - \sqrt[4]{b^3} =$ the multiplier.

Remark.—It may be shown in like manner that $\sqrt[4]{a^3} + \sqrt[4]{a^2b} + \sqrt[4]{ab^2} + \sqrt[4]{b^3}$ is the multiplier that will free $\sqrt[4]{a} - \sqrt[4]{b}$ from surds.

13. To find the multiplier which will free $\sqrt[4]{3} + \sqrt[4]{2}$ and $\sqrt[4]{4} - \sqrt[4]{3}$ from surds.

Ans. $\sqrt[4]{3^3} - \sqrt[4]{3^2 \times 2} + \sqrt[4]{3 \times 2^2} - \sqrt[4]{2^3}$ and $\sqrt[4]{4^3} + \sqrt[4]{4^2 \times 3} + \sqrt[4]{4 \times 3^2} + \sqrt[4]{3^3}$.

CASE X.

To reduce a fraction which contains surds to a fraction whose numerator or denominator shall be free from surds.

RULE.

Multiply the numerator and denominator by a multiplier which will free either the numerator or denominator (as required) from surds.

EXAMPLES.

1. To free the denominator of the fraction $\frac{a}{\sqrt{b}}$ from the surd. *Ans.* $\frac{a\sqrt{b}}{b}$.

2. To free the numerators of the fractions $\frac{\sqrt[3]{a^2}}{\sqrt{b}}$ and $\frac{\sqrt[4]{a}}{\sqrt[3]{b}}$ from surds. *Ans.* $\frac{a}{\sqrt[3]{a} \times \sqrt{b}} = \frac{a}{\sqrt[6]{a^2b^3}}$, and $\frac{a}{\sqrt[12]{a^9b^4}}$.

3. To free the denominators of $\frac{3}{\sqrt{5}}$ and $\frac{4}{\sqrt[3]{6}}$ from surds.

Ans. $\frac{3\sqrt{5}}{5}$ and $\frac{4\sqrt[3]{36}}{6}$.

4. Free the denominator of $\frac{5}{\sqrt{3}-\sqrt{2}}$ from surds.

Ans. $\frac{5(\sqrt{3}+\sqrt{2})}{3-2} = 5(\sqrt{3}+\sqrt{2})$.

5. Free the numerator of $\frac{\sqrt{c}+\sqrt{d}}{\sqrt{c}-\sqrt{d}}$ from surds.

Ans. $\frac{c-d}{(\sqrt{c}-\sqrt{d})^2} = \frac{c-d}{c+d-2\sqrt{cd}}$

6. Free the denominator of $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{8}+\sqrt{7}}$ from surds.

Ans. $(\sqrt{5}-\sqrt{2}) \times (\sqrt{8}-\sqrt{7})$.

7. Free the denominator of $\frac{3}{\sqrt[3]{5}-\sqrt[3]{2}}$ from surds.

Ans. $\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}$.

8. Free the denominator of $\frac{5}{\sqrt[3]{3}+\sqrt[3]{2}}$ from surds.

Ans. $\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4}$.

9. Free the denominator of $\frac{7}{\sqrt[3]{4}-\sqrt[3]{2}}$ from surds.

Ans. $\frac{7(\sqrt[3]{16}+\sqrt[3]{8}+\sqrt[3]{4})}{2}$.

10. Free the denominator of $\frac{\sqrt{3}}{\sqrt[4]{5}+\sqrt[4]{2}}$ from surds.

Ans. $\frac{\sqrt{3}}{3}(\sqrt[4]{125}-\sqrt[4]{50}+\sqrt[4]{20}-\sqrt[4]{8})$.

IMPOSSIBLE OR IMAGINARY ROOTS.

1st. Because all even powers of positive or negative numbers or quantities are positive, it follows that a negative number or quantity can not be an even power of a real number or quantity; consequently, when we consider and treat a negative number or quantity as being an even power, its root must be impossible in real numbers or quantities, or the root must be imaginary. Thus, $\sqrt{-1}$, $4\sqrt[4]{-2}$, $6\sqrt[6]{-3}$, $\sqrt[8]{-5}$, $\sqrt{-a}$, $\sqrt{-b}$, $-3\sqrt{-2}$, $-5\sqrt[4]{-6}$, $m\sqrt[4]{-b}$ are all impossible or imaginary, supposing a, b, and m to be real, and that a and b are positive.

2d. To show how imaginary numbers or quantities may occur in calculation, and to perceive their nature more fully, we will solve the following

QUESTION.

To find two numbers, whose sum is 8, and product 20.

Solution.—Because the square of the sum of two numbers equals the sum of their squares increased by twice their product, and the square of their difference equals the sum of their squares diminished by twice their product, it follows that $8^2 - 4 \times 20 = 64 - 80 = -16$ equals the square of the difference of the numbers; consequently, since $-16 = 16 \times -1$, by extracting the square root we get $\sqrt{-16} = 4\sqrt{-1}$ for the difference of the required numbers.

Hence, since the half sum of two numbers increased by their half difference equals the greater number, and that the half sum diminished by the half difference equals the less number, it follows that $\frac{8}{2} + \frac{4\sqrt{-1}}{2} = 4 + 2\sqrt{-1}$ and $\frac{8}{2} - \frac{4\sqrt{-1}}{2} = 4 - 2\sqrt{-1}$ will represent the sought numbers, which are imaginary; consequently, no real numbers can be found which will satisfy the conditions of the question.

Now, although we can not conceive $4 + 2\sqrt{-1}$ and $4 - 2\sqrt{-1}$ to represent any real numbers, yet they are expressions which, with certain conventions, will be found to satisfy the conditions of the question.

Thus, since $4 + 2\sqrt{-1}$ and $4 - 2\sqrt{-1}$ have been found by using the rules which we know to be applicable to real numbers and quantities, that is, since $4 + 2\sqrt{-1}$ and $4 - 2\sqrt{-1}$ have been found by treating them as being real numbers; reciprocally, when we apply them to see if they satisfy the conditions of the question, we must clearly use the rules which are applicable to real numbers or quantities, that is, we must treat $4 + 2\sqrt{-1}$ and $4 - 2\sqrt{-1}$ as if they were real numbers.

Hence, by adding $4 + 2\sqrt{-1}$ and $4 - 2\sqrt{-1}$ we get $4 + 2\sqrt{-1} + 4 - 2\sqrt{-1} = 8$; since, according to the rules which apply to real numbers or quantities, we must have $2\sqrt{-1} - 2\sqrt{-1} = 0$.

Also, the product of $4 + 2\sqrt{-1}$ and $4 - 2\sqrt{-1}$ is expressed by $(4 + 2\sqrt{-1}) \times (4 - 2\sqrt{-1}) = 16 + 8\sqrt{-1} - 8\sqrt{-1} - 4(\sqrt{-1})^2$; since, by the rules which are applicable to the multiplication of real numbers and quantities, the product of $2\sqrt{-1}$ and 4 is $8\sqrt{-1}$, and that of $-2\sqrt{-1}$ and 4 is $-8\sqrt{-1}$, also the product of $2\sqrt{-1}$ and $-2\sqrt{-1}$ must be expressed by $-2 \times 2 \times \sqrt{-1} \times \sqrt{-1} = -4(\sqrt{-1})^2$. Because $8\sqrt{-1} - 8\sqrt{-1} = 0$, our product is reduced to $16 - 4(\sqrt{-1})^2$; now because $\sqrt{-1}$ has been obtained from -1 by putting -1 under the sign of the square root, reciprocally we must take the square of $\sqrt{-1}$ by removing the sign of the square root; consequently, for $(\sqrt{-1})^2$ we must put -1, and our product reduces to $(4 + 2\sqrt{-1}) \times (4 - 2\sqrt{-1}) = 16 - 4 \times -1 = 16 + 4 = 20$. Hence, $4 + 2\sqrt{-1}$ and $4 - 2\sqrt{-1}$, when used according to the preceding conventions, or when used as if they were real numbers or quantities, have been found to satisfy the conditions of the question; noticing (particularly) that $\sqrt{-1}$ is squared by removing the sign of the root, or putting -1 for $(\sqrt{-1})^2$. It is clear that observations similar to the preceding will be applicable in all cases when imaginary numbers or quantities are used in calculation.

3d. It follows, from what has been done, that imaginary numbers or quantities must be added by the rules which are applicable to real numbers or quantities. Thus, the sum of $\sqrt{-1}$, $3\sqrt{-1}$, and $7\sqrt{-1}$, is $\sqrt{-1} + 3\sqrt{-1} + 7\sqrt{-1} = 11\sqrt{-1}$; observing that the coefficient of $\sqrt{-1}$ is 1. The sum of $\sqrt{-1}$, $8\sqrt{-1}$, $14\sqrt[3]{-1}$, and $-6\sqrt[3]{-1}$ equals $7\sqrt{-1} + 8\sqrt[3]{-1}$; noticing that $\sqrt{-1}$ and $\sqrt[3]{-1}$ are unlike expressions, because the indices of the roots are different. Similarly, the sum of $a\sqrt{-b}$ and that of $c\sqrt{-b}$ is expressed by $(a + c)\sqrt{-b} = (a + c) \times \sqrt{b} \times \sqrt{-1}$, and that of $m\sqrt[4]{-a}$, $n\sqrt[4]{-a}$, and $p\sqrt[6]{-c}$, is expressed by $(m + n) \times \sqrt[4]{a} \times \sqrt[4]{-1} + p\sqrt[6]{c} \times \sqrt[6]{-1}$.

Again, if we subtract $3\sqrt{-1}$ from $8\sqrt{-1}$, and $b\sqrt[4]{-1}$ from $a\sqrt[4]{-1}$, the remainders will be $5\sqrt{-1}$, and $(a - b)\sqrt[4]{-1}$; and in like manner if we subtract $m\sqrt[4]{-a}$ from $n\sqrt[6]{-a}$, the remainder will be $n\sqrt[6]{-a} - n\sqrt[4]{-a} = n\sqrt[6]{a} \times \sqrt[6]{-1} - n\sqrt[4]{a} \times \sqrt[4]{-1}$.

4th. If $2n$ stands for any positive whole number, a for any whole number or quantity, and A for any positive number or quantity, then since $-A = A \times -1$, it is clear that $a\sqrt[2n]{-A} = a\sqrt[2n]{A} \times \sqrt[2n]{-1}$ may be taken as the representative of any imaginary monomial.

5th. Multiplication and division of imaginary monomials, which result from the square root.

EXAMPLES.

1. To find the square of $\sqrt{-a}$ and the product of $\sqrt{-a}$ and $\sqrt{-b}$.

Ans. $(\sqrt{-a})^2 = (\sqrt{a} \times \sqrt{-1})^2 = -a$, and $\sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b} \times \sqrt{-1} = \sqrt{ab} \times (\sqrt{-1})^2 = -\sqrt{ab}$.

2. To find the product of $\sqrt{-a}$ and $-\sqrt{-b}$, and the product of $-\sqrt{-c}$ and $-\sqrt{-d}$. *Ans.* $\sqrt{ab}$ and $-\sqrt{cd}$.

3. To divide $\pm\sqrt{-a} = \pm\sqrt{a} \times \sqrt{-1}$ by $\pm\sqrt{-b} = \pm\sqrt{b} \times \sqrt{-1}$. *Ans.* $\sqrt{\frac{a}{b}}$.

4. To divide $\pm\sqrt{-a}$ by $\mp\sqrt{-b}$. *Ans.* $-\sqrt{\frac{a}{b}}$.

Hence, *the product of two imaginary monomials of the square root which have the same sign, equals minus the product of their coefficients multiplied by the product of the numbers or quantities under the surd, regarding them as real;* but if the monomials have unlike signs, *the product equals plus the product of the coefficients multiplied by the product of the numbers or quantities under the surd, regarding them as real.*

Hence, also, *the quotient of two imaginary monomials of the square root which have the same sign, equals plus the product of the quotient of the coefficients multiplied by the square root of the quotients of the numbers or quantities* (under the surd) *regarding them as real;* but if the imaginaries have unlike signs, *the quotient found* (as before) *must have the* negative *sign.*

EXAMPLES.

1. Multiply $5\sqrt{-7}$ by $3\sqrt{-2}$, and $-4\sqrt{-8}$ by $5\sqrt{-3}$.
Ans. $-15\sqrt{14}$, and $20\sqrt{24} = 40\sqrt{6}$.

2. Multiply $3\sqrt{-a}$ by $4\sqrt{-b}$, and $-2\sqrt{-c}$ by $8\sqrt{-d}$.
Ans. $-12\sqrt{ab}$, and $16\sqrt{cd}$.

3. Multiply $-5\sqrt{-3}$ by $-3\sqrt{-6}$, and $-a\sqrt{-c}$ by $-c\sqrt{-d}$.
Ans. $-45\sqrt{2}$, and $-ae\sqrt{cd}$.

4. Divide $\pm 4\sqrt{-10}$ by $\pm 2\sqrt{-5}$, and $-15\sqrt{-8}$ by $5\sqrt{-2}$.
Ans. $2\sqrt{2}$ and $-3\sqrt{4} = -6$.

5. Divide $30\sqrt{-8}$ by $-10\sqrt{-3}$.

Ans. $-3\sqrt{\frac{8}{3}} = -2\sqrt{6}$.

6. Find the second, third, and fourth powers of $\pm\sqrt{-1}$.
Ans. -1, $\mp\sqrt{-1}$, and 1.

7. Find the product of $2\sqrt{-1}$, $-5\sqrt{-1}$, and $-7\sqrt{-1}$.
Ans. $70(\sqrt{-1})^3 = -70\sqrt{-1}$.

8. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$, and find the square of $1\pm\sqrt{-1}$.

Ans. $(1+\sqrt{-1})\times(1-\sqrt{-1}) = 1+\sqrt{-1}-\sqrt{-1}-(\sqrt{-1})^2 = 2$, and $(1\pm\sqrt{-1})^2 = 1\pm 2\sqrt{-1}-1 = \pm 2\sqrt{-1}$.

9. Multiply $a+\sqrt{-b}$ by $a-\sqrt{-b}$, and find the cube of $c\pm\sqrt{-d}$.

Ans. $(a+\sqrt{-b})\times(a-\sqrt{-b}) = a^2 + a\sqrt{-b} - a\sqrt{-b} - (\sqrt{-b})^2 = a^2 + b$, and $(c\pm\sqrt{-d})^3 = c^3 \pm 3c^2\sqrt{-d} - 3cd \mp \sqrt{-d^3}$.

10. Divide $3 + 2\sqrt{-1}$ by $3 - 2\sqrt{-1}$, $a + \sqrt{-b}$ by $a - \sqrt{-b}$.

Ans. $\dfrac{3 + 2\sqrt{-1}}{3 - 2\sqrt{-1}} = \dfrac{(3 + 2\sqrt{-1})^2}{13} = \dfrac{5 + 12\sqrt{-1}}{13}$, and $\dfrac{a + \sqrt{-b}}{a - \sqrt{-b}} = \dfrac{(a + \sqrt{-b})^2}{a^2 + b} = \dfrac{a^2 - b + 2a\sqrt{-b}}{a^2 + b}$.

11. Find the value of $(\sqrt{3 + 4\sqrt{-1}} + \sqrt{3 - 4\sqrt{-1}})^{\frac{1}{2}}$.

Ans. $(3 + 4\sqrt{-1} + 3 - 4\sqrt{-1} + 10)^{\frac{1}{4}} = 2$, a real number; consequently, imaginary expressions are sometimes representatives of real numbers or quantities.

12. Divide $a^2 + b^2$ by $a + b\sqrt{-1}$, and 1 by $1 - \sqrt{-1}$.

Ans. $a - b\sqrt{-1}$, and $\dfrac{1 + \sqrt{-1}}{2}$.

6th. It is clear, from what has been done, that for $(\sqrt{-a})^2$ we may write $\sqrt{-a \times -a} = -a$, and that for $\sqrt{-a} \times \sqrt{-b}$ we may write $\sqrt{-a \times -b} = -\sqrt{ab}$.

REMARKS.

1. If we expand $\sqrt{1 - x}$ and $\dfrac{1}{\sqrt{1 - x}} = (1 - x)^{-\frac{1}{2}}$ by the binomial theorem, according to the ascending powers of x, we shall have $\sqrt{1 - x} = 1 - \dfrac{x}{2} - \dfrac{x^2}{2 . 4} - \dfrac{1 . 3x^3}{2 . 4 . 6} - \dfrac{1 . 3 . 5x^4}{2 . 4 . 6 . 8}$ $-$ etc., (1), and $\dfrac{1}{\sqrt{1 - x}} = 1 + \dfrac{x}{2} + \dfrac{1 . 3x^2}{2 . 4} + \dfrac{1 . 3 . 5x^3}{2 . 4 . 6} +$ etc., (2).

Supposing x to be positive, it is clear that (1) and (2) are true so long as x is not greater than 1; so that their right members may be taken as representatives of $\sqrt{1 - x}$ and $\dfrac{1}{\sqrt{1 - x}}$, and *vice versa*; consequently, if we put 1 for x in (1), we have $0 = 1 - \dfrac{1}{2} - \dfrac{1}{2 . 4} - \dfrac{1 . 3}{2 . 4 . 6} -$ etc., $= 0$, which gives $\dfrac{1}{2} + \dfrac{1}{2 . 4} + \dfrac{1 . 3}{2 . 4 . 6} + \dfrac{1 . 3 . 5}{2 . 4 . 6 . 8} +$ etc., $= 1$; and if we put 1 for x in (2), since $\frac{1}{0}$ denotes an unlimitedly great number, it results that the series $1+\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+$ etc., is unlimitedly great. Now if (for generality) we suppose (1) and (2) to be true, when x is greater than 1, it is clear that their first members will be impossible or imaginary (since $1-x$ will be negative, and that $\sqrt{1-x}$ expresses its square root), at the same time that the second member of (1) is negative, and that of (2) positive; consequently, *negative and positive expressions are sometimes signs of impossibility.*

2. Because the squares of the first members of (1) and (2) are $1-x$ and $\frac{1}{1-x}$ when x is not greater than 1, it follows that (according to the preceding generalization) if x is greater than 1, we must find the square of $\sqrt{1-x}$ by rejecting the sign of the root, which is in conformity to what has (previously) been shown.

3. If we divide the members of (1) by $1-x$ (since $\frac{\sqrt{1-x}}{1-x}=\frac{1}{\sqrt{1-x}}$), we shall have $\frac{1}{\sqrt{1-x}}=1+\frac{x}{2}+\frac{1.3x^2}{2.4}+$ etc., which is the same as (2); consequently (because when x is greater than 1), the right member of (1) being negative and that of (2) positive, it results that the sign of the imaginary expression $\sqrt{1-x}$ is changed by dividing it by the negative number $1-x$, which is in conformity to the rule of signs in Division.

4. Squaring (1) and (2), we have $1-x=\left(1-\frac{x}{2}-\frac{x^2}{2.4}-\text{etc.}\right)^2$ and $\frac{1}{1-x}=\left(1+\frac{x}{2}+\frac{1.3x^2}{2.4}+\text{etc.}\right)^2$; consequently, if x is greater than 1, the positive expressions $\left(1-\frac{x}{2}-\text{etc.}\right)^2$ and $\left(1-\frac{x}{2}+\text{etc.}\right)^2$ will be represented by the negative numbers $1-x$ and $\frac{1}{1-x}$.

SECTION XIII.

EQUATIONS.

(**1.**) AGREEABLY to Def. 6, p. 2, when the equality or equivalence of expressions is denoted by writing = (the sign of equality) between them, the result is called an equation; noticing that the expression which precedes the sign is called the first member, and that the remaining expression is called the second member; also, if the members are composed of monomials connected by either of the signs +, —, the monomials are called the terms of the equation. Thus, $x^2 - ax + b = c - d$ is an equation, having $x^2 - ax + b$ and $c - d$ for its first and second members, x^2, ax, b, c, and d being the terms of the equation.

(**2.**) Equations are (generally) composed of numbers or letters or both, which represent numbers or quantities; the first letters, a, b, c, etc., being used to stand for known numbers or quantities, while the last letters, x, y, z, etc., stand for unknown numbers or quantities. An equation which contains only one unknown letter is sometimes called a *single equation*.

(**3.**) If either member of an equation is the same as the other (under the same or a different form) or of a process indicated in the other, or if the terms of the equation are known, and given so as to make the members equal to each other, then the equation is called an *identical equation*. Thus, $3x + 5 = 2x + x + 5$, $(x + a)^2 = x^2 + 2xa + a^2$, $\dfrac{x^2 - a^2}{x - a} = x + a$, $5 + 3 - 1 = 4 + 3$, are identical equations.

(**4.**) If a single equation is such that it can not be satisfied, or its members made equal to each other, except by assigning one or more particular values to the unknown letter (contained in it), it is called a *determinate equation;* and any particular value of the unknown letter which satisfies the equation is called its *root*. The discovery of the roots is called the *solution of the equation*, and when the roots are found the equation is said to be *solved.* Thus, $2x + 3 = 5$ is a determinate equation, because 1 is the only value of x or root which satisfies the equation; similarly, the equation $x^2 - x = 6$ has 3 and -2 for its roots, and because they are the only values of x which can satisfy the equation, it follows that 3 and -2 are the only roots that the equation can have.

(**5.**) If an equation contains more than one unknown letter, and becomes a determinate equation when particular values are given to all the unknown letters but one, then we shall call the (given) equation an indeterminate equation. Thus, the equation $3x - 5y = 7$ is an indeterminate equation; for if 1, 2, 3, etc., are successively put for y, the equation will be reduced to the determinate equations $3x - 5 = 7$, $3x - 10 = 7$, $3x - 15 = 7$, etc.

(**6.**) If an equation contains one or more unknown letters, and is such that it can be reduced so as to consist of a finite number of monomial terms, into which the unknown letters enter only as factors with positive integral indices, it is called an *algebraic equation;* but if the equation can not be thus reduced, it is called a *transcendental equation.* Thus, $3x - 5 = 2x + 3$, $3x^2 + 2x = 12$, $x^3 - 2x^2 + 9x = 10$, $3x - 5y = 6$, $x^2 - 5xy = 3z + 20$, are algebraic equations; and $3^x = 10$, $x^y = 13$, $a^x = b$, $z^x = d$, are transcendental equations.

(**7.**) An algebraic equation which contains one or more unknown letters is said to be of the same degree as the greatest number which can be formed by adding the indices of the unknown letters which enter as factors into its separate terms, noticing that the equation is supposed to be reduced so as to consist of monomial terms, into which the unknown letters enter (only) as factors with positive integral indices. Thus, $3x - 7 = 20$, $4x + 3y = 14$ are of the first degree, $3x^2 - 4x = 8$, $2xy + 3y - 12 = 0$ are of the second degree, and so on.

(**8.**) An equation of the first degree is called a simple equation, an equation of the second degree which contains only one unknown letter is called a *quadratic equation,* or simply a *quadratic,* a single equation of the third degree is called a *cubic equation,* or simply a *cubic,* a single equation of the fourth degree is called a *biquadratic equation,* or simply a *biquadratic*, and generally a single equation of the n^{th} degree is said to be an equation of the n^{th} degree.

Thus, $2x + 3 = 7$, $4x - 3y = 5$ are simple equations, $x^2 + 2x = 4$ is a quadratic, $x^3 - 3x^2 + 4x = 15$ is a cubic, $4x^4 - 6x^2 + 5x = 40$ is a biquadratic, and generally, if n stands for any positive integer, the equation $ax^n + bx^{n-1} + cx^{n-2} +$ etc. $= p$ is said to be of the n^{th} degree or power, noticing that the letters a, b, c, etc., may one or more of them be negative, and that one or more of the letters b, c, etc., may be 0 if required.

(**9.**) If a single algebraic equation has terms into which all the positive integral powers of the letter, from the highest to the first, enter separately as factors, at the same time that the equation has a term (called the *absolute term*), into which the letter does not enter (unless it is conceived to have 0 for its index, which makes it an expression for 1), then the equation is said to be complete; but if any powers of the letter are wanting in the equation, *it is said to be incomplete;* and it is clear that the equation is incomplete, because the coefficients of the terms which are wanting equal 0, so that the deficient terms may be supplied by writing them with 0 for each of their coefficients. Thus, the equations $ax = b$, $ax^2 + bx = c$, $ax^4 + bx^3 + cx^2 + dx = e$ are complete equations; and $ax^2 = c$, $ax^3 + bx^2 = c$, $ax^3 + cx = d$ are incomplete, which are completed by writing $ax^2 + 0 \times x = c$, $ax^3 + bx^2 + 0 \times x = c$, etc., for them.

(**10.**) Equations which involve only numbers together with the unknown letter or letters are called *numeral*, or *numerical equations*, while those that involve known letters, or known letters and numbers, together with one or more unknown letters, are called *literal equations*. Thus, $3x - 5 = -x + 7$, $xy = 12$ are numerical equations, and $ax = b$, $3x^2 - 4ax = 30$ are literal equations.

(**11.**) Because the members of an equation must equal each other, if we subtract the right member from the first, then, by Ax. 6, p. 8, the result must equal 0; consequently, if the equation is a single equation, and x the unknown letter, then denoting the result of the subtraction by X, the equation will be reduced to $X = 0$, which may clearly be an algebraic or transcendental equation.

If the equation $X = 0$ is determinate, it is clear, from what has heretofore been shown, that its real roots can be found by the rule of Double Position in Arithmetic.

(**12.**) It is clear that the solution of any single determinate equation consists in transforming the equation in such a way that the unknown letter shall stand alone on one side, so that the known expression on the other side will be its value.

SOLUTION OF (SIMPLE) EQUATIONS CONTAINING ONE UNKNOWN LETTER.

CASE I.

1. Any term may be carried or *transposed* from one member of any equation to the other, by changing its sign, as is clear from Ax. 11.

2. Any term which is common to both members of any equation, and has the same sign in each, may be *erased* or *stricken out*. If the signs of all the terms of any equation are changed, it is clear that the equation will still exist.

EXAMPLES.

1. Given, $x - 5 = 3$ and $y + 4 = 9$, to find x and y.

Adding 5 to both members of the first, and subtracting 4 from both members of the second, they will become $x - 5 + 5 = 3 + 5$ and $y + 4 - 4 = 9 - 4$, or since $-5 + 5 = 0$, $3 + 5 = 8$, $4 - 4 = 0$, and $9 - 4 = 5$, we shall get $x = 8$ and $y = 5$. Putting 8 for x and 5 for y in the given equations, they become $8 - 5 = 3$ and $5 + 4 = 9$, which being identical equations, it follows that x and y have been correctly found.

2. Given, $x + 8 = 70$ and $y - 16 = 4$, to find x and y.
Ans. $x = 62$ and $y = 20$.

3. Given, $x + a + b = b + c$ and $y - d - e = f - e$, to find x and y.

Erasing b and e, and transposing a and d, the answers are $x = c - a$, $y = d + f$.

4. Given, $-x^3 + 2x^2 - 5x - 5 = -3x^2 - 17x + 9$, to

bring all the terms which contain x or its powers into the first member, and to make $-x^3$ positive.

Transposing the terms from the second member which contain x, uniting like terms, and then changing all the signs; the *Ans.* is $x^3 - 5x^2 - 14x = -14$.

5. Given, $3x^3 - 2x^2 + 6x = -x^3 - 7x^2 + 4x + 30$, to bring all the terms into the first member of the equation.

Ans. $4x^3 + 5x^2 + 2x - 30 = 0$.

CASE II.

1. The unknown letter in a simple equation may be freed from any multiplier by dividing all the other terms of the equation by the multiplier; and it may be freed from any divisor by multiplying all the other terms of the equation by the divisor, as is clear from Ax. II.

2. In like manner, the highest or any power of the unknown letter, in any equation, may be freed from any multiplier or divisor.

3. Hence any factor or divisor which is common to all the terms of any equation may be erased.

EXAMPLES.

1. Given, $3x + 5 = 20$, and $\frac{y}{4} - 7 = -2$, to find x and y.

Transposing 5 and 7, then dividing by 3 and multiplying by 4, we get $x = \frac{20 - 5}{3} = 5$ and $y = (7 - 2) \times 4 = 20$.

2. Given, $ax - b = c$, and $\frac{y}{e} + f = g$, to find x and y.

Ans. $x = \frac{b + c}{a}$ and $y = (g - f) \times e$.

3. Given, $ax^4 - bx^3 + cx^2 - dx = -e$, and $5y^3 - 10y^2 + 40y - 50 = 0$, to free x^4 and y^3 from their coefficients.

Ans. $x^4 - \frac{b}{a}x^3 + \frac{c}{a}x^2 - \frac{d}{a}x = -\frac{e}{a}$, and $y^3 - 2y^2 + 8y - 10 = 0$.

4. Given, $-\frac{x^2}{3} + \frac{2x}{3} = -\frac{7}{3}$, and $\frac{y^3}{a} + by^2 - cy = d$, to free x^2 and y^3 from their divisors, and to make $-x^2$ positive.

Ans. $x^2 - 2x = 7$, and $y^3 + aby^2 - acy = ad$.

CASE III.

Any equation may be freed from fractions by reducing all its terms to a common denominator, and then rejecting the common denominator, or by multiplying each term by the least common multiple of all the denominators, and then dividing the numerator of each fractional product by its denominator.

EXAMPLES.

1. Free the equation $\frac{x}{3} - \frac{5}{4} = \frac{x}{5}$ from fractions.

The terms reduced to a common denominator, give the equation $\frac{20x}{60} - \frac{75}{60} = \frac{12x}{60}$, or rejecting the denominator we have the equation $20x - 75 = 12x$, or $20x - 12x = 75$ or $x = \frac{75}{8} = 9\frac{3}{8}$.

2. Free the equation $\frac{5y}{6} + \frac{7}{3} = \frac{3}{4}$ from fractions.

Here we have 12 for the least common multiple of all the denominators; consequently, multiplying each term by 12 and reducing the fractional products, we get $10y + 28 = 9$, which gives $y = -\frac{19}{10} = -1.9$.

3. To free $\frac{2x^2}{3} - \frac{4x}{5} - 7 = 0$ from fractions.

Ans. $10x^2 - 12x - 105 = 0$.

4. To free $x^3 - \frac{ax^2}{b} + \frac{cx}{d} - \frac{e}{f} = 0$ from fractions.

Ans. $bdfx^3 - adfx^2 + bcfx - bde = 0$.

5. To free $\frac{3x-1}{4x-2} + 1 = \frac{5x-4}{8x+2}$ from fractions, and reduce the result to its most simple form, when the terms are brought into the first member and arranged according to the descending powers of x. *Ans.* $18x^2 + 8x - 7 = 0$.

Remark.—Supposing $Ax^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + An = 0$ to stand for an equation which is freed from fractions, A, A_1, A_2, etc., being its coefficients, and n a positive whole number; then, if we multiply all the terms of the equation by A^{n-1}, it can clearly be written in the form $(Ax)^n + A_1(Ax)^{n-1} + AA_2(Ax)^{n-2} + A^2A_3(Ax)^{n-3} + \ldots + A^{n-2}A_{n-1}(Ax) + A^{n-1}A_n = 0$, which by representing Ax by y becomes $y^n + A_1y^{n-1} + AA_2y^{n-2} + A^2A_3y^{n-3} + \ldots + A^{n-2}A_{n-1}y + A^{n-1}A_n = 0$, an equation in which the coefficient of y^n is 1.

Having found y from the preceding equation, since y is put for Ax, we shall clearly get $x = \frac{y}{A}$; consequently, the value of x is found.

Thus, the terms of the equation $2x^2 - x - 15 = 0$, when multiplied by 2, give the equation $(2x)^2 - 2x - 30 = 0$, which by using y for $2x$, becomes $y^2 - y - 30 = 0$, and since this equation is satisfied by putting 6 or -5 for y, it results that $x = 3$ and $x = -2.5$ are the roots of the proposed equation. Similarly, if the terms of the equation $3x^3 + 4x = 32$ are multiplied by $3^2 = 9$, and y put for $3x$, the equation is reduced to $y^3 + 12y = 288$; consequently, as this is satisfied by putting 6 for y, it results that $x = 2$ must be a root of the proposed equation.

CASE IV.

1. If any equation contains the unknown letter under the form of a surd; then having simplified the equation as required, by the preceding cases, we may order the terms so that the part under the surd shall stand alone on one side of the equation; consequently, if we raise both members to the power denoted by the index of the surd, the surd will be removed.

It is clear that we may (generally) proceed in a similar way to free an equation (in which the unknown letter is affected by two or more surds) from surds, by removing one surd at a time.

2. If the unknown letter in any equation is affected by any number of surds, we may put each surd part equal to a new letter, and free the equations thus obtained from surds; then substituting the letters for the surds in the proposed equation, we shall have as many equations as new letters and the unknown letter in the given equation; consequently, by the methods of elimination (to be given hereafter) we can

always reduce these equations to one equation containing one unknown letter, which being solved, we easily get the roots of the given equation.

EXAMPLES.

1. Given, $3\sqrt{x} - 2 = 4$, to find x.

The equation, by transposition and division, becomes $\sqrt{x} = 2$; consequently, by squaring, we get $(\sqrt{x})^2 = 4$, or $x = 4$.

2. Given, $\frac{2}{3}\sqrt[3]{x-3} + \frac{5}{6} = 4$, to find x.

Freeing the equation from fractions and transposing, we have $4\sqrt[3]{x-3} = 19$; consequently, by cubing, we have $(4\sqrt[3]{x-3})^3 = 19^3 = 6859$, or $64(x-3) = 6859$, which gives $x = \frac{7051}{64} = 110\frac{11}{64}$.

3. Given, $a + \sqrt{bx} = \sqrt{a^2 + cx}$, to find x.

Squaring both members, we get $a^2 + 2a\sqrt{bx} + bx = a^2 + cx$, or, erasing a^2 from both members, and transposing, we get $2a\sqrt{bx} = (c-b)x$; or, squaring, etc., we have $x = \left(\frac{2a}{c-b}\right)^2 \times b$. Since $2a\sqrt{bx} - (c-b)x = 0$ is satisfied by putting $x = 0$, it is clear that $x = 0$ is also a root of the proposed equation.

It is clear if a is finite, while $c - b$ is an infinitesimal (or very small), that $\frac{2a}{c-b}$ must be very great on account of the smallness of the divisor; consequently, when $\frac{2a}{c-b}$ is unlimitedly great, x must evidently be unlimitedly greater, provided b is finite.

4. Given, $\sqrt[3]{x} + 3 = 2\sqrt{x}$, to show how to find x.

By transposition and cubing, we get $x = (2\sqrt{x} - 3)^3$; consequently, putting $x = y^2$, we have $y^2 = (2y-3)^3 = 8y^3 - 36y^2 + 54y - 27$, or $8y^3 - 37y^2 + 54y - 27 = 0$; consequently, solving this equation, we shall get y, and then from the equation $x = y^2$ we shall get the values of x, which will satisfy the given equation.

Otherwise, squaring the members of the given equation, we have $(\sqrt[3]{x}+3)^2=4x$; consequently, putting $x=z^3$, we get $(z+3)^2=4z^3$, or $4z^3-z^2-6z-9=0$, an equation of a form more simple than the equation in y. Solving the equation in z, we get z, and thence x is found from the equation $x=z^3$.

CASE V.

1. In any equation, when the terms which contain the unknown letter are all brought to one side, and the terms which do not contain it are carried to the other side, if the side which contains the unknown letter is an exact power; then the equation will be simplified by extracting the root in both members of the equation.

2. If the unknown letter enters both members of an equation, and the terms can be so ordered that both members shall be an exact power, then the solution of the equation will be also simplified by extracting the root of both members.

EXAMPLES.

1. Given, $2x^2-3=x^2+6$, to find x.

The equation is easily reduced to $x^2=9$; consequently, by extracting the square root of both members, we get $x=\sqrt{9}=\pm 3$, that is, we may put either 3 or -3 for x in the given equation, and it will be satisfied, or, which is the same, 3 and -3 are the roots of the proposed equation.

2. Given $x^2+2x=8$, to find x.

Adding 1 to both members of the equation, it becomes $x^2+2x+1=8+1$, or $(x+1)^2=3^2$; consequently, extracting the square roots, we have $x+1=\pm 3$, or $x+1=3$, $x+1=-3$, which give $x=2$ and $x=-4$ for the roots of the proposed equation.

3. Given, $3x^2=2x+1$, to find x.

Adding x^2 to both members, we get $4x^2=x^2+2x+1$; consequently, extracting the square roots of both sides, we get $2x=x+1$, or $2x=-x-1$, which give $x=1$, or $x=-\frac{1}{3}$.

4. Given, $x + \frac{3}{x} = 5$, to find x.

Squaring the members, we get $x^2 + 6 + \frac{9}{x^2} = 25$; and subtracting 12 from both sides, we have $x^2 - 6 + \frac{9}{x^2} = \left(x - \frac{3}{x}\right)^2 = 13$; consequently, extracting the square roots (and taking the positive sign in the root of the right member), we get $x - \frac{3}{x} = \sqrt{13}$.

Hence, by taking the half sum and half difference of the given and preceding equation, we get $x = \frac{5 + \sqrt{13}}{2}$ and $\frac{3}{x} = \frac{5 - \sqrt{13}}{2}$ for the roots of the given equation.

5. Given, $x^2 + ax = b$, to find x.

Dividing by x, and transposing, we readily get $x - \frac{b}{x} = -a$; hence (as in the last question), we have $x + \frac{b}{x} = \sqrt{a^2 + 4b}$, and thence $x = \frac{\sqrt{a^2 + 4b} - a}{2}$ and $-\frac{b}{x} = -\frac{\sqrt{a^2 + 4b} + a}{2}$ are the roots of the given equation.

CASE VI.

1. If the unknown letter enters one or more of four arithmetical proportionals, we may put the sum of the first and fourth (or the extremes) equal to that of the second and third (or the means); and if the unknown letter enters one or more of three arithmetical proportionals, we may put the sum of the first and third terms (or the extremes) equal to twice the second (or the mean).

2. If the unknown letter enters one or more of four geometrical proportionals, put the product of the extremes equal to that of the means, and the proportion will be reduced to an equation; similarly, if the unknown letter enters one or more of three geometrical proportionals, we reduce the proportion to an equation, by making the product of the extremes equal to the square of the mean.

3. If the unknown letter enters one or more of three expressions in harmonical proportion, then *since the first must be to the third as the difference between the first and second is to the difference between the second and third*, we easily (by 2) reduce the harmonicals to an equation.

Similarly, if the unknown letter enters one or more of four harmonicals, because *the first must be to the fourth as the difference between the first and second is to the difference between the third and fourth*, we easily (by 2) reduce the harmonicals to an equation.

EXAMPLES.

1. Given, 3, 8, x, and $2x + 1$, in arithmetical proportion, to find x.

Since the sum of the extremes is $2x + 4$, and that of the means $x + 8$, we get the equation $2x + 4 = x + 8$; consequently, $x = 4$, and the arithmeticals become 3, 8, 4, and 9.

2. Given, 5, $7 + x$, and $3x$, in arithmetical proportion or progression, to find x.

Because the sum of the extremes is $5 + 3x$, and that twice the mean is $14 + 2x$, we have the equation $5 + 3x = 14 + 2x$; consequently, $x = 9$, and the progressionals are 5, 16, and 27.

3. Given, the geometrical proportion $3 : 4 :: x + 1 : x + 6$, to find x.

Because the product of the extremes is $3x + 18$, and that of the means is $4x + 4$, we get the equation $3x + 18 = 4x + 4$; consequently, $x = 14$, and the proportion becomes $3 : 4 :: 15 : 20$.

4. Given, the geometrical proportion $4 : x :: x : 9$, to find x. *Ans.* $x = 6$.

5. Given, 4, 3, and x, in harmonical proportion or *progression*, to find x.

From the proportion $4 : x :: 4 - 3 : 3 - x$ we get the equation $x = 12 - 4x$, or $x = 2.4$.

6. Given, 2, 3, x, and $x + 4$, in harmonical proportion, to find x.

The proportion $2 : x + 4 :: 3 - 2 : x + 4 - x$ gives the equation $x + 4 = 8$, or $x = 4$; consequently, the *harmonicals* are 2, 3, 4, and 8.

From the preceding cases we deduce the following rule for the solution of any (simple) determinate equation or proportion, which contains only one unknown letter (*which is usually called the unknown quantity*).

RULE.

1. *When the unknown letter is contained in an equation.*

1st. Free the equation from fractions by Case III., then unite like terms into one, by transposing according to Case I., when necessary.

2d. If the unknown letter enters the monomial terms of the equation with one or more negative indices; then change the signs of such indices, and write the letter with its index changed (in any term) for a divisor of the corresponding coefficient, and free the equation from the resulting fractional terms by Case III.

3d. Free the unknown letter from any surds which may affect it by Case IV., or transform the equation into a new equation containing a new unknown letter free from surds.

4th. Free the equation thus obtained from terms which contain the unknown letter with negative indices, as in 2d, and unite like terms into single terms as in 1st; noticing if the terms can be so arranged as to make the terms which contain the unknown letter a complete power, the root must be extracted as in Case V.

Hence, if the terms are all transposed to the first member, the equation will be reduced to the form $Ax^n + A_1x^{n-1} + A_2x^{n-2} + \dots + A_{n-1}x + A_n = 0$; in which x is the unknown letter, n a positive whole number, A, A_1, A_2, etc., known coefficients, and A_n the absolute term which is known; noticing that, by Case III., the preceding equation may be changed to another in which the coefficient of the highest power of the unknown letter equals 1; see the Remark.

2. *When the unknown letter enters one or more of the terms of a proportion.*

Reduce the proportion to an equation, by Case VI., then reduce the equation to its proper form, according to the preceding directions.

3. *If the final equation is a simple equation, it will clearly be of the form* $Ax + A_1 = 0$; *which evidently results from the equation given above, by putting* 1 *for n.*

Transposing A_1 in the preceding equation, and dividing by A, the coefficient of x, we shall, by Case II., get $x = -\frac{A_1}{A}$.

Hence, *if an equation after the required reductions is a simple equation, transpose its known terms to the right member, and its unknown terms (or those which contain the unknown letter) to the left member; then the unknown letter equals the quotient arising from the division of the right member by the coefficient (or multiplier) of the unknown letter; observing the rule of signs in Division.*

4. *Remarks.*—1st. Resuming the equation $x = -\frac{A_1}{A}$, we observe if $A_1 = 0$, while A is finite, that we shall clearly have $x = 0$.

2d. If $A = 0$, while A_1 is finite, it is clear that x must be unlimitedly great; because the divisor in $-\frac{A_1}{0}$ is 0 or an infinitesimal.

3d. If $A = 0$ and $A_1 = 0$, we have $x = -\frac{0}{0}$; which is of an undetermined form. If A and A_1 are reduced to 0, by assigning a particular value to some letter which is common to them, it is clear that they must have a common divisor. Hence, to get the true value of $x = -\frac{A_1}{A}$, we must reduce the fraction to its lowest terms (by dividing its numerator and denominator by their greatest common divisor), before the particular value is assigned to the common letter.

If, after freeing the terms of the fraction of their greatest common divisor, we still have $x = -\frac{0}{0}$, it is clear that x is indeterminate or arbitrary; but if x is not of the preceding form, it must clearly be either 0, finite, or unlimitedly great.

5. To illustrate the preceding rule take the following

EXAMPLES.

1. Given, $3x + 7 = 15 + x$, to find x. *Ans.* $x = 4$.

2. Given, $x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 25$, to find x. *Ans.* $x = 12$.

3. Given, $ax - b^2 = cx + d^2$, to find x.

Ans. $x = \frac{b^2 + d^2}{a - c}$.

4. Given, $ax + bx - c + d = e - fx$, to find x.

Ans. $x = \frac{c - d + e}{a + b + f}$.

5. Given, $\frac{3}{4x - 5} = \frac{7}{2x + 3}$ and $\frac{22y + 37}{17} = \frac{13y - 7}{9}$, to find x and y. *Ans.* $x = 2$, and $y = \frac{452}{23} = 19\frac{15}{23}$.

6. Given, $\frac{8x + 9}{5x - 2} + 2x = \frac{6x + 13}{3} - 7$, to find x.

Ans. $x = -\frac{11}{64}$.

7. Given, $\frac{x^2 - 2}{x + \sqrt{2}} = \frac{3x^2 - 4}{x\sqrt{3} + 2} - 8$, to find x.

Ans. $x = \frac{10 - \sqrt{2}}{\sqrt{3} - 1}$.

8. Given, $(3 + x) \times (5 + x) - 28 = x^2 + 27$ and $(a - 2y) \times (b - y) = 2y^2 - 7cd$, to find x and y.

Ans. $x = 5$ and $y = \frac{ab + 7cd}{a + 2b}$.

9. Given, $a + x = \frac{a^2 + 3b^2x}{a + x}$ and $a + y = \frac{a^2 + 3b^2y}{a - y}$, to find x and y. *Ans.* $x = 0$ or $3b^2 - 2a$, and $y = 0$ or $-3b^2$.

10. Given, $\sqrt{\frac{1}{x}} - \sqrt{x} = \sqrt{1 + x}$ and $\sqrt{y + 10} = 2 + \sqrt{y + 1}$, to find x and y. *Ans.* $x = \frac{1}{3}$, and $y = \frac{9}{16}$.

11. Given, $\sqrt{3 + x} + \sqrt{3 - x} = 3$ and $\sqrt{a^2 + \sqrt{y^2 - a^2}} = y$, to find x and y. *Ans.* $x = \pm\frac{3\sqrt{3}}{2}$, and $y = \pm\sqrt{a^2 + 1}$.

12. Given, $\sqrt{3 \pm x} = \sqrt[4]{9 \pm x}$, to find x.

Ans. $x = 0$ or ∓ 5.

13. Given, $\sqrt{5x + 7} = \sqrt[4]{49 + 3b^2x}$, to find x.

Ans. $x = 0$ or $\dfrac{3b^2 - 70}{25}$.

14. Given, $\dfrac{\sqrt{a + x} + \sqrt{b + x}}{\sqrt{a + x} - \sqrt{b + x}} = c$, to find x.

Ans. $x = \dfrac{a(1 - c)^2 - b(1 + c)^2}{4c}$.

15. "Given, $\sqrt[3]{a + x} + \sqrt[3]{a - x} = b$, to find x."

Cubing both members, we get $2a + 3[\sqrt[3]{a + x} + \sqrt[3]{a - x}] \times \sqrt[3]{a^2 - x^2} = 2a + 3b\sqrt[3]{a^2 - x^2} = b^3$; consequently, we easily get $x = \pm \sqrt{\left(a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3\right)}$.

16. "Given, $\sqrt[3]{(a + x)^2} + \sqrt[3]{(a - x)^2} = 3\sqrt[3]{a^2 - x^2}$, to find x."

Proceeding as in the last question, we easily get $(a + x)^2 + (a - x)^2 + 9\sqrt[3]{(a^2 - x^2)^3} = 2(a^2 + x^2) + 9(a^2 - x^2) = 27(a^2 - x^2)$; consequently, $x = \dfrac{2a}{\sqrt{5}} = \dfrac{2a}{5}\sqrt{5}$.

17. "Given, $\dfrac{3abc}{a + b} + \dfrac{a^2b^2}{(a + b)^3} + \dfrac{(2a + b)b^2x}{a(a + b)^2} = 3cx + \dfrac{bx}{a}$, to find x."

Since the equation can be put under the form $\dfrac{ab}{a + b}\left(3c + \dfrac{ab}{(a + b)^2}\right) = x\left(3c + \dfrac{b}{a} - \dfrac{(2ab + b^2)b}{(a + b)^2a}\right) = x\left(3c + \dfrac{ab}{(a + b)^2}\right)$, by rejecting the useless factor $3c + \dfrac{ab}{(a + b)^2}$, we shall have $x = \dfrac{ab}{a + b}$.

18. "Given, $\dfrac{bx}{2b - a} - \dfrac{(3bc + ad)x}{2ab(a + b)} - \dfrac{5ab}{3c - d} = \dfrac{(3bc - ad)x}{2ab(a - b)} - \dfrac{5a(2b - a)}{a^2 - b^2}$, to find x."

Because $-\dfrac{3bc + ad}{2ab(a + b)} - \dfrac{3bc - ad}{2ab(a - b)} = -\dfrac{3c - d}{a^2 - b^2}$, the equation can easily be put under the form $\left(\frac{b}{2b-a}-\frac{3c-d}{a^2-b^2}\right)x=$ $\left(\frac{b}{2b-a}-\frac{3c-d}{a^2-b^2}\right)\times\frac{5a(2b-a)}{3c-d}$; consequently, $x=\frac{5a(2b-a)}{3c-d}$.

Remarks.—1st. We have taken this and the preceding example from the Translation of Hirsch's Collection of Examples, Formulæ, etc., by Rev. J. A. Ross, A. M. (See the edition of the work published in 1827, vol. 1, page 131, examples 38 and 39). 2d. We have introduced the examples for the purpose of showing the great superiority which particular processes sometimes have over the most general.

19. Given, $\frac{ax-b}{\sqrt{ax}+\sqrt{b}}+c=2\sqrt{ax}-\sqrt{b}$, to find x.

Because any expression of the form $A-B$ is equivalent to $(\sqrt{A}+\sqrt{B})\times(\sqrt{A}-\sqrt{B})$, we easily get $x=\frac{c^2}{a}$.

20. Given, $18+\frac{3x-6}{7}+\frac{4x+9}{15}-\frac{7x-13}{50}=\frac{5x+10}{11}$ $-\frac{11x-38}{61}+2x+1$, to find x. *Ans.* $x=9$.

21. Given, $\left(5x+\frac{1}{x}\right)^2+1=\left(3x-\frac{1}{x}\right)^2+(4x+1)^2$, to find x. *Ans.* $x=2$.

22. Given, $4:1::4x^2-8x+3\sqrt{x}:x^2-x-\sqrt{x}$, to find x. *Ans.* $x=\frac{49}{16}$.

23. Given, $2:1::\sqrt[n]{a+x}+\sqrt[n]{a-x}:\sqrt[n]{a+x}-\sqrt[n]{a-x}$, to find x. *Ans.* $x=\frac{3^n-1}{3^n+1}\times a$.

24. Given, $x^2+3:x^2-5::x:4$, to find x.

The proportion gives the cubic equation $x^3-4x^2-5x-12=0$, which shows that x must be greater than 5.3 and less than 5.4; since by putting 5.3 for x in the equation, we get the negative result −1.983, and by putting 5.4 for x, we get the positive result 1.824.

25. Given, $-15x^{-\frac{1}{2}}+6x^{-\frac{1}{3}}+x^{\frac{1}{6}}-1=0$, to reduce the equation to a proper form for solution.

Multiply each of the terms of the equation by $x^{\frac{1}{2}} = x^{\frac{3}{6}}$, and it becomes $-15 + 6x^{\frac{1}{6}} + x^{\frac{4}{6}} - x^{\frac{3}{6}} = 0$; consequently, putting $x = y^6$, and ordering the equation according to the descending powers of y, we get $y^4 - y^3 + 6y - 15 = 0$, an equation of the fourth degree, whose solution gives y, and thence x can be found from $x = y^6$. The equation has two real and two imaginary roots, one of the real roots lying between -3 and -2, and the other between 1 and 2.

SOLUTION OF TWO OR MORE SEPARATE (SIMPLE) EQUATIONS, CONTAINING AS MANY UNKNOWN LETTERS AS THERE ARE EQUATIONS.

1. When two or more equations exist together (as supposed), they are sometimes called a *system of two or more equations;* and it is plain that their solution consists in reducing them to a single equation, or to an equation which contains only one unknown letter, whose solution will clearly give that of all the equations.

2. When a system of equations is reduced to a single equation, the unknown letters not contained in it are said to be *eliminated*, and the process used is called the *method of elimination.*

3. We will now proceed to give the common methods of elimination, together with some others, which will often be found useful in practice.

I. ELIMINATION BY SUBSTITUTION.

RULE.

Find the value of one of the unknown letters in one of the equations, and substitute *or put the value for the letter in each of the other equations; then we shall obtain a new system of equations, which will be one less in number than the given equations. Proceeding in a similar way with the equations thus obtained, their solution will be reduced to that of a system of equations one less in number (than the equations from which they have been deduced); and in like manner we may clearly continue the reduction of the number of equations, until a single equation is obtained, from whose solution we can get that of each of the given equations.*

EXAMPLES.

1. Given, $3x + 2y = 12$ and $4x + 3y = 17$, to find x and y.

From the first equation we get $y = \frac{12 - 3x}{2}$, and putting this value for y in the second equation, we have the single equation $4x + 3 \times \left(\frac{12 - 3x}{2.}\right) = 17$, or $8x + 36 - 9x = 34$, whose solution gives $x = 2$. Putting 2 for x in the equation $y = \frac{12 - 3x}{2}$, we get $y = \frac{12 - 6}{2} = 3$. Substituting 2 and 3 for x and y in the given equations, we get the identical equations $6 + 6 = 12$ and $8 + 9 = 17$; consequently, the values of x and y are correct.

2. Given, $4x - 13y = 11$ and $5x - 18y = 12$, to find x and y. *Ans.* $x = 6$ and $y = 1$.

3. Given, $ax + by = c$ and $a'x + b'y = c'$, to find x and y.

Ans. $x = \frac{b'c - bc'}{ab' - a'b}$ and $y = \frac{ac' - a'c}{ab' - a'b}$.

Remarks.—If $b'c - bc' = 0$, or $\frac{b'}{c'} = \frac{b}{c}$, we shall have $x = 0$ and $y = \frac{c}{b} = \frac{c'}{b'}$, as we clearly ought to do, since the given equations become $by = c$ and $b'y = c'$. If $ab' - a'b = 0$, while the numerators of the expressions for x and y are not equal to 0, it is clear that x and y will be unlimitedly great. If $b'c - bc' = 0$ and $ac' - a'c = 0$, then x and y will be reduced to the indeterminate forms $x = \frac{0}{0}$ and $y = \frac{0}{0}$. Since the equations are reducible to $ax + by = c$ and $ax + by = c$, which are equivalent to $ax + by = c$, it is clear that the preceding values of x and y show them to be really indeterminates. In like manner it may be shown that x and y will be indeterminates if $b'c - bc' = 0$, and $ab' - a'b = 0$.

4. Given, $x^2 + y = 11$ and $y^2 + x = 7$, to reduce the equations to a single equation.

From the second we get $x = 7 - y^2$, which, put for x in the first, gives the single equation $(7 - y^2)^2 + y = 11$, or $y^4 - 14y^2 + y + 38 = 0$, one of whose roots is 2. Putting 2 for y in $x = 7 - y^2$, we get $x = 3$; consequently, 3 and 2 are values of x and y which satisfy the given equations.

5. Given, $x^2 + y^2 = 13$ and $xy = 6$, to reduce the equations to a single equation.

From the second equation we get $y = \frac{6}{x}$, which, put for y in the first, gives the single equation $x^2 + \left(\frac{6}{x}\right)^2 = 13$, or $x^4 - 13x^2 + 36 = 0$, which is satisfied by putting $x = 3$, which gives $y = 2$.

6. Given, $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$, $\frac{1}{x} + \frac{1}{z} = \frac{3}{4}$, and $\frac{1}{y} + \frac{1}{z} = \frac{7}{12}$, to find x, y, and z.

From the first equation we get $\frac{1}{x} = \frac{5}{6} - \frac{1}{y}$, which, put for $\frac{1}{x}$ in the second, gives the two equations $\frac{5}{6} - \frac{1}{y} + \frac{1}{z} = \frac{3}{4}$ and $\frac{1}{y} + \frac{1}{z} = \frac{7}{12}$.

From the second of these equations we get $-\frac{1}{y} = \frac{1}{z} - \frac{7}{12}$, which, put for $-\frac{1}{y}$ in the first, gives the single equation $\frac{5}{6} + \frac{2}{z} - \frac{7}{12} = \frac{3}{4}$, or $\frac{2}{z} = \frac{3}{4} + \frac{7}{12} - \frac{5}{6} = \frac{9}{12} + \frac{7}{12} - \frac{10}{12} = \frac{6}{12}$, which gives $z = 4$.

Since $z = 4$, the equation $-\frac{1}{y} = \frac{1}{z} - \frac{7}{12}$ gives $-\frac{1}{y} = \frac{1}{4} - \frac{7}{12} = -\frac{1}{3}$, or $y = 3$; consequently, from $\frac{1}{x} = \frac{5}{6} - \frac{1}{y} = \frac{5}{6} - \frac{1}{3} = \frac{1}{2}$ we have $x = 2$. If we put 2, 3, and 4 for x, y, and z, in the given equations, they will be satisfied, or become identical equations; consequently, the values of x, y, and z are correct.

II. ELIMINATION BY COMPARISON.

RULE.

Find the value of one of the unknown letters in each of the given equations; then put any one of these values equal *to each of the others, and the solution of the given equations will be reduced to that of a system of equations, one less in number than the given equations.*

Proceed in like manner to reduce the solution of the system thus obtained to that of a system of equations containing one less in number than is contained in them, and so on, until a single equation is obtained, whose solution will evidently give the solution of each of the proposed equations.

EXAMPLES.

1. Given, $2x - y = 1$ and $5x - 2y = 4$, to find x and y.

From the first we get $y = 2x - 1$, and from the second we have $y = \frac{5x-4}{2}$; consequently, equating these values of y, we get the equation $\frac{5x-4}{2} = 2x - 1$, whose solution gives $x = 2$, and thence $y = 2x - 1$ gives $y = 3$.

2. Given, $\frac{x}{3} + \frac{y}{2} = 2$ and $3x - \frac{5y}{2} = 4$, to find x and y.

Freeing the equations from fractions, we have $2x + 3y = 12$ and $6x - 5y = 8$; consequently, from $x = \frac{12-3y}{2}$ and $x = \frac{8+5y}{6}$, we get $\frac{8+5y}{6} = \frac{12-3y}{2}$, or $8 + 5y = 36 - 9y$, whose solution gives $y = 2$; and hence, from $x = \frac{12-3y}{2}$, we have $x = 3$.

3. "Given, $x + y + z = 53$, $x + 2y + 3z = 105$, and $x + 3y + 4z = 134$, to find x, y, and z."

The equations, severally, give $x = 53 - (y + z)$, $x = 105 - (2y + 3z)$, $x = 134 - (3y + 4z)$; consequently, equating the first value of x to the second and third values, we get the equations $53 - y - z = 105 - 2y - 3z$ and $53 - y - z = 134 - 3y - 4z$. Solving these equations, we get $y = 6$ and $z = 23$; consequently, from $x = 53 - (y + z)$ we get $x = 24$.

4. Given, $x - y = 2$, $x - z = 7$, $y + z = 7$, to find x, y, and z.

From the first and second equations we get $x = y + 2$ and $x = z + 7$; consequently, the solution of the given equations is reduced to that of the equations $y+2=z+7$ and $y+z=7$. Solving these equations, we get $y=6$ and $z=1$; consequently, from $x=y+2$ we have $x=8$.

5. Given, $xy=a$, $xz=b$, $yz=c$, to find x, y, and z.

The first and second give $x=\frac{a}{y}$, $x=\frac{b}{z}$, and thence $\frac{a}{y}=\frac{b}{z}$, or $z=\frac{by}{a}$; consequently, as the third gives $z=\frac{c}{y}$, we have the single equation $\frac{by}{a}=\frac{c}{y}$, or $y^2=\frac{ac}{b}$, or $y=\sqrt{\frac{ac}{b}}$, and hence $z=\sqrt{\frac{bc}{a}}$, $x=\sqrt{\frac{ab}{c}}$.

6. Given, $x+y=a$ and $x^3+y^3=b$, to find x and y.

From the first $y=a-x$, or $y^3=a^3-3a^2x+3ax^2-x^3$, and from the second $y^3=b-x^3$; consequently, equating these values of y^3, we have $a^3-3a^2x+3ax^2-x^3=b-x^3$, or $3ax^2-3a^2x+a^3-b=0$; solving this equation, we get x, and thence y is found from the equation $y=a-x$.

7. Given, $7x+5y=37$ and $9x-11y=4$, to find x and y.

Ans. $x=3.5$ and $y=2.5$.

8. Given, $3x+4y=\frac{17}{6}$ and $5x-7y=\frac{1}{6}$, to find x and y.

Ans. $x=\frac{1}{2}$ and $y=\frac{1}{3}$.

9. Given, $x+y=a$ and $x-y=b$, to find x and y.

Ans. $x=\frac{a+b}{2}$ and $y=\frac{a-b}{2}$.

10. Given, $x+y=a$ and $x^2-y^2=b$; to find x and y.

Ans. $x=\frac{a^2+b}{2a}$ and $y=\frac{a^2-b}{2a}$.

11. Given, $x^2+y^2=a$ and $x^2-y^2=b$, to find x and y.

Ans. $x=\sqrt{\frac{a+b}{2}}$ and $y=\sqrt{\frac{a-b}{2}}$.

12. Given, $x^n+y^n=a$ and $x^n-y^n=b$, to find x and y.

Ans. $x=\sqrt[n]{\frac{a+b}{2}}$ and $y=\sqrt[n]{\frac{a-b}{2}}$.

13. Given, $xy = a$ and $\frac{x}{y} = b$, to find x and y.

Ans. $x = \sqrt{ab}$ and $y = \sqrt{\frac{a}{b}}$.

III. ELIMINATION BY ADDITION AND SUBTRACTION.

RULE.

1. If the given equations are simple, *multiply or divide (if necessary) the first two of them so as to make the coefficients of the same unknown letter the same in both; then add or subtract the equations, so that the terms thus found may destroy each other, and an equation will be obtained which does not contain the letter. In like manner eliminate the same letter from the third and one of the first two, and then from the fourth and one of the first three, and so on, until all the equations have been used; then it is clear that the solution of the given equations will be reduced to that of a system of equations one less in number than the given equations. Since the system of equations thus found admits of a similar reduction, and so on, it is clear that the solution of the given equations will in this way be reduced to that of a single equation.*

2. If the given equations are not simple, *it is clear that we can eliminate the different powers of a letter which are in one or both of the first two equations, and thence get an equation which is freed from the letter and its powers, and so on, until the unknown letter is eliminated from all the equations. Hence, we can clearly (as before) reduce the solution of the given equations to that of a single equation.*

EXAMPLES.

1. Given, $3x + 7y = 47$ and $8x - y = 27$, to find x and y. Multiplying the second by 7, we have $56x - 7y = 189$, and adding the first to this, we have $59x = 236$, or $x = 4$; consequently, putting 4 for x in the second of the given equations, we have $y = 32 - 27 = 5$.

2. Given, $3x + 2y = 12$ and $4x + 3y = 17$, to find x and y. Subtracting the first from the second, we have $x + y = 5$; consequently, subtracting twice this from the first, we get $x = 2$, and thence $y = 3$.

3. Given, $x + y = a$, $x + z = b$, $y + z = c$, to find x, y, and z.

Subtracting the third from the sum of the first two, we get $2x = a + b - c$ or $x = \frac{a + b - c}{2}$; and in like manner we get $y = \frac{a + c - b}{2}$ and $z = \frac{b + c - a}{2}$.

4. Given, $\frac{1}{x} + \frac{1}{y} = 5$, $\frac{1}{x} + \frac{1}{z} = 7$, $\frac{1}{y} + \frac{1}{z} = 8$, to find x, y, and z.

Ans. $x = \frac{1}{2}$, $y = \frac{1}{3}$, and $z = \frac{1}{5}$.

5. Given, $\frac{1}{x} - \frac{1}{y} = a$, $\frac{1}{x} - \frac{1}{z} = b$, $\frac{1}{y} + \frac{1}{z} = c$, to find x, y, and z.

Ans. $x = \frac{2}{a + b + c}$, $y = \frac{2}{b + c - a}$, $z = \frac{2}{a + c - b}$.

6. Given, $\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = a$, $\frac{1}{x} - \frac{1}{y} + \frac{1}{v} = b$, $\frac{1}{y} + \frac{1}{z} + \frac{1}{v} = c$, and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{v} = d$, to find x, y, z, and v.

Ans. $x = \frac{1}{d - c}$, $y = \frac{3}{a - b + c}$, $z = \frac{3}{3d - (2a + b + 2c)}$, $v = \frac{3}{a + 2b + 4c - 3d}$.

7. Given, $x^2 + y^2 = a$, $x^3 + y^3 = b$, to eliminate x from the equations.

Multiplying the first by x and subtracting the product from the second, we get $y^3 - xy^2 = b - ax$ or $x = \frac{y^3 - b}{y^2 - a}$; consequently, substituting the value of x in the first equation, we have $(y^3 - b)^2 + (y^2 - a)^3 = 0$, or $2y^6 - 3ay^4 - 2by^3 + 3a^2y^2 - (a^3 - b^2) = 0$, an equation of the sixth degree, whose solution will give y, and thence x can be found.

Otherwise—since the first equation gives $(x^2)^3 = x^6 = (a - y^2)^3$, and the second gives $(x^3)^2 = (b - y^3)^2$, we have $(y^3 - b)^2 + (y^2 - a)^3 = 0$, and so on, as before.

8. Given, $\sqrt[3]{x + a} + \sqrt{x + b} = c$, to solve the equation.

Put $x + a = y^3$ and $x + b = z^2$, and the equation becomes

$y + z = c$; consequently, eliminating x and z, we get $y^3 - y^2 + 2cy = a - b + c^2$.

Hence, solving this equation, we get y, and then x can be found from the equation $x = y^3 - a$.

9. Given, $ax + by + cz = d$, $a'x + b'y + c'z = d'$, $a''x + b''y + c''z = d''$, to find x, y, z.

Ans. $x = \dfrac{d(b'c'' - b''c') + d'(b''c - bc'') + d''(bc' - b'c)}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)}$,

$y = \dfrac{d(a''c' - a'c'') + d'(ac'' - a''c) + d''(a'c - ac')}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)}$, $z = \dfrac{d(a'b'' - a''b') + d'(a''b - ab'') + d''(ab' - a'b)}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)}$.

Thus, if $a = 2$, $b = 4$, $c = 5$, $d = 30$, $a' = 7$, $b' = 8$, $c' = 9$, $d' = 59$, $a'' = 11$, $b'' = 17$, $c'' = 7$, $d'' = 73$; then we shall get $x = 1$, $y = 2$, $z = 4$.

10. Given, $Ax^2 + Bx + C = 0$, $A'x^2 + B'x + C' = 0$, to eliminate x.

Multiplying the first by A′ and the second by A and subtracting, we get $(AB' - A'B)x + AC' - A'C = 0$; and in a similar way, if we multiply the first by C′ and the second by C, we get $(AC' - A'C)x + BC' - B'C = 0$.

Hence, eliminating x from these equations, we get $(AC' - A'C)^2 - (AB' - A'B) \times (BC' - B'C) = 0$; which is an equation of condition, that must be satisfied in order that the given equations may have one or more common values of x; noticing, if the terms of the equation mutually destroy each other, that the equations will be identical; since all the values of x which satisfy one of them will also satisfy the other.

Remarks.—1. It is clear that the existence of the preceding equation of condition is requisite, in order that the given equations may have a common factor in terms of x, which being put equal to 0, the resulting values of x shall satisfy both equations.

2. Hence, if we proceed with the equations (as in finding the greatest common divisor) so as to eliminate x, then by putting the first remainder which does not contain x equal to 0, we shall clearly obtain the equation of condition. Indeed, if we multiply the second equation by A and divide the product by the first equation, we shall get $(AB' - A'B)x + AC'$

$- A'C$ for the first remainder; and if we write the terms of the equations in a contrary order, multiply the second by C and then divide the product by the first, we shall get $(AC' - A'C)x + BC' - B'C$ for the first remainder.

Hence, eliminating x from these remainders, we shall get the same equation of condition as before.

3. Because we may in like manner eliminate an unknown letter and its powers from any two algebraic equations, it is clear that we shall have another method of elimination, which is called *Elimination by the method of the greatest Common Divisor.* On account of the striking analogy between this method and that of Addition and Subtraction, we do not think it important to take any further notice of it.

IV. ELIMINATION BY ARBITRARY MULTIPLIERS.

RULE.

Multiply all the equations but one by separate arbitrary multipliers, and add the resulting equations and the remaining equation together. Then (in the sum) put the coefficients of all the unknown letters but one (in the given equations) separately equal to 0, *and the given equations will be reduced to a single equation which contains one of the unknown letters (of the given equations) and the multipliers; consequently, after the multipliers have been found from the assumed equations, the letter may be found from the reduced equation.*

EXAMPLES.

1. Given, $5x - 8y = -4$, $11x - 10y = 14$, to find x and y.

Multiplying the terms of the first equation by m, and adding the corresponding terms of the second, we get $(5m + 11)x - (8m + 10)y = -4m + 14$.

Assuming $8m + 10 = 0$ or $m = -\frac{5}{4}$, we shall have $x = \frac{-4m + 14}{5m + 11} = 4$; and if we assume $5m + 11 = 0$ or $m = -\frac{11}{5}$, we shall have $y = -\frac{-4m + 14}{8m + 10} = 3$.

2. Given, $3x + 2y = 12$ and $4x + 3y = 17$, to find x and y.

Multiplying the terms of the first by m, and adding the corresponding terms of the second, we shall have $(3m + 4)x + (2m + 3)y = 12m + 17$.

Assuming $3m + 4 = 0$ or $m = -\frac{4}{3}$, we get $y = \frac{12m + 17}{2m + 3} = 3$; and assuming $2m + 3 = 0$ or $m = -\frac{3}{2}$, we have $x = \frac{12m + 17}{3m + 4} = 2$.

Remark.—If we assume $12m + 17 = 0$ or $m = -\frac{17}{12}$, we shall get $(3m + 4)x + (2m + 3)y = 0$ or $\frac{x}{y} = -\frac{2m + 3}{3m + 4} = \frac{2}{3}$ = the ratio of x to y; consequently, if we know y we can easily get x, and *vice versa.*

3. Given, $4x - 3y + 2z = 14$, $6x + 7y - 11z = 25$, and $9x + 2y + 3z = 62$, to find x, y, and z.

Multiplying the terms of the first by m, those of the second by n, and adding their corresponding terms and those of the third, we get $(4m + 6n + 9)x + (-3m + 7n + 2)y + (2m - 11n + 3)z = 14m + 25n + 62$.

Assuming $4m + 6n + 9 = 0$, $-3m + 7n + 2 = 0$, we get $m = -\frac{51}{46}$ and $n = -\frac{35}{46}$; consequently, $z = \frac{14m + 25n + 62}{2m - 11n + 3} = 3$.

Assuming $4m + 6n + 9 = 0$ and $2m - 11n + 3 = 0$, we have $m = -\frac{117}{56}$ and $n = -\frac{3}{28}$; consequently, $y = \frac{14m + 25n + 62}{-3m + 7n + 2} = 4$.

From the values of y and z, we shall, from either of the given equations, get $x = 5$; or we can get x by assuming the equations $-3m + 7n + 2 = 0$; $2m - 11n + 3 = 0$, which give $x = \frac{14m + 25n + 62}{4m + 6n + 9} = 5$.

Since this method of elimination is not often used in practice, the preceding examples must suffice for its illustration.

V. PARTICULAR METHODS OF ELIMINATION, ILLUSTRATED BY EXAMPLES.

1st.—1. Given, $3x + 2y = 68$ and $2x + 3y = 72$, to find x and y.

Multiplying the terms of the first by 72, and those of the second by 68, we shall get $(3x + 2y) \times 72 = 68 \times 72$ and $(2x + 3y) \times 68 = 72 \times 68$; consequently, since the second members of these equations are equal, we may equate their first members, and shall thence get $(3x + 2y) \times 72 = (2x + 3y) \times 68$ or $(3x + 2y) \times 18 = (2x + 3y) \times 17$; which is easily reduced to $54x + 36y = 34x + 51y$ or $x = \frac{3y}{4}$.

Putting $\frac{3y}{4}$ for x in either of the given equations, we get $y = 16$, and thence $x = \frac{3y}{4}$ gives $x = 12$.

Remarks.—1. If we put $x = yz$, and substitute yz for x in the given equations, and eliminate y from the resulting equations, we shall get an equation in z, whose solution will give $z = \frac{3}{4}$, or $\left(\text{since } z = \frac{x}{y}\right)$ we have $x = \frac{3y}{4}$, as above; consequently, our method comes to the same thing as to put yz for x in the given equations, and then to eliminate y and find z from the resulting equation.

2. It is easy to perceive that we may, in a similar way, solve any number of simple equations.

2. Given, $x + y = 7$ or $x^2 + 2xy + y^2 = 49$ and $x^2 + y^2 = 25$, to show how to find x and y.

Here we have $(x^2 + 2xy + y^2) \times 25 = (x^2 + y^2) \times 49$, or $25xy = 12x^2 + 12y^2$; which is easily reduced to $\left(\frac{x}{y}\right)^2 - \frac{25}{12}\left(\frac{x}{y}\right) = -1$, and putting $\frac{x}{y} = z$, we have the equation $z^2 - \frac{25}{12}z = -1$, whose solution will give z, and thence x and y can easily be found.

3. "Given, $x^2 + xy = 56$ and $xy + 2y^2 = 60$, to find x and y."

Here we have $(x^2 + xy) \times 60 = (xy + 2y^2) \times 56$ or $15x^2 + 15xy = 14xy + 28y^2$; which is easily reduced to $\left(\frac{x}{y}\right)^2 + \frac{1}{15}\left(\frac{x}{y}\right) = \frac{28}{15}$, or putting $\frac{x}{y} = z$ we have the equation $z^2 + \frac{z}{15} = \frac{28}{15}$. Solving the equation we get z, and thence we easily obtain the values of y and x.

It is hence clear if we have two equations which contain two unknown letters, as x and y, such that each of them is *homogeneous* in terms of x and y, that by putting yz for x in the equations, and eliminating y from them, we shall get an equation whose solution will give z, and thence y and x can be found.

2d.—1. Given, $x + y = 7$ and $x^2 + y^2 = 25$, to find x and y.

Assume $x = z + v$ and $y = z - v$, then, by substitution, the equations will be reduced to $z + v + z - v = 7$ and $(z + v)^2 + (z - v)^2 = 25$, or to $z = \frac{7}{2}$ and $2v^2 + 2z^2 = 25$.

Putting $\frac{7}{2}$ for z in the second equation, we get $v^2 = \frac{25}{2} - \frac{49}{4} = \frac{1}{4}$, or taking the square root, we have $v = \frac{1}{2}$. Hence, we have $x = z + v = \frac{7}{2} + \frac{1}{2} = 4$ and $y = z - v = \frac{7}{2} - \frac{1}{2} = 3$.

2. Given, $x^2 + y^2 = 25$ and $xy = 12$, to find x and y.

Putting $x = z + v$ and $y = z - v$, the equations will be reduced to $2z^2 + 2v^2 = 25$ and $z^2 - v^2 = 12$, or to $z^2 + v^2 = \frac{25}{2}$ and $z^2 - v^2 = 12$.

Taking the half sum and half difference of the preceding equations, we get $z^2 = \frac{49}{4}$ and $v^2 = \frac{1}{4}$; consequently, taking the square roots, we have $z = \frac{7}{2}$, $v = \frac{1}{2}$; which give $x = 4$ and $y = 3$.

Remark.—Because $x = z + v$ and $y = z - v$ give $z = \frac{x + y}{2}$ and $v = \frac{x - y}{2}$, and that $xy = z^2 - v^2$, we shall get $xy = \left(\frac{x + y}{2}\right)^2 - \left(\frac{x - y}{2}\right)^2$; *consequently, we perceive how to reduce any product to the difference of two squares.* Thus, since $209 = 19 \times 11$, if we put 19 for x and 11 for y, we get $\frac{x + y}{2} = 15$ and $\frac{x - y}{2} = 4$; consequently, we shall have $209 = 15^2 - 4^2$; *and it is clear that any odd integer* (*except* 1) *may be regarded as being the difference of the squares of two integers.*

3. Given, $(x-y)(x^2-y^2)=a$ and $(x+y)(x^2+y^2)=b$, to find x and y.

Putting $x=z+v$ and $y=z-v$, we get $x-y=2v$, $x^2-y^2=4zv$, $x+y=2z$, and $x^2+y^2=2z^2+2v^2$; consequently, substituting these values in the given equations, they will be reduced to $8zv^2=a$ and $4z^3+4zv^2=b$. Subtracting the first of these equations from twice the second, we get $8z^3=2b-a$, whose cube root gives $2z=\sqrt[3]{2b-a}$; consequently, dividing the first by four times this value of $2z$, we have $v^2=\dfrac{a}{4\sqrt[3]{2b-a}}$, whose square root gives

$$v=\frac{\sqrt{a}}{2\sqrt[6]{2b-a}}.$$

Hence, we have $x=\dfrac{\sqrt[3]{2b-a}}{2}+\dfrac{\sqrt{a}}{2\sqrt[6]{2b-a}}=\dfrac{\sqrt{2b-a}+\sqrt{a}}{2\sqrt[6]{2b-a}}$

and $y=\dfrac{\sqrt{2b-a}-\sqrt{a}}{2\sqrt[6]{2b-a}}$.

It is clear from the preceding examples that this method of elimination will generally be very useful in solving two equations which contain two unknown letters, as x and y, such that they are *symmetrically* involved in the equations; or (which is the same) when the equations remain the same, when x is put for y, and y for x, in them.

3d.—1. Given $x+y=7$ and $xy=12$, to find x and y.

Squaring the first and subtracting four times the second from it, we get $x^2+2xy+y^2-4xy=x^2-2xy+y^2=(x-y)^2=1$, whose square root gives $x-y=1$.

Hence, we easily get $x=4$ and $y=3$, since the half sum and half difference of the equations $x+y=7$ and $x-y=1$ will clearly give the values of x and y.

2. Given, $x-y=7$ and $xy=60$, to find x and y.

Here we have $(x-y)^2+4xy=(x+y)^2=289$, or $x+y=17$; consequently, $x=12$ and $y=5$.

3. Given, $x+y=a$ and $x^3+y^3=b$, to find x and y.

By division we have $\dfrac{x^3+y^3}{x+y}=x^2-xy+y^2=\dfrac{b}{a}$, and we have $(x+y)^2=x^2+2xy+y^2=a^2$; consequently, by subtraction, we have $x^2 + 2xy + y^2 - (x^2 - xy + y^2) = 3xy = a^2 - \frac{b}{a} = \frac{a^3 - b}{a}$, or $xy = \frac{a^3 - b}{3a}$. Hence, the solution of the given equations is reduced to that of the equations $x + y = a$ and $xy = \frac{a^3 - b}{3a}$.

Remark.—If we have $x - y = d$ and $x^3 - y^3 = c$, we may in a similar way reduce the solution to that of $x - y = d$ and $xy = \frac{c - d^3}{3d}$.

4. Given, $x + y = a$ and $x^4 + y^4 = b$, to find x and y.

Squaring the first we get $x^2 + 2xy + y^2 = a^2$, which gives $x^2 + y^2 = a^2 - 2xy$; and, squaring this, we get $x^4 + 2x^2y^2 + y^4 = a^4 - 4a^2xy + 4x^2y^2$, or we have $2x^2y^2 - 4a^2xy + a^4 = x^4 + y^4$. Hence, since $x^4 + y^4 = b$, the solution of the given equations is reduced to that of the quadratic $x^2y^2 - 2a^2xy = \frac{b - a^4}{2}$ (which gives xy), and the equation $x + y = a$.

Remark.—It may be shown in a similar way that the solution of $x - y = a$ and $x^4 + y^4 = b$ is reducible to that of the equations $x^2y^2 + 2a^2xy = \frac{b - a^4}{2}$ and $x - y = a$.

5. Given, $x + y = a$ and $x^5 + y^5 = b$, to find x and y.

By division we have $\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4$ $= x^4 + y^4 + x^2y^2 - xy(x^2 + y^2) = \frac{b}{a}$; consequently, since $x^4 + y^4 + x^2y^2 = 3x^2y^2 - 4a^2xy + a^4$ and $x^2 + y^2 = a^2 - 2xy$, we shall get $3x^2y^2 - 4a^2xy + a^4 - xy(a^2 - 2xy) = 5x^2y^2 - 5a^2xy + a^4 = \frac{b}{a}$ or $x^2y^2 - a^2xy = \frac{b - a^5}{5a}$.

Solving the quadratic, we shall get xy, and of course the solution of the given equations is reduced to that of $x + y = a$, and an equation of the form $xy = c$.

Remark.—If $x - y = a$ and $x^5 - y^5 = b$ are treated in a similar way, we shall get $x^2y^2 + a^2xy = \frac{b - a^5}{5a}$, and the solution depends on $x - y = a$, and an equation of the form $xy = d$.

6. Given, $x^2 + xy + y^2 = a$ and $x^4 + x^2y^2 + y^4 = b$, to find x and y.

Since the first gives $x^2 + y^2 = a - xy$, or $x^4 + 2x^2y^2 + y^4 = a^2 - 2axy + x^2y^2$, we get $a^2 - 2axy = x^4 + x^2y^2 + y^4 = b$, or $xy = \frac{a^2 - b}{2a}$. From this and the first equation we get

$x^2 + 2xy + y^2 = a + \frac{a^2 - b}{2a} = \frac{3a^2 - b}{2a}$, or $x + y = \sqrt{\frac{3a^2 - b}{2a}}$;

and from $3xy = \frac{3a^2 - 3b}{2a}$ and the first equation, we get $x^2 - 2xy + y^2 = \frac{3b - a^2}{2a}$, or $x - y = \sqrt{\frac{3b - a^2}{2a}}$; hence, x and y are easily found.

Remark.—Since the division of the second by the first equation gives $x^2 - xy + y^2 = \frac{b}{a}$, it is clear that the solution of the given equations is reduced to that of the equations $x^2 + xy + y^2 = a$ and $x^2 - xy + y^2 = \frac{b}{a}$.

7. Given, $xy = a$ and $x^n + y^n = b$, to find x and y.

If from the square of the second we subtract four times the n^{th} power of the first, we shall get $(x^n + y^n)^2 - 4x^ny^n = (x^n - y^n)^2 = b^2 - 4a^n$, or $x^n - y^n = \sqrt{b^2 - 4a^n}$; consequently, we can easily find x and y.

Remark.—If in the cubic equation $x^3 + ax = b$ we put $x = y + z$ (since $x^3 = (y + z)^3 = y^3 + 3y^2z + 3yz^2 + z^3 = y^3 + z^3 + 3yz(y + z)$), we shall get $y^3 + z^3 + 3yz(y + z) + a(y + z) = b$, or $y^3 + z^3 + (3yz + a)(y + z) = b$. Since there are two unknown letters in the preceding equation, it is clear that we may determine them by assuming $3yz + a = 0$, or $yz = -\frac{a}{3}$, which reduces the equation to $y^3 + z^3 = b$. Hence, if, as in the question, we determine y and z from the equations $yz = -\frac{a}{3}$ and $y^3 + z^3 = b$, we shall get $x = y + z$ = the root of the equation $x^3 + ax = b$; and the result will be what is called Cardan's Rule for the Solution of Cubic Equations.

8. Given, $x^3 + y^3 = a$ and $x^2y + xy^2 = b$, to find x and y.

If three times the second is added to the first, we shall get $x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3 = a + 3b$; consequently, extracting the cube root, we have $x + y = \sqrt[3]{a + 3b}$.

Hence, if we divide the second by the corresponding members of the preceding equation, we shall get $xy = \frac{b}{\sqrt[3]{a + 3b}}$; consequently, knowing $x + y$ and xy, it is easy to find x and y.

It is clear from the preceding examples, if we have two equations containing two unknown letters, as x and y, that it will often be of great importance to notice the form in which the letters enter the equations, so that they may be reduced to simpler forms by division, or the reduction of them to exact squares, cubes, etc.

4th.—Given, $3x - 2y = 5$, $2x - y = 3$, and $x + 2y = 7$, to find x and y, so that they shall nearly satisfy all the equations.

If no one of the equations is a consequence of the other two, it is clear that x and y can not be found so as exactly to satisfy each equation; nevertheless, it is plain that such values may be found for x and y as shall satisfy the equations more nearly than any other values of x and y.

Thus, the equations give, severally, $3x = 2y + 5$, $2x = y + 3$, and $x = 7 - 2y$; consequently, by adding these equations, we have $6x = y + 15$, or $x = \frac{y + 15}{6}$, which we shall regard as the average or mean value of x, as determined from the given equations. Similarly, from the given equations we have $2y = 3x - 5$, $y = 2x - 3$, and $2y = 7 - x$, which give $y = \frac{4x - 1}{5}$ for the mean value of y.

Hence, assuming the average values of x and y for their required values, we have $6x = y + 15$ and $5y = 4x - 1$; consequently, the given equations are reduced to two equations, containing two unknown letters.

Solving the equations, we get $x = 2.846$ and $y = 2.076$, which, being put for x and y in the proposed equations, give

$3x - 2y = 4.386$, $2x - y = 3.616$, and $x + 2y = 6.998$, which do not much differ from 5, 3, and 7, the results in the question.

Modification of the Method.—From the given equations we have $x = \frac{2y + 5}{3}$, $x = \frac{y + 3}{2}$, and $x = -2y + 7$, whose sum gives $3x = \frac{2y + 5}{3} + \frac{y + 3}{2} - 2y + 7$, or $18x = -5y + 61$, and in a similar way the values of y give $3y = 3x - 2$.

Solving the equations $18x = -5y + 61$ and $3y = 3x - 2$, we get $x = 2.797$ and $y = 2.130$; and putting these values for x and y in the given equations, they give $3x - 2y = 4.131$, $2x - y = 3.464$, and $x + 2y = 7.057$; these results show that the second equation is more nearly satisfied than by the preceding method.

It is clear that this method, founded on the principle that the proposed equations ought to contribute equally to the determination of x and y, ought to be very exact.

Another Solution.—Multiplying each of the proposed equations by the coefficient of x in it (according to its sign), and adding the products, we get the equation $14x - 6y = 28$, or $7x - 3y = 14$; and proceeding in like manner to multiply by the coefficients of y, etc., we get $-6x + 9y = 1$.

Solving the equations $7x - 3y = 14$ and $-6x + 9y = 1$, we get $x = 2.866$ and $y = 2.022$; and putting these values for x and y in the proposed equations, they give $3x - 2y = 4.555$, $2x - y = 3.711$, and $x + 2y = 6.910$.

From a comparison of the results obtained by these methods, it is clear that the proposed equations are as nearly satisfied by the first two as by the last.

Remarks.—1. It is plain that any number of simple equations which contain a less number of unknown letters than there are equations, may (by either method) in like manner be reduced to the same number of equations as there are unknown letters, and thence the values of the letters may be found.

2. Equations of the kind here considered often occur in Physical and Astronomical Inquiries; and the most common way of solving them is called *Gauss' Method of Minimum Squares*, which is essentially the same as our last method.

SOLUTION OF PROBLEMS, OR QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. Questions which admit of algebraic solutions must evidently involve expressed and implied conditions, which are sufficient to enable us to find any number or quantity required, from the numbers or quantities which are known or given.

2. Hence, if in the solution of any question we use one or more of the last letters of the alphabet, as x, y, z, etc., to stand for the thing or things required, then, if we use these unknown letters in conjunction with the things that are given, as we should do to verify the question if we had solved it, we shall (generally) obtain the equation or equations requisite for the solution of the question. Where it may be noticed, that when it is necessary to deduce new conditions from those which are given or implied, in order to obtain the requisite equation or equations, then no general rule can be given; and the proper mode of proceeding must be left to the skill and experience of the student.

3. Having reduced a question to an equation or equations, if we solve the equation or equations (by the methods heretofore given) we shall have solved the question, as required.

Remark.—It is generally best to use the smallest number of unknown letters possible, by expressing some of them in terms of the others, according to the conditions of the question; because the solution will (generally) thereby be simplified, and the skill of the student (in the management of difficult questions) increased.

EXAMPLES.

1. To find a number, such that if it is multiplied by 4, and 11 added to the product, the result shall equal 47.

If x stands for the number, $4x$ will stand for 4 times the number, and adding 11 we get $4x + 11$, which, by the conditions of the question, must equal 47; consequently, the question is reduced to the equation $4x + 11 = 47$.

To solve the equation, we transpose 11 and divide by 4, which gives $x = \frac{47 - 11}{4}$ or 9; and putting 9 for x in $4x +$

$11 = 47$, it becomes $4 \times 9 + 11 = 36 + 11 = 47$, an identical equation; consequently, 9 is the required number.

2. Divide \$968 between A and B, so that A may have 6 times as much as B and \$93 more.

Denote B's money by x, then the question gives the equation $(6 + 1)x + \$93 = \$968$.

Hence B must have \$125 and A must have \$843.

3. To find a number, such that if it is multiplied by 6, and 7 subtracted from the product, the remainder shall equal 23.

Let x stand for the number, then, by the question, we have the equation $6x - 7 = 23$. Hence $x = 5$.

4. Divide \$1900 between A and B, so that A shall have 7 times as much as B and \$100 less.

Represent B's money by x, then, from the conditions of the question, we shall have the equation $(7 + 1)x - \$100 = \$1900$.

Hence B must have \$250 and A must have \$1650.

5. Divide \$1800 between A, B, and C, so that their shares shall be as the numbers 2, 3, 4.

The shares may clearly be represented by $2x$, $3x$, and $4x$; consequently, by the conditions of the question, we shall have the equation $2x + 3x + 4x = 9x = \$1800$ or $x = \frac{\$1800}{9} = \$200$.

Hence, the shares of A, B, and C are \$400, \$600, and \$800.

Remark.—If we have to divide the number or quantity, a, into parts, which shall be as the numbers m, n, p, q, etc.; then, since mx, nx, px, qx, etc., may clearly represent the parts, we shall have $\frac{ma}{m + n + p + \text{etc.}}$, $\frac{na}{m + n + p + \text{etc.}}$, $\frac{pa}{m + n + \text{etc.}}$, for the parts; results that are in conformity to the rule of Fellowship or Partnership in Arithmetic.

6. Divide \$3600 between A, B, and C, so that A and B together shall have 3 times as much as C, and that A and C together shall have half as much as B.

From the implied (or implicit) relations between the shares of A, B, and C, it results that their shares may be expressed by x, $8x$, and $3x$.

Hence, by the question, we have $12x = \$3600$ or $x = \$300$; consequently, the shares of A, B, and C are severally expressed by \$300, \$2400, and \$900.

7. To find the interest on any principal, P, for any number of days, T, at six per cent. per annum, simple interest; supposing the year to consist of 365.25 days.

Let x stand for the interest; then 0.06 expresses the interest of a unit of P for 365.25 days, and of course $0.06 \times P$ is the interest of P for the same time. Because the interest on P is proportional to the times, we have $365.25 : T :: 0.06P : x$ or $x = \frac{0.06PT}{365.25} = \frac{0.08PT}{487}$; which is the interest, as required.

Hence, *the calculation of the interest on* P, *at six per cent. for the time*, T (*supposing the year to consist of* 365.25 *days*), *is reduced to the calculation of the interest on* P *at eight per cent. per annum, supposing the year to consist of* 487 *days*. Thus, the interest on \$47.25 for $23\frac{1}{2}$ days, as found from $x = \frac{0.08PT}{487}$, by putting \$47.25 for P, and $23\frac{1}{2}$ for T, will be found to give $x = \$0.1824$ for the required interest.

Remark.—Because $487 = 480 + 7 = 480\left(1 + \frac{7}{480}\right)$, we shall have $x = \frac{0.08PT}{487} = \frac{0.01PT}{60\left(1 + \frac{7}{480}\right)} = \frac{0.001 \times PT}{6}\left[1 - \frac{7}{480} + \left(\frac{7}{480}\right)^2 - \left(\frac{7}{480}\right)^3 + \text{etc.}\right]$; consequently, putting $\frac{0.001 \times PT}{6} = A$, $\frac{7A}{480} = B$, $\frac{7B}{480} = C$, and so on; we shall have $x = A - B + C - D + \text{etc.}$; which can be put under the forms $x = A - (B - C + D - E + \text{etc.}) = A - B + (C - D + E - F + \text{etc.}) = A - B + C - (D - E + F - G + \text{etc.}) = \text{etc.}$; consequently, since each term is less than the next preceding term, it follows that x is less than A and greater than $A - B$, and that it is less than $A - B + C$ and greater than $A - B + C - D$, and so on. If we apply the preceding series to find the interest of \$47.25 for $23\frac{1}{2}$ days, we shall have A = \$0.18506, which is too great, and A − B = \$0.18506 − \$0.00269 = \$0.18237, which is too small, and so on; and it is clear that we may take \$0.1824 for the sought interest.

8. "A hare is 50 leaps before a greyhound, and she takes 4 leaps while the greyhound takes 3; but 2 greyhound's leaps are equal to 3 of the hares. How many leaps must the greyhound make to overtake the hare?"

Let x stand for the number of leaps of the hound, then $\frac{3x}{2}$ are clearly the corresponding leaps of the hare. Also, $\frac{3x}{2} - 50$ are the number of leaps of the hare, while those of the hound are x. Hence, by the question, we have the proportion $4 : 3 :: \frac{3x}{2} - 50 : x$, or $4x = 3\left(\frac{3x}{2} - 50\right)$; consequently, $x = 300$.

9. To find a number, such that if from the sum of its third and fourth parts we subtract its fifth part, the result shall equal 69.

Let x denote the number, then from the conditions of the question we shall have $\frac{x}{3} + \frac{x}{4} - \frac{x}{5} = 69$; consequently, $x = 180$.

Otherwise, to avoid fractions, put $60y$ for the number, and we shall have $\frac{60y}{3} + \frac{60y}{4} - \frac{60y}{5} = 69$, or $20y + 15y - 12y = 23y = 69$, or $y = 3$; consequently, $60y = 180$, as required.

10. To find two numbers, such that if the first is divided by 3, and the second by 4, the sum of their quotients shall equal 89; and if the first is divided by 5, and the second by 13, the difference of their quotients shall equal 18.

To avoid fractions, let $15x$ and $52y$ stand for the numbers, then by the conditions of the question we shall have the equations $5x + 13y = 89$ and $3x - 4y = 18$. Multiplying the members of the first equation by 3, and those of the second by 5, they give $15x + 39y = 267$ and $15x - 20y = 90$; consequently, subtracting the members of the second of these from the corresponding members of the first, we get $59y = 177$, or $y = 3$; and thence $x = 10$.

Hence, $15x$ and $52y$ become 150 and 156, which are the required numbers.

11. "If A and B together can perform a piece of work in 8 days, A and C together in 9 days, B and C together in 10 days, how many days will it take each person to perform the same work alone?"

Because A and B do the work in 8 days, it is clear that they do $\frac{1}{8}$ of the work in one day; and in like manner A and C do $\frac{1}{9}$ of it in one day; B and C do $\frac{1}{10}$ of it in one day; consequently, by adding and dividing by 2, it follows that A, B, and C together will do the part of the work expressed by $\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) \div 2$, in one day.

If from the preceding result we subtract the work done by B and C in one day, we shall get $\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) \div 2 - \frac{1}{10} = \left(\frac{1}{8} + \frac{1}{9} - \frac{1}{10}\right) \div 2 = \frac{49}{720}$ for the part of the work done by A in one day, and thence $\frac{720}{49} = 14\frac{34}{49}$ equals the number of days that it will take A to do the work; and in like manner B can do the work in $17\frac{23}{41}$ days; C can do it in $23\frac{7}{31}$ days.

12. If A can do a piece of work in a days, and B can do it in b days, C in c days, and so on, it is required to find how long it will take them jointly to perform the work.

If 1 stands for the work and t for the time, and if we represent $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} +$ etc., by m, then we shall have $t = \frac{1}{m}$.

Thus, if A can mow a piece of grass in 2 days, B in 3 days, and C in 4 days, then, since $m = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$, we shall have $t = \frac{12}{13}$ of a day, for the time in which A, B, and C together can mow the piece of grass.

13. "Five gamblers, A, B, C, D, E, play together on this condition, that he who loses shall give to all the rest as much as they already have. First A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the fifth game they have all the same sum, viz., each \$32. How much had each before they began to play?"

If we use a to represent the money each had at the end of the play, $5a$ will stand for what the gamblers all had; consequently, if x stands for A's money at the commencement of the play, $5a - x$ will equal the money which all the rest had at the same time.

Since A loses $5a - x$ in the first game, $x - (5a - x) = 2x - 5a$ will equal his money at the end of the first game; and as A's money at the end of the first game is doubled at the end of the second game, which is doubled again at the end of the third game, and so on, it follows from the question that we shall have $2^4(2x - 5a) = a$, or $x = \frac{5a}{2} + \frac{a}{32}$; consequently, putting \$32 for a, we get $x =$ \$81.

Because B's money is doubled at the end of the first game, and that he loses at the end of the second game, it follows, if we change $2^4(2x - 5a) = a$ into $2^3(2^2x - 5a) = a$, that x will represent B's money at the commencement of the play; consequently, by solving the equation, we get $x = \frac{5a}{4} + \frac{a}{32} =$ \$41.

In a similar way, it appears, if there are n gamblers, that the equation $x = \frac{na}{2^m} + \frac{a}{2^n}$ will give the money of the m^{th} gambler at the commencement of the play. Thus, putting $n = 5$ and m equal to 3, 4, 5, successively, we get \$21, \$11, and \$6 for what C, D, and E severally had at the commencement of the play.

Remarks.—1. We have taken this question from Ross' Translation of Hirsch's Collection of Examples, etc., see Vol. I., page 219, question 66, in which we have put $a =$ \$32, instead of £32.

2. We have given the solution of the question, because it is sometimes considered as difficult, and for the purpose of

making the solution applicable to any number of gamblers; also, for the purpose of showing how to derive certain expressions from others, according to the laws by which they are connected.

14. "The hour and minute-hand of a clock are exactly together at 12 o'clock; when are they next together?"

Since the minute-hand makes twelve revolutions while the hour-hand makes one, it results that from 12 o'clock at noon to 12 o'clock at night, or in 12 hours, the minute-hand will pass the hour-hand 10 times, and coincide with it at 12 o'clock at night; consequently, if x denotes the sought time (since the hands move uniformly), we shall clearly have $x = \frac{12 \text{ hours}}{11} = 1$ hour and $\frac{1}{11}$ of an hour, or the hands coincide at $\frac{1}{11}$ of an hour after 1 o'clock. Hence, the hands will be together at 1 o'clock and $\frac{1}{11}$, 2 o'clock and $\frac{2}{11}$, 3 o'clock and $\frac{3}{11}$, and so on.

Remarks.—1. If any two bodies, A and B revolve around a common center in given times, P and P′, and that whether their angular motions are uniform or not; then, if 1 stands for the angular motion of either body in one revolution, we shall have $\frac{1}{P}$ and $\frac{1}{P'}$ for the mean angular motions of the bodies in a common unit of the times.

2. If P is less than P′, and the bodies are moving in the same direction, $\frac{1}{P} - \frac{1}{P'}$ evidently expresses the excess of the mean angular motion of A over that of B; consequently, if a stands for the angular distance of the bodies at any time T, and t for the time reckoned from T, when (at a mean rate of motion) the angular distance of the bodies equals 0, we shall clearly have $t = \frac{a}{\frac{1}{P} - \frac{1}{P'}} = \frac{aP'}{\frac{P'}{P} - 1} = \frac{aP}{1 - \frac{P}{P'}} = \frac{aPP'}{P' - P}$, (1). If the bodies move in contrary directions, it is plain that (1) becomes $t = \frac{a}{\frac{1}{P} + \frac{1}{P'}} = \frac{aP'}{\frac{P'}{P} + 1} = \frac{aP}{1 + \frac{P}{P'}} = \frac{aPP'}{P' + P}$,

(2). Thus, if $a = \frac{1}{12}$, $P = 1$ hour, and $P' = 12$ hours, then, from $t = \frac{aP'}{\frac{P'}{P} - 1}$ we get $t = \frac{1}{11}$ of an hour, and the hands of a clock coincide at $\frac{1}{11}$ of an hour after 1 o'clock.

If the hands move in contrary directions, then, on the same suppositions, the hands will be together at $\frac{1}{13}$th of an hour after 1 o'clock, as is clear from $t = \frac{aP'}{\frac{P'}{P} + 1}$.

3. If for a in (1) we put 1, they will be adapted to what are called the synodic motions of the planets around the sun. Thus, if $P = 224.700787$ days, and $P' = 365.256374$ days, which are the periodic times of the revolutions of Venus and the earth around the sun; then, since $\frac{P}{P'} = 0.615186$ and $1 - \frac{P}{P'} = 0.384814$, we shall, from $t = \frac{aP}{1 - \frac{P}{P'}} = \frac{P}{1 - \frac{P}{P'}}$, get $t = 583.920509$. days, which is called the *synodic period* of Venus's revolution around the sun, or it is the mean apparent time of its revolution around the sun to an observer at the earth's center.

15. Two bodies, A and B, are moving uniformly forward in the same direction, in a right line, at such a rate that A passes over the distance m in a unit of the times, and B over the distance n in the same time; then, supposing A and B to be at the distance a from each other, at the time T, it is proposed to find the time t reckoned from T, when A and B will be together.

If A is in advance of B, and n greater than m, it is clear that we shall have $t = \frac{a}{n - m}$, because B gains the distance $n-m$ on A in each unit of the time. If A is in advance of B, and m greater than n, it is clear that A and B must have been together before the time T, and that the interval is expressed by $t = \frac{a}{m-n} = -\frac{a}{n-m}$.

If we add $t = \frac{a}{n-m}$ to T, we shall get $\text{T} + t = \text{T} + \frac{a}{n-m} = \frac{n\text{T} - (m\text{T} - a)}{n-m}$ (1), and if we subtract $t = -\frac{a}{n-m}$ from T, we shall have $\text{T} - t = \frac{n\text{T} - (m\text{T} - a)}{n-m} = \frac{m\text{T} - (n\text{T} + a)}{m-n}$, (2). By comparing (1) and (2), it is clear that the first form of (2) differs from (1) in no respect excepting that the sign of t is changed from + to −; consequently, it is clear that (1), by changing the sign of t, embraces the solution of the question in its fullest extent.

If the bodies move in contrary directions, we must clearly change the sign of m, and we shall have $t = \frac{a}{n+m}$; consequently, (1) becomes $\text{T} + t = \frac{n\text{T} + m\text{T} + a}{m+n} = \frac{(m+n)\text{T} + a}{m+n}$, (3).

If m equals n, and the bodies are moving in the same direction, $t = \frac{a}{n-m}$ is reduced to $t = \frac{a}{0} = \infty$ = an infinite time, which clearly shows, if a is finite, that the bodies can never be together; which is also clear from the nature of the case, since the bodies move equally fast.

Again, if $m = n$, and $a = 0$, or an infinitesimal, $t = \frac{a}{n-m}$ becomes $t = \frac{0}{0}$, which is of an indeterminate form; and it clearly shows t to be indeterminate, since the bodies moving equally fast, and being together at the time T, it is clear that they will always be together and move forward as one body.

Also, if n and m are not equal to each other, while $a = 0$; then $t = 0$, and it is plain that the bodies, being together at the time T, can not be together at any other time.

Hence, the bodies can be together in all cases excepting when a is finite, and $n = m$, or when $t = \frac{a}{0} = \infty$, which shows the conditions of the question to be such that the bodies can never be together, or, as it is sometimes said, it indicates an absurdity in the question.

Remarks.—1. When n is greater than m, we have found $t = \frac{a}{n-m}$, and when m is greater than n, we have found $t = \frac{a}{m-n}$; and thence we have obtained $T + t = T + \frac{a}{n-m}$ and $T - t = T - \frac{a}{m-n} = T + \frac{a}{n-m}$ for the times when the bodies are actually together. It is hence clear that the actual time when the bodies are together is properly expressed by $T + \frac{a}{n-m}$, and that whether n is greater or less than m; consequently, since the expression is true when there is any difference, however small, between m and n, it clearly ought to be regarded as true when $m = n$, or when $t = \frac{a}{0} = \infty =$ an infinite time.

2. From the expressions $t = \frac{a}{n-m}$ and $t = \frac{a}{m-n}$, we get the actual times when the bodies are together, by adding the first to T and subtracting the second from it; consequently, if we consider the first as positive, and write the expressions in isolated forms, we shall have $t = \frac{a}{n-m}$ and $t = -\frac{a}{m-n}$; noticing, that the sign $-$ which precedes $\frac{a}{m-n}$ means that $\frac{a}{m-n}$ is to be subtracted, and that t in the first member which is under a positive form is essentially negative.

Hence we perceive the use of isolated negative quantities in calculation; for to get the actual times when the bodies are together from $t = -\frac{a}{m-n}$, we must subtract $\frac{a}{m-n}$ from T, and we shall get $T + \frac{a}{n-m}$, as required; hence the reasonableness of the remarks made in Section I., pp. 2 and 3, is manifest.

16. Two travelers, A and B, are 150 miles apart, and going in the same direction, A following B who travels uniformly 30 miles a day. If A overtakes B in 50 days, it is proposed to find the rate of A's daily travel. *Ans.* 33 miles.

17. A person went a distance of 600 miles in 20 days. Some days he went 40 miles, while during the remaining days his daily progress was only 15 miles. It is proposed to find the number of days in which he traveled 40 miles, and the number of days in which he made only a distance of 15 miles. *Ans.* 12 days at the rate of 40, and 8 at 15.

18. A, B, and C started together for a place 300 miles distant, and traveled uniformly; and their daily travel was such that B completed the distance 10 days earlier than A, and C 20 days later than A; moreover, B's daily travel was to that of C as 5 to 2. How many days were A, B, and C on the road, and how many miles did they travel per day?

Ans. The times were 30, 20, and 50 days; and the daily rates of travel were 10, 15, and 6 miles.

19. A and B start together from the same place and travel in the same direction; A goes a miles the first day, $a+b$ miles the second day, $a+2b$ miles the third day, and so on, in arithmetical progression; also, B goes a' miles the first day, $a'+b'$ miles the second day, $a'+2b'$ the third day, and so on, in arithmetical progression. It is hence proposed to find when A and B will be together again, after they commence their journey.

Ans. If t stands for the time in days, we have $t = \frac{2(a'-a)}{b-b'} + 1$. Thus, if $a=1$, $b=2$, and $a'=40$, $b'=1$, we get $t=79$ days.

20. A person has two bins; in one of which is a mixture of corn and oats, such that the bin contains 25 bushels of corn and 37 bushels of oats, the whole being worth $34.29; the other bin contains a mixture of 80 bushels of corn and 9 bushels of oats, worth $63.78. Supposing the corn and oats in the bins to constitute mixtures such that the corn and oats in one bin are of the same value per bushel that they are in the other bin; it is proposed to find the worth of the corn and oats per bushel.

Ans. The corn is worth $0.75 per bushel, and the oats are worth $0.42 per bushel.

21. A paid a debt of £25 15*s.* by using a certain number of guineas and sovereigns; and B by interchanging

the denominations of the pieces paid a debt of £25 10*s*. Supposing the guinea to be 21 shillings, and the sovereign 20 shillings, how many pieces of each kind were used?

Ans. A used 15 guineas and 10 sovereigns, and B used 10 guineas and 15 sovereigns.

22. A person had a certain number of dollars, which, being counted out by 8 dollars at a time, there remained nothing over, but being counted out by 13 dollars at a time there remained 9 dollars; then how many dollars did the person have, supposing that the number of counts amounted to 51? *Ans.* 256.

23. There is a number consisting of two digits, such that if the number is increased by the sum of its digits the sum is 102, and if the number is diminished by the sum of its digits the remainder is 72; what is the number? *Ans.* 87.

24. What number is that whose unit's digit is 2 times the ten's digit, and is such that the number equals 4 times the sum of its two digits?

Ans. 12, 24, 36, and 48 satisfy the question.

25. There is a number consisting of two digits, such that the number equals 4 times the sum of its digits, and that the product of the digits equals 2 times their sum; what is the number? *Ans.* 36.

26. Why are the numbers 21, 42, 63, and 84 severally equal to 7 times the sum of their digits?

27. The difference of two numbers is 3, and the difference of their squares is 219; what are the numbers?

Ans. 38 and 35.

28. The difference of two numbers or quantities is a, and the difference of their squares is b; what are they?

Ans. $\frac{b+a^2}{2a}$ and $\frac{b-a^2}{2a}$.

29. The product of two numbers is 35, and the sum of their squares is to the difference of their squares as 37 to 12; what are they? *Ans.* 7 and 5.

30. The product of two numbers or quantities is a, and the sum of their squares is to the difference of their squares

as m to n; it is hence required to find the numbers or quantities. *Ans.* $\sqrt[4]{\frac{(m+n)a^2}{m-n}}$ and $\sqrt[4]{\frac{(m-n)a^2}{m+n}}$.

31. A put a certain sum at interest, at 6 per cent. per annum. At the end of two years he received back \$196 as interest and principal; and at the end of two years more he received \$284 as interest and principal; also, at the end of the next two years he received \$480 as interest and principal. He then found that there was only \$100 due him; it is hence required to find the principal. *Ans.* \$800.

32. If P denotes any principal, r the rate per cent., and t the time in years; it is required to find t, such that the interest shall equal P^n.

Ans. $t=\frac{100P^{n-1}}{r}$; thus, if $n=2$, $P=6$, $r=6$, then $t=$ 100 years.

33. Let any principal, P, put at simple interest for the time, T, at the rate of interest, r, on a unit of P in a unit of the time, amount to P′; then, supposing P′, r, and T to be known, it is required to find P.

Ans. $P=\frac{P'}{1+rT}$; in which P *is called* the present *worth or value* of P′.

34. Suppose the sums P, P′, P″, etc., severally, are to be paid at the times T, T′, T″, etc.; then supposing r to be the interest on a unit of the money in a unit of the times, it is proposed to find the time, t, at which the moneys P, P′, P″, etc., may be paid at once, and that without loss or gain.

Ans. Let p, p', p'', etc., be the present values of P, P′, P″, etc., severally, and represent $p+p'+p''+$ etc., by S, and $P+P'+P''+$ etc., by S′; then $\frac{S'}{1+rt}$ is the present value of S′, which is also represented by S; consequently, we must have $S=\frac{S'}{1+rt}$, or $t=\frac{S'-S}{rS}$; where it may be noticed that t is called the *equated time* of the payments. Thus, if $P=\$200$, $P'=\$600$, $P''=0$, etc., $T=4$ years, $T'=12$ years, $r=\$0.06$; then we shall have $p=\$161.290$, $p'=\$348.839$, $S'=\$800$, $S=\$510.129$; consequently, $t=$

9.47 years. Calculating the amount of \$510.129 at the rate $r = \$0.06$ for 9.47 years, we find it to be \$799.984; consequently, since this differs from \$800 (only) by \$0.016, it follows that our process must be very exact.

Remarks.—1. Since $S = S' - (S' - S)$, it is plain that we shall have $t = \frac{S' - S}{r[S' - (S' - S)]} = \frac{S' - S}{rS'}\left(1 + \frac{S' - S}{S'} + \left(\frac{S' - S}{S'}\right)^2 + \text{etc.}\right)$; consequently, if $\frac{S' - S}{S'}$ is very small in comparison to 1, we may approximately put 1 for the series, and we shall get $t = \frac{S' - S}{rS'}$ nearly.

2. Since $p = \frac{P}{1 + rT} = P(1 - rT + r^2T^2 - \text{etc.})$, $p' = P'(1 - rT' + r^2T'^2 - \text{etc.})$, and so on, we shall have $p + p' + \text{etc.} = P + P' + \text{etc.} - r(PT + P'T' + \text{etc.}) + r^2(PT^2 + P'T'^2 + \text{etc.}) - r^3(PT^3 + \text{etc.}) + \text{etc.}$; consequently, if T, T', are so small that the terms which involve r^2, r^3, etc., may be rejected, we shall approximately have $p + p' + \text{etc.} = S = P + P' + \text{etc.} - r(PT + P'T' + \text{etc.}) = S' - r(PT + P'T' + \text{etc.})$, or $S' - S = r(PT + P'T' + \text{etc.})$; hence, by substituting the value of $S' - S$, t is reduced to $t = \frac{PT + P'T' + \text{etc.}}{S'}$ nearly, which is in accordance with the common rule for finding the equated time.

3. If we calculate the value of t by the preceding expression, which corresponds to the example given above, we shall get $t = 10$ years, instead of 9.47 years, as we have shown (above).

4. If, instead of putting p, p', etc., for $\frac{P}{1 + rT}$, $\frac{P'}{1 + rT'}$, etc., we put them for $\frac{P}{(1 + r)^{T}}$, $\frac{P'}{(1 + r)^{T'}}$, etc., then they will stand for the present values of P, P', at compound interest.

Also, instead of the equation $S = \frac{S'}{1 + rt}$, we shall have the equation $S = \frac{S'}{(1 + r)t}$ or $(1 + r)^t = \frac{S'}{S}$; consequently, by the properties of logarithms, we shall get $t = \frac{\log. S' - \log. S}{\log. (1 + r)}$ for the *equated time of payment.*

35. A and B traded with a joint stock of \$3000, and gained \$2400. Now A's stock, increased by \$100, was found to be to that of B, when diminished by \$100, as 7 to 3. What were the stocks of A and B, and what part of the gain belonged to each?

Ans. The stocks were \$2000 and \$1000, and their shares of the gain were \$1600 and \$800.

36. The stocks of A and B are such, that if each of them is increased by \$10000 they will be to each other as 3 is to 4; and if each stock is diminished by \$2000, they will be to each other as 3 to 8. What are the stocks?

Ans. \$5000 and \$10000.

37. It has been found by experiment that the weights of a cubic foot of water, cork, and iron are severally 1000, 240, and 7600 avoirdupois ounces, at the temperature of about 60 degrees of Fahrenheit's thermometer. It is hence required to find how many cubic feet of iron must be attached to 10 cubic feet of cork, so that when the compound is put into water the cork shall be just immersed in the water, admitting that the water does not enter the pores of the cork and iron.

Ans. If a denotes the number of cubic feet of cork, and x those of the iron, we have $x = \frac{38a}{330} = \frac{19a}{165}$; and putting $a = 10$, we get $x = 1\frac{5}{33}$ cubic feet.

38. Representing the weight of a certain bulk of water by 1, *we call the proportional weights of equal bulks of other substances their specific gravities.* Now if we (in round numbers) use 19 to stand for the specific gravity of gold, and suppose that there is a substance such that if we attach to it a quantity of gold whose bulk is $\frac{1}{5}$ of that of the substance, and find $3.366\dot{6}$. for the specific gravity of the compound; then it is required to find the specific gravity of the substance.

Ans. 0.240, consequently the substance is cork.

39. Let x and y stand for the bulks of two substances which are mixed together, and are such that the sum of the bulks before mixture is sensibly the same that it is after

mixture. Then if a, b, c are the specific gravities of x, y, and the mixture, it is proposed to find the ratio of x to y, and also the ratio of their weights.

Ans. $\frac{x}{y} = \frac{c-b}{a-c}$, and $\frac{ax}{by} = \frac{a(c-b)}{b(a-c)} =$ the ratio of the weights.

Remarks.—1. The conditions of the question give the equation $ax + by = c(x+y)$, or $c = \frac{ax+by}{x+y}$; consequently, if a represents the worth of a unit of x, and b that of a unit of y, it follows that $ax + by$ will be the worth of $x + y$; consequently, c will be the worth of a unit of $x + y$ in the mixture; hence, the reasonableness of the rule of Alligation Medial in Arithmetic becomes evident.

2. From $\frac{x}{y} = \frac{c-b}{a-c}$ it follows, if we represent x by $c - b$, that we must represent y by $a - c$. Hence, since a, b, and c may severally represent the worth of a unit of x, y, and of the mixture, the reason of the rule of Alligation Alternate in Arithmetic is manifest.

40. Given, 10 and 26, the sums of the first and second terms, and of the third and fourth terms of an arithmetical progression, to find the terms of the progression.

Ans. 3, 7, 11, and 15.

41. Given, 15 and 60, the sums of the first and second terms, and of the second and third terms of a geometrical progression, to find the progression.

Ans. 3, 12, 48, 192, 768, etc.

42. To find a number, which, being added to 4, 16, and 40, the results shall be in geometrical progression. *Ans.* 8.

43. Given, the sum and sum of the squares of three numbers in geometrical progression, equal to 7 and 21, to find the progression. *Ans.* 1, 2, and 4.

*44. "A father, who has three children, bequeaths his property by will in the following manner: To the eldest son he leaves a sum a, together with the n^{th} part of what remains; to the second he leaves a sum $2a$, together with the

n^{th} part of what remains after the portion of the eldest and $2a$ have been subtracted; to the third he leaves a sum $3a$, together with the n^{th} part of what remains after the portions of the two other sons and $3a$ have been subtracted. The property is found to be entirely disposed of by this arrangement. Required the amount of the property."

Ans. It is clear that $3a$ is the property received by the third son, since there is no remainder; and as the second son received $2a$ and an n^{th} part of the remainder, if we represent the remainder by x, we shall clearly have $\frac{n-1}{n}x = 3a$, or $x = \frac{3na}{n-1}$; consequently, $\frac{x}{n} + 2a = \frac{(2n+1)a}{n-1}$ is the property received by the second son.

Again, $x + 2a = \frac{3na}{n-1} + 2a = \frac{(5n-2)a}{n-1}$ is the remainder of the father's property after the first son received his share, which by the question must be the $\frac{n-1}{n}$th part of the remainder after he received the portion a; consequently, if y denotes the remainder, we shall get $\frac{n-1}{n}y = \frac{(5n-2)a}{n-1}$, or $y = \frac{(5n^2-2n)a}{(n-1)^2}$; hence, $\frac{y}{n} + a = \frac{(5n-2)a}{(n-1)^2} + a = \frac{(n^2+3n-1)a}{(n-1)^2}$ is the portion of the eldest son; and it is clear that $y + a = \frac{(5n^2-2n)a}{(n-1)^2} + a = \frac{(6n^2-4n+1)a}{(n-1)^2}$ expresses the father's property.

45. The same things being supposed as in the preceding question, with the addition that the father has $n - 1$ children instead of three, and that the eldest receives a and an n^{th} part of the remainder, the next $2a$ and an n^{th} part of the remainder, and so on to the last, who receives $(n-1)a$; then it is proposed to show that $(n-1)a$ is the portion of each child, and that the father's estate equals $(n-1)^2a$.

Ans. We have (as before) $\frac{n-1}{n}x = (n-1)a$, or $x = na$; consequently, $\frac{x}{n} + (n-2)a = a + (n-2)a = (n-1)a =$ the portion of the child whose number is $n-2$. Also, $x + (n-2)a = 2(n-1)a$ gives the equation $\frac{n-1}{n}y = 2(n-1)a$, or $y = 2na$; consequently, $\frac{y}{n} + (n-3)a = 2a + (n-3)a = (n-1)a$ is the portion of the child whose number is $n-3$; and in like manner it may be shown that the child whose number is $n-4$ receives $(n-1)a$ for his portion, and so on; consequently, $(n-1)^2a =$ the father's estate.

Remarks.—We have extracted this and the preceding question from Davies' Translation of Bourdon's Algebra. [See p. 79, etc., of the work; published by Wiley & Long in 1835.] We have given the questions, and a slight sketch of the methods to be used in their solution; because the methods used by Bourdon appear to us to be unnecessarily long and embarrassing.

46. Two market-women, A and B, had (each) the same number of eggs; A sold hers in parcels consisting of 9 eggs each, and received 8 cents for each parcel sold; also, B sold hers in parcels consisting of 13 eggs each, and received 12 cents for each parcel sold; and the money received by A and B together was $10.40. A sold more eggs than B, and received less money by twice as many cents as A and B had eggs that remained unsold; moreover, B had three times as many eggs remaining as A had. It is proposed to find the number of eggs which A and B had, and what they each received for the eggs sold, and the number of eggs that remained to each.

Ans. A and B each had 578 eggs; A received $5.12, and B received $5.28; A had two eggs left, and B had six.

47. "Two artillerymen together filled 1000 cartridges, and used the same quantity of powder. One says to the other, 'If I had filled as many cartridges as you have, I should have made use of 18 cwt. of powder.' Then the other answers him, 'Had I filled as many cartridges as you have, I should have used only 8 cwt. of powder.' How many cartridges did each fill? and how much powder did each use?

"*Ans.* The first filled 400, the second 600 cartridges, and each used 12 cwt. of powder."

APPENDIX TO (SIMPLE) EQUATIONS.

Since in finding the limits of numbers or quantities it is frequently necessary to use those which are not equal to each other, we here propose to give the general principles of reasoning which are applicable to

INEQUALITIES OR INEQUATIONS.

1. If A and B stand for two numbers or quantities of the same kind (which may consist of one or more terms), such that A is greater than B, then, according to 18, of Sec. I., we signify that A is greater than B, or B less than A, by writing $A > B$, or $B < A$, which are called *inequalities* or *inequations*, noticing that A and B are called the first and second members of the first inequality, while B and A are those of the second. The first inequality is read by saying that A is greater than B, or (which is often preferable) by saying that A is not less than B; also, the second inequality is read by saying that B is less than A, or B is not greater than A.

Inequalities whose first members are all greater or less than their second members, are said to exist in the same sense; but inequalities whose first members are not all greater or less than their second members, are not all said to exist in the same sense. Thus, $3>2$ and $7>4$ exist in the same sense; also, $5<6$ and $10<14$ exist in the same sense; but $8<15$ and $9>6$ exist in a contrary sense.

2. If $\pm C$ is added to the members of the preceding inequalities, they will clearly become $A \pm C > B \pm C$, or $B \pm C < A \pm C$, which exist in the same sense as the given inequalities. Thus, if 5 is added to the members of $x - 5 > 2$, it becomes $x > 7$; and subtracting 3 from the members of $x + 3 < 9$, it becomes $x < 6$.

Hence, *any term in either member of an inequality* (as in an equation) *may be transposed into the other member by changing its sign, without altering the sense of the inequality;* also, *any term which is common to both members, and has the same sign in each, may be erased or stricken out of the inequality, without affecting its sense.* Thus, taking $A \pm C > B \pm C$ or $B \pm C < A \pm C$, and erasing $\pm C$, the inequalities become $A > B$ or $B < A$; also, transposing A and B, the inequality $A > B$ or $B < A$ will be changed to $-B > -A$ or $-A < -B$; consequently, $A > B$ and $-A < -B$ exist in a contrary sense. Hence, *when the signs of both members of any inequality are changed (or multiplied by -1), the sense of the inequality will be changed.*

3. Adding the corresponding members of $A>B$ or $B<A$, and $C > D$ or $D < C$, the sums clearly give $A + C>B + D$ or $B + D < A + C$.

Consequently, *the sum of any number of inequalities which exist in the same sense will exist in the same sense as the inequalities.* Thus, from the addition of $x + 3 > 5$ and $y - 3 > 7$ there results $x + y > 12$.

If from $8> 5$ or $5 < 8$ we take $6 > 4$ or $4 < 6$, we get $2 > 1$ or $1 < 2$, which exist in the same sense as the given inequalities. But if from $8 > 5$ or $5 < 8$ we take $7 > 3$ or $3 < 7$, we get $1 < 2$ or $2 > 1$, which do not exist in the same sense as the given inequalities.

Hence, *the difference of two inequalities which exist in the same sense is not always an inequality which exists in the same sense as the given inequalities.*

4. Taking the inequality $\frac{x}{2} - 3 > \frac{x}{3} + 1$ or $\frac{x}{3} + 1 < \frac{x}{2} - 3$, and reducing its terms to a common denominator, and then rejecting the denominator, there results the inequality $3x - 18 > 2x + 6$ or $2x + 6 < 3x - 18$, which gives $x > 24$ or $24 < x$.

Hence, *considering and treating the denominators of the fractional terms of any inequality as being positive, we free the inequality from fractions in the same way that we free an equation from fractions, and the sense of the inequality will remain unchanged.*

Also, any positive factor which is common to the terms of an inequality may be erased, and *vice versa*, and that without altering the sense of the inequality. Thus, from $8 > 6$ and $\frac{1}{2} < 1$ we get $4 > 3$ and $1 < 2$.

5. If we have the inequalities $A > B$ or $B<A$ and $C>D$ or $D < C$, such that their members are essentially positive, then by taking the product of their corresponding members, we shall plainly have $AC > BD$ or $BD < AC$.

Hence, *the product of any number of inequalities which exist in the same sense, and whose members are essentially positive, is an inequality which exists in the same sense as the given inequalities.*

Hence, too, *if the members of such an inequality are raised to any power whose index is a positive integer, the result will be an inequality which exists in the same sense as the given inequality; but if the index of the power is a negative integer, it is plain that the result will be an inequality which exists in a sense contrary to that of the given inequality.* Thus, from $x > 3$ and $y > 2$ we get $xy > 6$; and from $x^{\frac{1}{3}} < 3$ and $y^{-\frac{1}{2}} > 2$ we get $x < 27$ and $(y^{-\frac{1}{2}})^{-2} < 2^{-2}$ or $y < \frac{1}{4}$.

6. If we take the product of the (corresponding) members of the inequalities $-A < -B$ or $-B > -A$ and $-C < -D$ or $-D > -C$, we get $AC > BD$ or $BD < AC$, an inequality which exists in a sense contrary to that of the given inequalities. If we multiply $AC > BD$ or $BD < AC$ by $-E < -F$ or $-F > -E$, the result will be $-ACE < -BDF$ or $-BDF > -ACE$.

Hence, *the product of an even number of inequalities which exist in the same sense, and whose members are essentially negative, is an inequality which exists in a sense contrary to that of the given inequalities; and the product of an odd number of such inequalities is an inequality which exists in the same sense as the given inequalities.*

Hence, also, *any even positive integral power of such an inequality will be an inequality which will exist in a sense contrary to that of the given inequality; and if we take the index of the power with the negative sign, the power will be an inequality which will exist in the same sense as the given inequality; but if the members of such an inequality are raised to any odd positive integral power, the result will be an inequality which will exist in the same sense as the given inequality, while, if the index of the power is taken with the negative sign, the result will be an inequality which will exist in a sense contrary to that of the given inequality.*

Thus, from $-4 < -3$ and $-5 < -2$ we get $20 > 6$; and from $-2 > -3$, $-4 > -5$, and $-6 > -7$ we have

$-48 > -105$. Also, from $-2 > -3$ we get $(-2)^2 < (-3)^2$ and $(-2)^{-2} > (-3)^{-2}$ or $4 < 9$ and $\frac{1}{4} > \frac{1}{9}$; and from $-3 < -2$ we have $(-3)^3 < (-2)^3$ and $(-3)^{-3} > (-2)^{-3}$, or $-27 < -8$ and $-\frac{1}{27} > -\frac{1}{8}$.

7. If we take $A > -B$ or $-B < A$, and suppose A to be positive, while $-B$ is negative, then we shall have $A^2 > B^2$ or $B^2 < A^2$ if B is numerically less than A, and the reverse when B is numerically greater than A. Also, from $A > -B$ we clearly have $A^3 > -B^3$ or $-B^3 < A^3$.

Hence, *if the members of an inequality have unlike signs, and are raised to a power whose index is an even positive integer, the powers will form an inequality which will exist in the same sense as the given inequality, if the negative member is numerically less than the positive member, and* vice versa *if the negative member is numerically greater than the positive member; and the reverse will be true in each case if the index of the power is taken with the negative sign.*

Also, *if the members of such an inequality are raised to a power whose index is an odd positive or negative integer, the result will be an inequality which will exist in the same sense as the given inequality.*

Thus, from $3 > -2$ and $4 > -5$ we get $(3)^2 > (-2)^2$ and $(4)^2 < (-5)^2$ or $9 > 4$ and $16 < 25$; and $(3)^{-2} < (-2)^{-2}$, $(4)^{-2} > (-5)^{-2}$, or $\frac{1}{9} < \frac{1}{4}$ and $\frac{1}{16} > \frac{1}{25}$. Also, from $3 > -2$ we get $(3)^3 > (-2)^3$ and $(3)^{-3} > (-2)^{-3}$, or $27 > -8$ and $\frac{1}{27} > -\frac{1}{8}$.

8. From $A > B$ or $B < A$ we get $A^{\frac{1}{n}} > B^{\frac{1}{n}}$ when n is an odd positive integer, or $A^{-\frac{1}{n}} < B^{-\frac{1}{n}}$; and the same results hold good if n is an even integer, when the positive roots are used; but if we take the negative roots (n being even), we shall get $-A^{\frac{1}{n}} < -B^{\frac{1}{n}}$ and $-A^{-\frac{1}{n}} > -B^{-\frac{1}{n}}$.

Also, if we have $A > -B$ and $-A < -B$; then if n is an odd positive integer, we shall have $(A)^{\pm\frac{1}{n}} > (-B)^{\pm\frac{1}{n}}$ and $(-A)^{\frac{1}{n}} < (-B)^{\frac{1}{n}}$ and $(-A)^{-\frac{1}{n}} > (-B)^{-\frac{1}{n}}$.

Thus, from $27 > 8$ we get $(27)^{\frac{1}{3}} > (8)^{\frac{1}{3}}$ and $(27)^{-\frac{1}{3}} < (8)^{-\frac{1}{3}}$

or $3 > 2$ and $\frac{1}{3} < \frac{1}{2}$; and from $9 > 4$ there results $(9)^{\frac{1}{2}} > (4)^{\frac{1}{2}}$ $(9)^{-\frac{1}{2}} < (4)^{-\frac{1}{2}}$, or $3 > 2$ and $\frac{1}{3} < \frac{1}{2}$; also, $-(9)^{\frac{1}{2}} < -(4)^{\frac{1}{2}}$ and $-(9)^{-\frac{1}{2}} > -(4)^{-\frac{1}{2}}$, or $-3 < -2$ and $-\frac{1}{3} > -\frac{1}{2}$; and from $8 > -27$ and $-8 > -27$ we get $(8)^{\pm\frac{1}{3}} > (-27)^{\pm\frac{1}{3}}$ or $2^{\pm 1} > (-3)^{\pm 1}$, and $(-8)^{\frac{1}{3}} > (-27)^{\frac{1}{3}}$, $(-8)^{-\frac{1}{3}} < (-27)^{-\frac{1}{3}}$, or $-2 > -3$ and $-\frac{1}{2} < -\frac{1}{3}$.

Hence, *we can extract any root of the members of any inequality when the index of the root is an odd positive integer, and the result will be an inequality which will exist in the same sense as the given inequality; and if the members of the given inequality are both positive, and we extract any even positive root of its members, the same conclusions hold true when the positive roots are used; but if the positive root is taken in one member and the negative in the other, the positive root will be greater than the negative root.*

9. If we have $A = B + C$, and suppose B and C to be positive, then by rejecting C we have the inequality $A > B$ or $B < A$; and if we have $A = -B + C$, $-B$ being negative and C positive, then by rejecting C, we get the inequality $A > -B$. Hence, *we perceive how inequalities may be conceived to result from equations.*

10. To illustrate what has been done, take the following

EXAMPLES.

1. Given, $2x - 5 > 1$, to find the limit of x.

Transposing 5 and dividing by 2, we have $x > 3$; consequently, since x must be greater than 3, we may call 3 the *inferior limit* of x.

2. Given, $2x - 3 < 4$, to find the limit of x.

Proceeding as before, we get $x < 3\frac{1}{2}$; consequently, since x must be less than $3\frac{1}{2}$, we may call $3\frac{1}{2}$ the *superior limit* of x.

3. Given, $2x - 3 > 1$ and $3x + 1 < 16$, to find the limits of x.

From the first inequality we get $x > 2$, and from the second we have $x < 5$; consequently, 2 and 5 are the *inferior* and *superior limits* of x.

4. Given, $x + \frac{x}{2} - \frac{x}{3} > \frac{1}{6}$, and $\frac{x}{3} - \frac{x}{4} < \frac{1}{12}$, to find the limits of x.

Freeing the inequalities from fractions, etc., we get $x > \frac{1}{7}$ and $x < 1$.

5. Given, $x^2 - 3 > 1$ and $x^3 + 2 < 29$, to find the limits of x.

The inequalities are equivalent to $x^2 > 4$ and $x^3 < 27$; consequently, by extracting the roots we get $x > 2$ and $x < 3$.

6. Given, $-x^2 < -9$ and $-x^3 > -64$, to find the limits of x.

By changing the signs of the inequalities, we get $x^2 > 9$ and $x^3 < 64$; consequently, from the extraction of the roots, we have $x > 3$ and $x < 4$.

7. Given, $3x + 2y > 7 - 2x + y$ and $5x - 7y < 2$, to find the limits of x and y.

The inequalities give $x > \frac{7 - y}{5}$ and $x < \frac{7y + 2}{5}$; consequently, we must have $7y + 2 > 7 - y$, which gives $y > \frac{5}{8}$. Hence assuming $y = \frac{5}{8} + z = \frac{5 + 8z}{8}$, we get $x > \frac{51 - 8z}{40}$ and $x < \frac{51 + 56z}{40}$; in which any positive number may be substituted for z.

8. A person's estate being increased by its third part and then diminished by \$4000 is greater than \$3900; also, the estate being diminished by its fourth part and then by \$4400 is less than \$150. What are the limits of the estate.

Ans. \$5925 and \$6066⅔.

9. A person sells two valuable horses for a sum which is not less than \$600, and the difference of the prices of the

horses is not greater than \$80. What are the limits of the money received for each horse?

Ans. Not less than \$340 for the most valuable, and not less than \$260 for the other.

SECTION XIV.

QUADRATIC EQUATIONS.

(**1.**) THERE are two kinds of quadratic equations.

1st. Those which involve only the square of the unknown letter, which are called *incomplete, simple, or pure quadratics*, such as $x^2 = 9$, $5x^2 - 7 + x^2 = -x^2 + 21$, $\frac{5x^2}{2} - \frac{x^2}{3} - 5 = 8 + \frac{x^2}{6}$, and $ax^2 + b = c$. They are sometimes called quadratic equations of *two terms*, because, by transposition, etc., they can always be reduced to the form $ax^2 = b$, or $x^2 = \frac{b}{a}$. Thus, $5x^2 - 7 + x^2 = -x^2 + 21$ by transposition becomes $5x^2 + x^2 + x^2 = 21 + 7$, and uniting like terms, and dividing by 7, we have $x^2 = 4$. Similarly, the equation $\frac{5x^2}{2} - \frac{x^2}{3} - 5 = 8 + \frac{x^2}{6}$, when freed from fractions, is easily reduced to $2x^2 = 13$ or $x^2 = \frac{26}{4}$.

Having reduced the equation to the form $x^2 = \frac{b}{a}$, if we extract the square root, both in the arithmetical and algebraic senses, we get $x = \sqrt{\frac{b}{a}}$ and $x = -\sqrt{\frac{b}{a}}$ for these roots, which may both be expressed by the form $x = \pm\sqrt{\frac{b}{a}}$, by using + for ± for one of them, and − for ± for the other;

if we substitute $\pm\sqrt{\frac{b}{a}}$ for x in $x^2 = \frac{b}{a}$, we get $\left(\pm\sqrt{\frac{b}{a}}\right)^2 = \frac{b}{a}$ or $\frac{b}{a} = \frac{b}{a}$ an identical equation; consequently, the equation $x^2 = \frac{b}{a}$ has *two roots*, which are expressed by $\pm\sqrt{\frac{b}{a}}$, or, which is the same, the equation admits of *two solutions*, one of which is obtained by putting $\sqrt{\frac{b}{a}}$ for x, and the other by putting $-\sqrt{\frac{b}{a}}$ for x.

2d. Those equations which involve the square and first power of the unknown letter—which are called *adfected, affected, or complete quadratics*—such as $3x^2 - 4x - 3 = 18 - 2x - x^2$, $\frac{x^2}{3} - \frac{8x}{5} + \frac{4}{3} = 40$, and $ax^2 + bx = c$. They are sometimes called quadratic equations of *three terms*, because, by transposition, etc., they can always be reduced to the form $ax^2 + bx = c$, or $x^2 + \frac{b}{a}x = \frac{c}{a}$. Thus $3x^2 - 4x - 3 = 18 - 2x - x^2$ by transposition becomes $3x^2 + x^2 - 4x + 2x = 18 + 3$; and uniting like terms, we have $4x^2 - 2x = 21$, or dividing by the coefficient of x^2, we get $x^2 - \frac{x}{2} = \frac{21}{4}$; also, $\frac{x^2}{3} - \frac{8x}{5} + \frac{4}{3} = 40$, when freed from fractions, and reduced as before, becomes $5x^2 - 24x = 580$, or $x^2 - \frac{24x}{5} = \frac{580}{5} = 116$.

Extracting the square root of $x^2 + \frac{b}{a}x$ by the common rule, we get

$$\begin{array}{r|l} & x^2 + \frac{b}{a}x \left(x + \frac{b}{2a} \right. \text{ root.} \\ & x^2 \\ \hline 2x + \frac{b}{2a} & \frac{b}{a}x \\ & \frac{b}{a}x + \frac{b^2}{4a^2} \\ \hline & \quad -\frac{b^2}{4a^2} \text{ the remainder;} \end{array}$$

consequently, we have $x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}$, or $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$.

Hence, having reduced the equation to the form $x^2 + \frac{b}{a}x = \frac{c}{a}$, by adding $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ to its members, we shall get $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$, or $\left(x + \frac{b}{2a}\right)^2 = \frac{c}{a} + \frac{b^2}{4a^2} = \frac{4ac + b^2}{4a^2}$; consequently, extracting the square root (as in the case of a pure quadratic), we get $x + \frac{b}{2a} = \pm\sqrt{\frac{c}{a} + \frac{b^2}{4a^2}} = \pm\sqrt{\frac{4ac+b^2}{(2a)^2}}$, or $x = -\frac{b}{2a} \pm \sqrt{\frac{c}{a} + \frac{b^2}{4a^2}} = \frac{-b \pm \sqrt{4ac + b^2}}{2a}$, which shows that the equation has *two roots*, or admits of two solutions, one of which results from taking + for ±, and the other from taking − for ±.

Hence, a complete quadratic equation may be solved by the following

RULE.

1. *By transposition, etc., reduce the equation to the form* $x^2 + \frac{b}{a}x = \frac{c}{a}$; *then add* $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ (= *the square of half the coefficient of the first power of* x, *which is called* completing the square) *to both members, and the first member will be* $\left(x + \frac{b}{2a}\right)^2$, *or the square of* $x + \frac{b}{2a}$.

2. *Extracting the square root, we get* $x + \frac{b}{2a} = \pm\sqrt{\frac{c}{a} + \frac{b^2}{4a^2}}$, *which, being an equation of the first degree, is solved by transposing* $\frac{b}{2a}$.

Thus, if we take the equation $x^2 - 8x = 9$, and compare its terms to those of $x^2 + \frac{b}{a}x = \frac{c}{a}$, we shall get − 8 and 9 for the representatives of $\frac{b}{a}$ and $\frac{c}{a}$; consequently, $x + \frac{b}{2a} = \pm$

$\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}$ becomes $x-4=\pm\sqrt{9+16}=\pm 5$, which gives 9 and -1 for the roots of $x^2-8x=9$; and since 9 and -1, when put for x, reduce it to $9=9$ an identical equation, it follows that the roots have been correctly found.

Again, if we compare the equation $x^2+\frac{x}{2}=\frac{15}{16}$ to $x^2+\frac{b}{a}x=\frac{c}{a}$, we shall have $\frac{1}{2}$ and $\frac{15}{16}$ for the representatives of $\frac{b}{a}$ and $\frac{c}{a}$; and of course $x+\frac{b}{2a}=\pm\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}$ becomes $x+\frac{1}{4}=\pm\sqrt{\frac{15}{16}+\frac{1}{16}}=\pm 1$, which gives $x=\frac{3}{4}$ and $x=-\frac{5}{4}$ for the roots of $x^2+\frac{x}{2}=\frac{15}{16}$.

(2.) The equations $x^2=\frac{b}{a}$ and $x^2+\frac{b}{a}x=\frac{c}{a}$ can be reduced to the same general forms in x.

For, dividing their terms by x, and transposing, they become $x-\frac{b}{ax}=0$, and $x-\frac{c}{ax}=-\frac{b}{a}$, whose first members are of like forms, and 0 is the right member of the first, while $-\frac{b}{a}$ is that of the second equation, which is the only difference in the forms of the equations.

Because x and $-\frac{b}{ax}$ enter the first equation in the same way, it is clear that the values of x and $-\frac{b}{ax}$ must be the roots of the equation; and in like manner, it is clear that the values of x and $-\frac{c}{ax}$ must be the roots of the second equation.

To find the roots, we square the members of the equations, and get $\left(x-\frac{b}{ax}\right)^2=0$, $\left(x-\frac{c}{ax}\right)^2=\frac{b^2}{a^2}$, or $x^2-\frac{2b}{a}+\frac{b^2}{a^2x^2}=0$, and $x^2-\frac{2c}{a}+\frac{c^2}{a^2x^2}=\frac{b^2}{a^2}$; and by adding $\frac{4b}{a}$ to the members of the first of these equations, and $\frac{4c}{a}$ to those of the second, they become $x^2 + \frac{2b}{a} + \frac{b^2}{a^2x^2} = \frac{4b}{a}$ and $x^2 + \frac{2c}{a} + \frac{c^2}{a^2x^2} = \frac{b^2}{a^2} + \frac{4c}{a}$, or $\left(x + \frac{b}{ax}\right)^2 = \frac{4b}{a}$ and $\left(x + \frac{c}{ax}\right)^2 = \frac{b^2}{a^2} + \frac{4c}{a}$; and extracting the square roots of the members of these equations in the arithmetical sense, we get $x + \frac{b}{ax} = 2\sqrt{\frac{b}{a}}$ and $x + \frac{c}{ax} = \sqrt{\frac{4c}{a} + \frac{b^2}{a^2}}$.

If we take the half sum and half difference of the equations $x - \frac{b}{ax} = 0$ and $x + \frac{b}{ax} = 2\sqrt{\frac{b}{a}}$, we get $x = \sqrt{\frac{b}{a}}$ and $-\frac{b}{ax} = -\sqrt{\frac{b}{a}}$ for the roots of the equation $x^2 = \frac{b}{a}$.

Hence we clearly perceive the reason why the square root of $x^2 = \frac{b}{a}$ is usually written in the form $x = \pm\sqrt{\frac{b}{a}}$; viz., because the sign + in ± is taken to express the arithmetical root x in $x - \frac{b}{ax} = 0$, while the sign − is used to express the algebraic root $-\frac{b}{ax}$.

Again, if we take the half sum and half difference of the equations $x - \frac{c}{ax} = -\frac{b}{a}$ and $x + \frac{c}{ax} = \sqrt{\frac{4c}{a} + \frac{b^2}{a^2}}$, we shall have $x = -\frac{b}{2a} + \frac{1}{2}\sqrt{\frac{4c}{a} + \frac{b^2}{a^2}} = -\frac{b}{2a} + \sqrt{\frac{c}{a} + \frac{b^2}{4a^2}}$ and $-\frac{c}{ax} = -\frac{b}{2a} - \sqrt{\frac{c}{a} + \frac{b^2}{4a^2}}$, for the roots of the equation $x^2 + \frac{b}{a}x = \frac{c}{a}$; and it is clear that these roots are the same that we have found in (1).

If we put $y = -\frac{c}{ax}$ or $xy = -\frac{c}{a}$, the equation $x - \frac{c}{ax} = -\frac{b}{a}$ becomes $x + y = -\frac{b}{a}$; consequently, the solution of the equation $x^2 + \frac{b}{a}x = \frac{c}{a}$ is reduced to that of the equations $x +$

$y = -\frac{b}{a}$ and $xy = -\frac{c}{a}$; noticing that the values of x and y are the roots of the equation $x^2 + \frac{b}{a}x = \frac{c}{a}$. Because $(x + y)^2 - 4xy = (x - y)^2 = \frac{b^2}{a^2} + \frac{4c}{a}$, we get $x - y = \sqrt{\frac{4c}{a} + \frac{b^2}{a^2}}$; consequently, the solution of the equations $x + y = -\frac{b}{a}$ and $xy = -\frac{c}{a}$ is reduced to that of the equations $x + y = -\frac{b}{a}$ and $x - y = \sqrt{\frac{4c}{a} + \frac{b^2}{a^2}}$, and the values of x and y will be found to give the same values of the roots of the equation $x^2 + \frac{b}{a}x = \frac{c}{a}$ as we have previously found.

Thus, if we compare the equation $x^2 + 2x = 15$ to $x^2 + \frac{b}{a}x = \frac{c}{a}$, $\frac{b}{a}$ and $\frac{c}{a}$ will be represented by 2 and 15; consequently, the equations $x + y = -\frac{b}{a}$ and $xy = -\frac{c}{a}$ will become $x + y = -2$ and $xy = -15$, and thence we shall have $x - y = \sqrt{\frac{4c}{a} + \frac{b^2}{a^2}} = \sqrt{60 + 4} = 8$; hence, by taking the half sum and half difference of the equations $x + y = -2$ and $x - y = 8$, we get $x = 3$ and $y = -5$ for the roots of the equation $x^2 + 2x = 15$.

If we represent the roots of the equation $x^2 + \frac{b}{a}x = \frac{c}{a}$ or of its equivalent $x^2 + \frac{b}{a}x - \frac{c}{a} = 0$ by r and r'; then, since $r + r' = -\frac{b}{a}$ and $rr' = -\frac{c}{a}$, we shall clearly have $x^2 + \frac{b}{a}x - \frac{c}{a} = (x - r)(x - r') = 0$; and the expression $x^2 + \frac{b}{a}x - \frac{c}{a}$ is resolved into the factors $x - r$ and $x - r'$, agreeably to what has heretofore been proved.

Moreover, by taking the first derived equation of $x^2 + \frac{b}{a}x = \frac{c}{a}$, we get $2x + \frac{b}{a} = 0$, or $x = -\frac{b}{2a}$; which being equal to half the sum of the roots of $x^2 + \frac{b}{a}x = \frac{c}{a}$, it is clear if the roots are real that $-\frac{b}{2a}$ must lie between them; which is in conformity what has previously been shown.

Hence, if $-\frac{b}{2a}$ does not lie between the roots of $x^2 + \frac{b}{a}x = \frac{c}{a}$ or $x^2 + \frac{b}{a}x - \frac{c}{a} = 0$, it follows that they must be impossible or imaginary.

Thus, if we take the equation $x^2 + 2x + 3 = 0$ for an example; we have $-\frac{b}{2a} = -1$, which being put for x in $x^2 + 2x + 3$, gives the positive result 2. Also, if we put $x = \pm \infty = \pm$ an unlimitedly great number for x, in $x^2 + 2x + 3$; the results are also positive. Hence, since $x^2 + 2x + 3$ does not change its sign between $x = -1$ and $x = \pm \infty$, it follows that the roots of $x^2 + 2x + 3 = 0$ are impossible or imaginary; indeed, the solution of the equation gives $x = -1 \pm \sqrt{-2}$; which are impossible roots.

(3.) The solution of the equation $x^2 + \frac{b}{a}x = \frac{c}{a}$, can be reduced to that of a pure quadratic.

For, put $x + \frac{b}{2a} = y$ or $x = y - \frac{b}{2a}$, and we shall get $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = y^2$; consequently, since $x^2 + \frac{b}{a}x = \frac{c}{a}$, we shall have $y^2 = \frac{c}{a} + \frac{b^2}{4a^2}$, or $y = \pm\sqrt{\frac{c}{a} + \frac{b^2}{4a^2}}$. Hence, $x = y - \frac{b}{2a}$ gives $x = -\frac{b}{2a} \pm \sqrt{\frac{c}{a} + \frac{b^2}{4a^2}}$ for the roots of $x^2 + \frac{b}{a}x = \frac{c}{a}$.

Remarks.—1. If we have the general equation $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + Px + Q = 0$, and put $x + \frac{A}{n} = y$ or $x = y - \frac{A}{n}$; then if we put $y - \frac{A}{n}$ for x in the given equation, it will be changed into an equation of the form $y^n + B'y^{n-2} + C'y^{n-3} + \text{etc.} = 0$; an equation which wants its second term. For, by the binomial formula, we have $x^n = \left(y - \frac{A}{n}\right)^n = y^n - Ay^{n-1} + \frac{n(n-1)}{1\,.\,2}y^{n-2} \times \left(\frac{A}{n}\right)^2 - \text{etc.}$, $Ax^{n-1} = A\left(y - \frac{A}{n}\right)^{n-1} = Ay^{n-1} - \text{etc.}$, and so on. Hence, by addition, we get $x^n + Ax^{n-1} + Bx^{n-2} + \text{etc.} = y^n + B'y^{n-2} + C'y^{n-3} + \text{etc.} = 0$, since the coefficient of y^{n-1} is 0.

2. It is clear if $n = 2$, that the preceding transformation will change $x^2 + \frac{b}{a}x - \frac{c}{a} = 0$ into the pure quadratic equation $y^2 - \frac{b^2}{4a^2} - \frac{c}{a} = 0$; which gives $y^2 = \frac{c}{a} + \frac{b^2}{4a^2}$, as above.

3. It may be noticed, that, in the reduction of a question to a quadratic equation, we can often so represent whatever is to be found, as immediately to obtain a pure quadratic.

Thus, if we wish to divide 100 into two such parts, that the sum of their square roots may be 14; then if we represent the roots by $7 + x$ and $7 - x$, their sum will equal 14, and we shall get $(7 + x)^2 + (7 - x)^2 = 100$, or $98 + 2x^2 = 100$ or $x^2 = 1$ and $x = 1$. Hence, $7 + x$ and $7 - x$ become 8 and 6, which are the roots of the sought squares, for $8 + 6 = 14$, and $8^2 + 6^2 = 64 + 36 = 100$.

(4.) Any equation of the form $x^{2n} + ax^n = b$, which is such that the index of x in one of the terms is twice as great as it is in the other term, is of such a form that x^n may be found by the solution of a quadratic.

For if we put $x^n = y$, we get $x^{2n} = y^2$; consequently, the proposed equation is reduced to the quadratic $y^2 + ay = b$.

Hence, if r and r' denote the roots of the quadratic, we shall have $x^n = r$ and $x^n = r'$; consequently, if we extract the n^{th} roots of these equations, we shall have the required values of x, or those values which will satisfy the equation $x^{2n} + ax^n = b$.

Thus, if $n = 2$, $a = 2$, and $b = 24$, the equation in x becomes $x^4 + 2x^2 = 24$, and thence the equation in y becomes $y^2 + 2y = 24$, whose roots are 4 and -6. Hence, the equations $x^n = r$ and $x^n = r'$ become $x^2 = 4$ and $x^2 = -6$; consequently, extracting the square roots of these, we get $x = \pm 2$ and $x = \pm\sqrt{-6}$ for the roots of the proposed equation; the first two being real and the last two impossible or imaginary.

Remark.—An equation which is not given in the form $x^{2n} + ax^n = b$, can often be reduced to it. Thus, if we take the equation $x + \sqrt{2x + 9} = 13$; then if we multiply its members by 2 and add 9 to the products, we shall get $2x + 9 + 2\sqrt{2x + 9} = 35$, which is of the requisite form.

For, by putting $\sqrt{2x + 9} = y$, we have $y^2 + 2y = 35$; whose roots give $y = 5$ and $y = -7$. Hence we have $\sqrt{2x + 9} = 5$ and $\sqrt{2x + 9} = -7$; which, by squaring and reducing, give $x = 8$ and $x = 20$; noticing, that $x = 20$ requires the radical in the proposed equation to be taken with the sign $-$.

(5.) When two or more separate equations, containing as many unknown letters as there are equations, are to be solved, then the general method of proceeding (whatever may be the degree of one or more of the equations) consists in reducing their solution by the methods of elimination given in Simple Equations, to that of a single equation containing one unknown letter. Hence, if the single equation can be solved by the rule for the solution of quadratic equations, the solution of the given equations is reduced to that of a quadratic.

Thus, if we take the equations $x^2 + xy = a$ and $y^2 - xy = b$ for an example, then, since the second gives $x = \frac{y^2 - b}{y}$, the first, by the substitution of this value of x, is easily reduced to $\left(\frac{y^2 - b}{y}\right)^2 + y^2 - b = a$, or $y^4 - \frac{a + 3b}{2}y^2 = -\frac{b^2}{2}$; which, by putting $y^2 = z$, becomes the complete quadratic $z^2 - \frac{a + 3b}{2}z = -\frac{b^2}{2}$; consequently, the solution of the given equations is reduced to that of a quadratic.

When two equations containing two unknown letters are proposed for solution, there are particular methods of elimination (such as those given in Simple Equations, which result from the forms of expressions) that are often preferable to the general methods. Since the solution of $x^2 + \frac{b}{a}x = \frac{c}{a}$ has in (2) been reduced to that of the equations $x + y = -\frac{b}{a}$ and $xy = -\frac{c}{a}$, it is clear that one of the most important of these methods consists in reducing (when it is possible) the solution of the given equations to that of two equations of the forms $x + y = a$ and $xy = p$; in which a and p are given, while x and y are the unknown letters.

For an example of this method, let the equations $x + y = a$ and $x^3 + y^3 = b$ be given, to find x and y.

By division we get $\frac{x^3 + y^3}{x + y} = x^2 - xy + y^2 = (x + y)^2 - 3xy = \frac{b}{a}$, or $a^2 - 3p = \frac{b}{a}$; consequently, we have $p = \frac{a^3 - b}{3a}$, and the solution of the given equations is reduced to that of the equations $x + y = a$ and $xy = p = \frac{a^3 - b}{3a}$.

For another example, take the equations $x + y = a$ and $x^4 + y^4 = b$.

From the second equation we get $x^4 + y^4 - 2x^2y^2 = b - 2p^2$, or $(x^2 - y^2)^2 = (x + y)^2(x - y)^2 = a^2(x^2 - 2xy + y^2) = a^2[(x + y)^2 - 4xy] = b - 2p^2$, or $a^2[a^2 - 4p] = b - 2p^2$, which is easily reduced to the quadratic $p^2 - 2a^2p = \frac{b - a^4}{2}$. Completing the square, etc., we have $p^2 - 2a^2p + a^4 = a^4 + \frac{b - a^4}{2} = \frac{a^4 + b}{2}$, or $p - a^2 = \pm\sqrt{\frac{a^4 + b}{2}}$, or $p = a^2 \pm \sqrt{\frac{a^4 + b}{2}}$; noticing that for ± we must use —; supposing x and y are both positive. Hence, the solution of the given equations is reduced to that of $x + y = a$ and $xy = p = a^2 - \sqrt{\frac{a^4 + b}{2}}$.

To illustrate what has been done, take the following

EXAMPLES.

1. Given, $4x^2 - 30 = 33 - 3x^2$, to find x.

By transposition, etc., we get $x^2 = 9$; consequently, the roots of the given equation are expressed by $x = \pm 3$, or by $x = 3$ and $x = -3$.

2. Given, $\frac{3x^2}{4} - \frac{x^2}{5} - 221 = \frac{x^2}{3} + 559$, to find x.

Freeing the equation from fractions, etc., we have $x^2 = 3600$; consequently, by extracting the square roots, we have $x = \pm 60$.

3. Given, $3(x^2 - 8) + 2(4x^2 - 5) = 8(x^2 - 1) + 22$, to find x.
Ans. $x = \pm 4$.

4. Given, $\sqrt{x^2 - 25} = 12$, to find x.
Squaring the members of the equation so as to remove the surd sign, we have $x^2 - 25 = 144$; consequently, $x = \pm 13$.

5. Given, $\frac{2ax + b^2x^2}{\sqrt{a^2 + x^2}} = \sqrt{a^2 + x^2}$ to find x.

Freeing the equation from fractions, etc., we have $b^2x^2 = (a - x)^2$; consequently, extracting the square root, we get $bx = \pm (a - x)$, or $bx = a - x$ and $bx = -a + x$, which give $x = \frac{a}{b - 1}$ and $x = -\frac{a}{b - 1}$.

Remarks.—If $b = 1$, the first value of x becomes $x = \frac{a}{2}$, and the second becomes $x = -\frac{a}{0} = -\infty = -$ infinity; also, if $b = -1$, the first value of x becomes $x = \frac{a}{0} = \infty =$ infinity, and the second value becomes $x = \frac{a}{2}$.

Now, since the given equation is equivalent to $2ax + b^2x^2 = a^2 + x^2$, if we put ± 1 for b, it will become $2ax + x^2 = a^2 + x^2$. Regarding x as finite, this equation, by erasing x^2 from both members, reduces to $2ax = a^2$, or $x = \frac{a}{2}$, which accounts for one of the values found for x when $b = \pm 1$. If x equals $\pm \infty = \pm$ infinity, we must reject the terms $2ax$ and a^2 (since a is supposed to be finite) from the equation $2ax + x^2 = a^2 + x^2$, on account of their comparative smallness; consequently, the equation is satisfied by $x = \pm \infty$, since it becomes $x^2 = x^2$.

6. Given, $\sqrt{x^2 - a^2} + x = \dfrac{x^2}{\sqrt{x^2 - a^2} - x}$, to find x.

The equation is easily reduced to $-a^2 = x^2$, which gives $x = \pm\sqrt{-a^2} = \pm a\sqrt{-1}$, which are imaginary values, since a is supposed to be real.

7. Given, $\dfrac{x + a}{x^2 - ax} + \dfrac{a - x}{x^2 + ax} = \dfrac{b}{x^2 - c}$, to find x.

Ans. $x = \pm\left(\dfrac{4ac - ba^2}{4a - b}\right)^{\frac{1}{2}}$.

8. Given, $\sqrt{a + x} + \sqrt{a - x} = \sqrt{2b}$, to find x.

Squaring the members of the equation, etc., we easily get $\sqrt{a^2 - x^2} = b - a$; then squaring the members of this, etc., we get $x = \pm\sqrt{2ab - b^2}$.

Otherwise, by putting $a + x = y^2$ and $a - x = z^2$, the equation becomes $y + z = \sqrt{2b}$; and by taking the half sum and half difference of the equations $a + x = y^2$ and $a - x = z^2$, we get $y^2 + z^2 = 2a$ and $x = \dfrac{y^2 - z^2}{2}$. Squaring the members of $y + z = \sqrt{2b}$, we get $y^2 + 2yz + z^2 = 2b$, which, subtracted from $2y^2 + 2z^2 = 4a$, gives $y^2 - 2yz + z^2 = 4a - 2b$, or, extracting the square root, we have $y - z = \pm\sqrt{4a - 2b}$. Taking half the products of the corresponding members of $y + z = \sqrt{2b}$ and $y - z = \pm\sqrt{4a - 2b}$, we get $x = \dfrac{y^2 - z^2}{2} = \pm\sqrt{2ab - b^2}$; consequently, we have the same values of x as before.

Remarks.—If a and b have like signs, and a is numerically not less than $\dfrac{b}{2}$, it is clear that the values of x will be real, and that they will be 0 when $a = \dfrac{b}{2}$; but if a and b have unlike signs, or if they have the same signs, and a is numerically less than $\dfrac{b}{2}$, the values of x will always be imaginary, since the expression under the surd will be negative.

9. Given, $\frac{(a+x)^2}{a} + \frac{(a+x)^2}{x} = \frac{a^2x^2}{b^3}$, to find x.

Ans. $x = \frac{ab}{a-b}$.

10. Given, $\frac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}} = m$, to find x.

Ans. $x = \pm a\sqrt{\frac{1+m^2}{2m}}$.

11. Given, $(a^2-c)x^2 + abx + b^2 = d^2 + (2d\sqrt{c} - ab)x$, to find x.

Ans. $x = \frac{\pm d - b}{a \mp \sqrt{c}}$.

12. Given, $\frac{x^4+a^4}{x^4-a^4} = 2$, to find x. *Ans.* $x = \pm a\sqrt[4]{3}$.

13. Given, $3x^6 - 3120 = 525 - 2x^6$, to find x.
Ans. $x = 3$ and $x = -3$ are the only real values of x which satisfy the equation.

14. Given, $3x^5 - 81x^2 = 0$, to find the values of x.
Since the equation is equivalent to $(x^3-27)x \times x = 0$, it is satisfied by putting $x = 0$, $x = 0$, and $x^3 - 27 = 0$; consequently, 0, 0, and 3 are the only real roots of the equation.

15. Given, $x^6 = a^6$, to find x.
To simplify the equation, put $x = ay$, then we shall (by substitution) get $a^6y^6 = a^6$, or $y^6 = 1$; consequently, by extracting the square root, we have $y^3 = \pm 1$, or $y^3 = 1$ and $y^3 = -1$, which give $y = 1$ and $y = -1$ for the only real values of y; hence, $x = ay$ gives $x = a$ and $x = -a$ for the only real values of x which satisfy the given equation.

Remarks.—If we put $y = z + 1$ and $y = z - 1$ in the equations $y^3 = 1$ and $y^3 = -1$, they will become $z^3 + 3z^2 + 3z + 1 = 1$ and $z^3 - 3z^2 + 3z - 1 = -1$, which are easily reduced to $(z^2 + 3z + 3)z = 0$ and $(z^2 - 3z + 3)z = 0$. These equations are satisfied by putting $z = 0$, which give $y = 1$ and $y = -1$, as before; consequently, we shall get $x = a$ and $x = -a$.

Again, the equations are also satisfied by putting $z^2 + 3z$

$+ 3 = 0$ and $z^2 - 3z + 3 = 0$, which are complete quadratics. Dividing these equations by z, and putting them under the forms $z + \frac{3}{z} = -3$ and $z + \frac{3}{z} = 3$, we get $\left(z + \frac{3}{z}\right)^2 - 12 = \left(z - \frac{3}{z}\right)^2 = = -3$ and $\left(z + \frac{3}{z}\right)^2 - 12 = \left(z - \frac{3}{z}\right)^2 = -3$; consequently, by extracting the square roots in the arithmetical sense, we get $z - \frac{3}{z} = \sqrt{-3}$ and $z - \frac{3}{z} = \sqrt{-3}$, which are identical in their forms.

By taking the half sum and half difference of the equations $z + \frac{3}{z} = -3$ and $z - \frac{3}{z} = \sqrt{-3}$, we get $= \frac{-3 + \sqrt{-3}}{2}$ and $\frac{3}{z} = \frac{-3 - \sqrt{-3}}{2}$ for the roots of $z^2 + 3z + 3 = 0$; and in a similar way, from the equations $z + \frac{3}{z} = 3$ and $z - \frac{3}{z} = \sqrt{-3}$, we get $z = \frac{3 + \sqrt{-3}}{2}$ and $\frac{3}{z} = \frac{3 - \sqrt{-3}}{2}$ for the roots of the equation $z^2 - 3z + 3 = 0$.

Hence, we have $y = z + 1 = \frac{-3 + \sqrt{-3}}{2} + 1 = \frac{-1 + \sqrt{-3}}{2}$ and $y = z + 1 = \frac{-3 - \sqrt{-3}}{2} + 1 = \frac{-1 - \sqrt{-3}}{2}$ for two of the roots of the equation $y^3 = 1$; and in a similar way, from $y = z - 1$ we get $y = \frac{1 + \sqrt{-3}}{2}$ and $y = \frac{1 - \sqrt{-3}}{2}$ for two of the roots of the equation $y^3 = -1$.

It results from what has been done that 1, $\frac{-1 + \sqrt{-3}}{2}$, and $\frac{-1 - \sqrt{-3}}{2}$ (which are sometimes called *the cube roots* of 1) are the three roots of $y^3 = 1$; and in like manner, -1, $\frac{1 + \sqrt{-3}}{2}$, and $\frac{1 - \sqrt{-3}}{2}$ are the roots of $y^3 = -1$.

If we multiply the preceding roots by a, we shall get a,

$-a$, $a\left(\frac{-1+\sqrt{-3}}{2}\right)$, $a\left(\frac{-1-\sqrt{-3}}{2}\right)$, $a\left(\frac{1+\sqrt{-3}}{2}\right)$, and $a\left(\frac{1-\sqrt{-3}}{2}\right)$ for the six roots of the equation $x^6 = a^6$; noticing that the first two of these roots are real, while the remaining four are imaginary.

16. Given, $x^2 + 4x - 13 = 8$, to find x.

Ans. $x = 3$ and $x = -7$.

17. Given, $3x^2 - 5x + 8 = 160$, to find x.

Ans. $x = 8$ and $x = -\frac{19}{3}$.

18. Given, $-2x^2 + 7x = -4$, to find x.

Changing the signs of all the terms, so as to make the term which contains x^2 positive (which it must always be before the square is completed), we get the equation $2x^2 - 7x = 4$, whose roots are easily found to be 4 and $-\frac{1}{2}$.

19. Given, $\frac{x+2}{x-3} + \frac{2}{x} + \frac{1}{2} = 7$, to find x.

Ans. $x = 4$ and $x = \frac{3}{11}$.

20. Given, the proportion $4x + 7 : 3x + 1 :: 9x + 3 : 7x - 9$, or the equivalent equation $\frac{4x+7}{3x+1} = \frac{9x+3}{7x-9}$, to find x.

Ans. $x = 11$ and $x = -6$.

21. Given, $\frac{x^2-3}{x-1} + \frac{5-x^2}{x+3} = \frac{4x+9}{x-2}$, to find x.

If we free the equation from fractions we shall get the quadratic $23x^2 + 24x = 55$ or $x^2 + \frac{24}{23}x = \frac{55}{23}$; consequently, completing the square, etc., we have $x^2 + \frac{24}{23}x + \left(\frac{12}{23}\right)^2 = \frac{55}{23} + \left(\frac{12}{23}\right)^2 = \frac{1409}{(23)^2}$ or $x + \frac{12}{23} = \pm\frac{\sqrt{1409}}{23}$, which gives $x = \frac{-12 \pm \sqrt{1409}}{23}$, whose approximate values are $x = 1.110$ and $x = -2.153$.

22. Given, $x + \frac{6}{x} = 5$, to find x.

Ans. $x = 2$ and $x = 3$.

23. Given, $\frac{x+2}{x-3} + \frac{6(x-3)}{x+2} = 5$, to find x.

Ans. $x = 8$ and $x = 5\frac{1}{2}$.

24. Given, $x^{-2} - 3x^{-1} = -2$, to find x.

Ans. $x^{-1} = 1$ and $x^{-1} = 2$, or $x = 1$ and $x = \frac{1}{2}$.

25. Given, $\frac{3x^2+5}{4} - \frac{2x^2}{x^2-6} = 2$, to find x.

The equation, when freed from fractions, etc., is equivalent to $3x^4 - 29x^2 = -18$, or $x^4 - \frac{29x^2}{3} = -6$, which, by putting $y = x^2$, is reduced to the quadratic $y^2 - \frac{29}{3}y = -6$. Hence, adding $\left(\frac{29}{3} \div 2\right)^2 = \frac{841}{36}$ to both members of the equation, we get $y^2 - \frac{29}{3}y + \frac{841}{36} = \left(y - \frac{29}{6}\right)^2 = -6 + \frac{841}{36} = \frac{625}{36}$; consequently, extracting the square roots of the members of this equation, we shall have $y - \frac{29}{6} = \pm\frac{25}{6}$, or $y = \frac{29 \pm 25}{6}$.

If we take + for ± in the preceding equation, we get $y = x^2 = \frac{54}{6} = 9$; and if we take − for ±, we get $y = \frac{4}{6}$, or $x^2 = \frac{4}{6}$; and from $x^2 = 9$ and $x^2 = \frac{6}{9}$, by taking the square roots, we get $x = \pm 3$ and $x = \pm\frac{\sqrt{6}}{3}$ for the roots of the proposed equation.

26. Given, $\frac{3}{x} + \frac{4}{x-1} - \frac{15}{x+2} = 0$, to find x.

Ans. $x = 3$ and $x = \frac{1}{4}$.

27. "Given, $4a^2x^2 + 4a^2c^2x + 4abd^2x - 9cd^2x^2 + (ac^2 + bd^2)^2 = 0$, to find x."

Because the equation is equivalent to $(2ax + ac^2 + bd^2)^2 = 9cd^2x^2$, we get (by extracting the square roots of the members of this equation) $2ax + ac^2 + bd^2 = \pm\, 3d\sqrt{c} \times x$.

Hence, we easily get $x = -\dfrac{ac^2 + bd^2}{2a + 3d\sqrt{c}}$ and $x = -\dfrac{ac^2 + bd^2}{2a - 3d\sqrt{c}}$.

28. "Given, $mx^2 - 2mx\sqrt{n} = nx^2 - mn$, to find x."

Since the equation is equivalent to $(x\sqrt{m})^2 - 2\sqrt{m^2n}\,.\,x + mn = (x\sqrt{m} - \sqrt{mn})^2 = nx^2$, we get (as in the last question), $x\sqrt{m} - \sqrt{mn} = \pm\, x\sqrt{n}$; and thence $x = \dfrac{\sqrt{mn}}{\sqrt{m} + \sqrt{n}}$ and $x = \dfrac{\sqrt{mn}}{\sqrt{m} - \sqrt{n}}$.

29. "Given, $\dfrac{5a + 10ab^2}{9b^2 - 3a^2b^2}x^2 - \left(\dfrac{5\sqrt{a+b}}{3b^3} + \dfrac{(1 + 2b^2)cd\sqrt{c}}{3 - a^2}\right)x + \dfrac{cd}{ab}\sqrt{(a+b)c} = 0$, to find x."

Putting $\dfrac{5a + 10ab^2}{9b^2 - 3a^2b^2} = \mathrm{A}$ and dividing by $\mathrm{A}x$, the equation (by transposition) is easily put under the form $x + \dfrac{cd}{ab\mathrm{A}}\,\dfrac{\sqrt{(a+b)c}}{x} = \dfrac{5\sqrt{a+b}}{3b^3\mathrm{A}} + \dfrac{(1 + 2b^2)cd\sqrt{c}}{(3 - a^2)\mathrm{A}}$; which (according to what has heretofore been shown) is such that the values of x and $\dfrac{cd}{ab\mathrm{A}}\,\dfrac{\sqrt{(a+b)c}}{x}$ are the roots of the proposed equation. To find these values, we square both members of the preceding equation and subtract $\dfrac{4cd}{ab\mathrm{A}}\sqrt{(a+b)c}$ from the squares, and then extracting the square roots of the results in the arithmetical sense, we get $x - \dfrac{cd}{ab\mathrm{A}}\,\dfrac{\sqrt{(a+b)c}}{x} = \dfrac{5\sqrt{a+b}}{3b^3\mathrm{A}} - \dfrac{(1 + 2b^2)cd\sqrt{c}}{(3 - a^2)\mathrm{A}}$. Hence, by taking the half sum and half difference of this and the preceding equation, we get $x =$

$\frac{5\sqrt{a+b}}{3b^3\text{A}} = \frac{(3-a^2)\sqrt{a+b}}{ab(1+2b^2)}$ and $\frac{cd\sqrt{(a+b)c}}{ab\text{A} \quad x} = \frac{(1+2b^2)cd\sqrt{c}}{(3-a^2)\text{A}}$ $= \frac{3b^2cd\sqrt{c}}{5a}$, for the roots or values of x in the proposed equation.

30. Given, $ax^2 + bx = c$, to find x.

If we multiply the terms of the equation by $4a$ (or four times the coefficient of the square of the unknown letter), and add b^2 (or the square of the coefficient of the simple power of the unknown letter) to both sides, we shall get $4a^2x^2 + 4abx + b^2 = b^2 + 4ac$ or $(2ax + b)^2 = b^2 + 4ac$; consequently, by extracting the square root in both members, we shall get $2ax + b = \pm\sqrt{b^2 + 4ac}$ or $x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$.

Remarks.—1. Resuming $x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$, then ($a$ being positive) if c is positive, since $b^2 + 4ac$ is not less than b^2, it follows that $\sqrt{b^2 + 4ac}$ is not less than b; consequently, when c is greater than 0, the values of x must be real and have contrary signs; and when $c = 0$, one of the values of x is 0, and the other equals $\frac{-b}{a}$.

2. If c is negative, we have $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; which shows if $4ac$ is not greater than b^2, that the values of x must have the same sign as $-b$; and that they will be equal when $4ac = b^2$, and that $\frac{-b}{2a}$ is their common value.

If $4ac$ is greater than b^2, since $b^2 - 4ac$ is negative, it follows that the values of x will be impossible or imaginary.

3. Resuming the equation $ax^2 + bx = c$, then if c is finite and a an infinitesimal, by supposing $x = \infty$, or to be infinitely great, we must reject c on account of its comparative minuteness, and we shall have $ax^2 + bx = 0$ or $x = -\frac{b}{a} = -\infty =$ an unlimitedly great result.

Again, still supposing a to be an infinitesimal, if we regard x as being finite, then on account of the minuteness of a, we must reject ax^2, and shall thence get $bx = c$ or $x = \frac{c}{b}$.

31. Given, $x^{2n} - 2ax^n = b$, to find x.

Adding a^2 to both members of the equation, we get $x^{2n} - 2ax^n + a^2 = (x^n - a)^2 = a^2 + b$; consequently, by extracting the square roots, etc., we get $x^n = a \pm \sqrt{a^2 + b}$ or $x = \sqrt[n]{a \pm \sqrt{a^2 + b}}$.

Remark.—If the given equation had been $x^n - 2ax^{\frac{n}{2}} = b$, we should have found $x = \sqrt[n]{(a \pm \sqrt{a^2 + b})^2}$.

32. Given, $\sqrt{2x^2 + 7} = x^2 - 4$, to find x.

By transposition, the equation can be put under the form $x^2 - \sqrt{2x^2 + 7} = 4$, then multiplying the members by 2 and adding 7 to the products, we get $2x^2 + 7 - 2\sqrt{2x^2 + 7} = 15$; consequently, putting $y = \sqrt{2x^2 + 7}$, the equation is reduced to the quadratic $y^2 - 2y = 15$. Solving the quadratic, we get $y = 1 \pm 4$ or $y = 5$ and $y = -3$; consequently, by restoring the value of y, we shall get $\sqrt{2x^2 + 7} = 5$ and $\sqrt{2x^2 + 7} = -3$. If we suppose the radical in the given equation to be positive, it is clear that the second of the preceding equations must be rejected, and that the values of x must be found from the equation $\sqrt{2x^2 + 7} = 5$. Squaring the members of this equation, we get $2x^2 + 7 = 25$ or $x^2 = 9$, whose square root gives $x = \pm 3$; consequently, if x is required to be positive, we must reject the equation $x = -3$ and put $x = 3$.

33. Given, $x^4 - 4x^3 + 8x^2 - 8x = 1365$, to find the positive value of x.

By extracting the square root of the first member of the equation, it is easy to put it under the form $(x^2 - 2x)^2 + 4(x^2 - 2x) = 1365$; consequently, putting $x^2 - 2x = y$, the equation is reduced to the quadratic $y^2 + 4y = 1365$. Solving the quadratic, we get $y = x^2 - 2x = 35$, and thence $x = 7$.

34. Given, $x^2 - 3 = x\sqrt{\frac{5}{x^2} - 1}$, to find x, supposing the radical is positive.

The equation being equivalent to $x^2 - 3 = \sqrt{5 - x^2}$, is easily reduced to $5 - x^2 + \sqrt{5 - x^2} = 2$; consequently, $x = 2$.

35. Given, $\sqrt{3x-5}+\sqrt{2x-1}=4+\sqrt{5x-22}$, to find x, regarding the radicals as being positive. *Ans.* $x=7$.

36. Given, $\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}=(x+4)^2$, to find the real values of x. *Ans.* $x=\pm\sqrt{15}$.

37. "Given, $\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}=x$, to find x."

Transposing $\sqrt{1-\frac{1}{x}}$ and squaring, we get $\left(\sqrt{x-\frac{1}{x}}\right)^2=\left(x-\sqrt{1-\frac{1}{x}}\right)^2$, or $x=x^2-2\sqrt{x^2-x}+1$, which is easily reduced to $x^2-x-2\sqrt{x^2-x}+1=0$, whose square root gives $\sqrt{x^2-x}-1=0$, or $x^2-x=1$; solving this, we get $x=\dfrac{1\pm\sqrt{5}}{2}$.

Otherwise.—If we put $y^2=x-\frac{1}{x}$ and $z^2=1-\frac{1}{x}$, the given equation will be reduced to $y+z=x$.

Subtracting the second of the assumed equations from the first, we get $y^2-z^2=(y+z)(y-z)=x(y-z)=x-1$, which gives $x=\dfrac{1}{1+z-y}$.

Hence, we get $y+z=\dfrac{1}{1+z-y}$, or $y+z+z^2-y^2=1$; and from $z^2=1-\frac{1}{x}=1-(1+z-y)=y-z$, the equation is reduced to $2y-y^2=1$, or $y^2-2y+1=0$, or $y=1$. Hence, the equation $y+z+z^2-y^2=1$ reduces to $z^2+z=1$, whose solution gives $z=\dfrac{-1\pm\sqrt{5}}{2}$; consequently, we have $x=y+z=1+\dfrac{-1\pm\sqrt{5}}{2}=\dfrac{1\pm\sqrt{5}}{2}$.

38. Given, $\sqrt{a+x}+\sqrt{b+x}=\sqrt{c+2x}$, to find x.

Put $y^2=a+x$, $z^2=b+x$, and $v^2=c+2x$; then the equation becomes $y+z=v$, or $y^2+2yz+z^2=v^2$.

From the assumed equations we have $y^2+z^2=a+b+2x=a+b-c+v^2$, which reduces the preceding equation to $2yz=c-(a+b)$.

Also, from the assumed equations we have $y^2 - z^2 = a - b$, whose square gives $y^4 - 2y^2z^2 + z^4 = (a - b)^2$, and adding $4y^2z^2 = [c - (a + b)]^2$ to this, we have $y^4 + 2y^2z^2 + z^4 = (a - b)^2 + [c - (a + b)]^2$, and denoting the square root of this by A, we have $y^2 + z^2 = \mathrm{A}$. Hence, from $x = \frac{y^2 + z^2 - (a + b)}{2}$ we get $x = \frac{\mathrm{A} - (a + b)}{2}$, as required.

39. Given, $x + y = a$ and $xy = b$, to find x and y.

Ans. $x = \frac{a \pm \sqrt{a^2 - 4b}}{2}$ and $y = \frac{a \mp \sqrt{a^2 - 4b}}{2}$.

40. Given, $ax + by = c$ and $xy = d$, to find x and y.

Ans. $x = \frac{c \pm \sqrt{c^2 - 4abd}}{2a}$ and $y = \frac{c \mp \sqrt{c^2 - 4abd}}{2b}$.

41. Given, $x^2 + y^2 = a$ and $bx^2 - cy^2 = d$, to find x and y.

Ans. $x = \pm\sqrt{\frac{d + ac}{b + c}}$ and $y = \pm\sqrt{\frac{ab - d}{b + c}}$.

42. Given, $x + y = a$ and $\frac{x^2}{y} + \frac{y^2}{x} = b$, to find x and y.

Multiplying the terms of the second by xy, and dividing by the corresponding members of the first, we get $\frac{x^3 + y^3}{x + y} = x^2 - xy + y^2 = \frac{b}{a}xy$, or $x^2 - \frac{a + b}{a}xy + y^2 = 0$.

Because the first equation gives $x^2 + 2xy + y^2 = a^2$, we easily reduce the preceding equation to $\frac{3a + b}{a}xy = a^2$, or $xy = \frac{a^3}{3a + b}$.

Hence, the solution of the given equations is reduced to that of the equations $x + y = a$ and $xy = \frac{a^3}{3a + b}$.

If $a = 12$, and $b = 18$, we shall have $xy = \frac{1728}{54} = 32$; consequently, from $x + y = 12$ and $xy = 32$ we get $x = 8$ and $y = 4$, or $x = 4$ and $y = 8$.

43. Given, $x^2 + xy = a$ and $xy + 2y^2 = b$, to find x and y.

From the first $y = \frac{a - x^2}{x}$, which, being substituted in the second, we get $2\left(\frac{a-x^2}{x}\right)^2 + a - x^2 = b$, or $x^4 - (3a+b)x^2 = -2a^2$; whose solution gives $x = \pm\left(\frac{3a+b\pm\sqrt{a^2+6ab+b^2}}{2}\right)^{\frac{1}{2}}$; consequently, having found x, we get y from the equation $y = \frac{a-x^2}{x}$.

If $a = 56$ and $b = 60$, we get $\sqrt{a^2+6ab+b^2} = 164$ and $x = \pm\sqrt{114 \pm 82} = \pm 14$, $x = \pm 4\sqrt{2}$.

Hence, if we take the rational values, we have $x = \pm 14$, and $y = \mp 10$.

44. "Given, $\frac{xyz}{x+y} = a$, $\frac{xyz}{y+z} = b$, $\frac{xyz}{x+z} = c$, to find x, y, and z."

Dividing the numerators and denominators of the first members of these equations by xyz, the equations become $\frac{1}{\frac{1}{yz}+\frac{1}{xz}} = a$, $\frac{1}{\frac{1}{xz}+\frac{1}{xy}} = b$, $\frac{1}{\frac{1}{yz}+\frac{1}{xy}} = c$, which are easily reduced to $\frac{1}{yz}+\frac{1}{xz} = \frac{1}{a}$, $\frac{1}{xz}+\frac{1}{xy} = \frac{1}{b}$, $\frac{1}{yz}+\frac{1}{xy} = \frac{1}{c}$.

If from the half sum of these equations we subtract the equations severally, we shall get $\frac{1}{xy} = \frac{ab+ac-bc}{2abc}$, $\frac{1}{yz} = \frac{ab+bc-ac}{2abc}$, $\frac{1}{xz} = \frac{ac+bc-ab}{2abc}$.

Dividing the second of these by the product of the first and third, and extracting the square root of the result, we have $x = \pm\sqrt{\frac{2abc(ab+bc-ac)}{(ab+ac-bc).(ac+bc-ab)}}$; and in a similar way we get $y = \pm\sqrt{\frac{2abc(ac+bc-ab)}{(ab+ac-bc).(ab+bc-ac)}}$, $z = \pm\sqrt{\frac{2abc(ab+ac-bc)}{(ab+bc-ac).(ac+bc-ab)}}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers, such that the ratio of the greater to the less shall be 5, and their product 845.

The numbers may be represented by x and $5x$, since the greater is 5 times the less; consequently, we must have $5x^2 = 845$, or $x^2 = 169 = 13^2$, which gives $x = \pm 13$. Hence, supposing the numbers to be positive, we shall have 13 and 65 for the numbers.

2. To find two numbers, such that their sum shall be 20 and their product 75.

If $10 + x$ and $10 - x$ stand for the numbers, then one of the conditions of the question is satisfied, since their sum is 20; and to satisfy the other condition, we must have $(10 + x).(10 - x) = 100 - x^2 = 75$, or $x^2 = 25$; consequently, by extracting the arithmetical square root of 25, we get $x = 5$, so that $10 + x$ and $10 - x$ become 15 and 5.

3. Let $2a$ and p stand for the sum and product of two numbers or quantities; then it is required to find them. The first condition of the question is satisfied by representing the numbers or quantities by $a + x$ and $a - x$, whose sum is $2a$; and to satisfy the second condition we must have $(a+x)(a - x) = a^2 - x^2 = p$, or $x^2 = a^2 - p$, whose arithmetical root gives $x = \sqrt{a^2 - p}$. Hence, $a + x$ and $a - x$ become $a + \sqrt{a^2 - p}$ and $a - \sqrt{a^2 - p}$, which satisfy the conditions of the question.

4. To find two numbers whose difference is 8 and product 240.

Represent the numbers by $x + 4$ and $x - 4$, and their difference will be 8, agreeably to the conditions of the question. To satisfy the other condition of the question, we must have $(x + 4)(x - 4) = x^2 - 16 = 240$ or $x^2 = 256$; whose arithmetical square root gives $x = 16$.

Hence, $x + 4$ and $x - 4$ become 20 and 12, which are the sought numbers.

5. To find two numbers or quantities whose difference is $2d$ and whose product is p.

Representing the numbers or quantities by $x + d$ and $x - d$, we have $x^2 - d^2 = p$ or $x = \sqrt{p + d^2}$; consequently, we shall have the numbers or quantities represented by $\sqrt{p + d^2} + d$ and $\sqrt{p + d^2} - d$; which satisfy the conditions of the question.

6. If $2a$ and b^2 represent the sum, and sum of the squares

of two numbers or quantities, then it is required to find them.

If $a+x$ and $a-x$ represent the numbers or quantities, then one of the conditions of the question is satisfied; and to satisfy the other condition we must have the equation $(a+x)^2+(a-x)^2=2a^2+2x^2=b^2$ or $x^2=\frac{b^2-2a^2}{2}$; whose arithmetical root gives $x=\sqrt{\frac{b^2-2a^2}{2}}$.

Hence, we have $a+x$ and $a-x$ expressed by $a+\sqrt{\frac{b^2-2a^2}{2}}$ and $a-\sqrt{\frac{b^2-2a^2}{2}}$; which will be real if b^2 is not less than $2a^2$, and positive if $\sqrt{\frac{b^2-2a^2}{2}}$ is not greater than a.

7. Given, $2d$ and b^2, the difference and sum of the squares of two numbers or quantities, to find them.

Representing the numbers or quantities by $x+d$ and $x-d$, by proceeding as in the last question, we shall have $\sqrt{\frac{b^2-2d^2}{2}}+d$ and $\sqrt{\frac{b^2-2d^2}{2}}-d$ for the sought numbers or quantities; which will be real if $2d^2$ is not greater than b^2, and positive if $\sqrt{\frac{b^2-2d^2}{2}}$ is not less than d.

8. To find a number or quantity, such that if it is subtracted from $2a$, and the remainder multiplied by the number or quantity, the product shall equal b.

If we represent it by $a+x$, the remainder will be $a-x$, and *vice versa;* consequently, we shall have $(a+x)(a-x)=a^2-x^2=b$ or $x^2=a^2-b$, which gives $x=\pm\sqrt{a^2-b}$.

Hence, the sought number or quantity is $a\pm\sqrt{a^2-b}$; since $[2a-(a\pm\sqrt{a^2-b}].(a\pm\sqrt{a^2-b})=(a\mp\sqrt{a^2-b}).(a\pm\sqrt{a^2-b})=a^2-(a^2-b)=b$.

9. Given, $2a$ and b, the sum and the sum of the cubes of two numbers or quantities, to find them.

Representing them by $a+x$ and $a-x$, the first condition is satisfied; and to satisfy the second condition, we have

$(a+x)^3+(a-x)^3=2a^3+6ax^2=b$ or $x^2=\frac{b-2a^3}{6a}$, which gives $x=\sqrt{\frac{b-2a^3}{6a}}$. Hence, $a+\sqrt{\frac{b-2a^3}{6a}}$ and $a-\sqrt{\frac{b-2a^3}{6a}}$ are the sought numbers or quantities; which will be real if $2a^3$ is not greater than b, and positive if $\sqrt{\frac{b-2a^3}{6a}}$ is not greater than a.

10. Given, $2d$ and b, the difference and difference of the cubes of two numbers or quantities, to find them.

Representing them by $x+d$ and $x-d$, we have $(x+d)^3-(x-d)^3=6x^2d+2d^3=b$. Hence $\sqrt{\frac{b-2d^3}{6d}}+d$ and $\sqrt{\frac{b-2d^3}{6d}}-d$, are the sought numbers or quantities; which will be real if $2d^3$ is not greater than b, and positive if d is not greater than $\sqrt{\frac{b-2d^3}{6d}}$.

11. To divide the number or quantity $2a$ into two parts, such that their product shall be to the sum of their squares, in the ratio of m to n.

Since the parts may be expressed by $a+x$ and $a-x$, their product and the sum of their squares will be expressed by a^2-x^2 and $2a^2+2x^2$. Hence, we have the proportion $a^2-x^2:2a^2+2x^2::m:n$, or the equivalent equation $2ma^2+2mx^2=na^2-nx^2$; which gives $x=a\sqrt{\frac{n-2m}{n+2m}}$.

Hence, the sought numbers or quantities are expressed by $a+a\sqrt{\frac{n-2m}{n+2m}}$ and $a-a\sqrt{\frac{n-2m}{n+2m}}$; which will be real and positive if $2m$ is not greater than n; moreover, if q^2-p^2 and $2(p^2+q^2)$ are put for m and n (supposing q greater than p), they will become $a\times\frac{q+p}{q}$ and $a\times\frac{q-p}{q}$; which will be rational, if a, p, and q are rational.

12. "A company at a tavern had £8 15*s.* to pay for their reckoning, but before the bill was settled two of them left the room, and then those who remained had 10*s.* apiece more to pay than before; how many were there in company?"

Let $x+1$ represent the number, then $x-1$ will represent the number after two had left; and since $\frac{£8\ 15s.}{x+1}$ and $\frac{£8\ 15s.}{x-1}$ evidently express the money paid by an individual before and after two had left the room, it follows, from the question, that we must have the equation $\frac{£8\ 15s.}{x+1} + 10s. = \frac{£8\ 15s.}{x-1}$.

Freeing the equation from fractions, we have $(£8\ 15s.)(x-1) + 10s. \times (x^2-1) = (£8\ 15s.) \times (x+1)$; which is easily reduced to $10s.x^2 = £18 = 360s.$, or $x^2 = 36$ and $x = 6$; consequently, $x+1 = 7$ is the sought number; noticing that in getting x from the equation $x^2 = 36$ we have rejected the negative root expressed by $x = -6$; because it is clearly not applicable to the question.

Otherwise.—Representing the sought number by x, we shall have $x-2$ for the representative of the number after two had left the room; consequently, we shall, as before, have the equation $\frac{£8\ 15s.}{x} + 10s. = \frac{£8\ 15s.}{x-2}$. Freeing the equation from fractions, etc., we get the equation $x^2 - 2x = 35$; then completing the square we have $(x-1)^2 = 36$, whose positive square root gives $x-1 = 6$, and thence $x = 7$, as before.

Remarks.—1. It is clear that the first of these solutions is preferable to the second; because it gives a simple instead of a complete quadratic, so that the unknown letter is found by extracting the square root, without the necessity of previously completing the square.

2. It is hence clear that in solving questions, we ought always (for the sake of simplicity) to avoid the use of complete quadratics, when we can so express whatever is required as to obtain the solution by simple quadratics.

3. We will add, that all the questions we have given, with the exception of the first, are ordinarily reduced to complete quadratics, or to equations containing two unknown letters; processes which are entirely unnecessary, as is clear from the solutions we have given.

13. "A person bought a quantity of cloth for £33 15*s.*, which he sold again at £2 8*s.* per piece, and gained by the

bargain as much as one piece cost him; required the number of pieces."

If x stands for the number of pieces, it is clear that $\frac{£33\ 15s.}{x} = \frac{675s.}{x}$ and $(£2\ 8s.)x = 48s. \times x$ will equal the price of each piece, and what was received for all the pieces; consequently, we must have the equation $48x - \frac{675}{x} = 675$ or $x - \frac{225}{16x} = \frac{225}{16}$; noticing, that the values of x and $\frac{-225}{16x}$ are the roots of this equation, agreeably to what has been previously shown. From $\left(x - \frac{225}{16x}\right)^2 + 4 \times \frac{225}{16} = \left(x + \frac{225}{16x}\right)^2 = \left(\frac{225}{16}\right)^2 + \frac{4 \times 225}{16} = \frac{225}{16}\left(\frac{225}{16} + 4\right) = \frac{225}{16} \times \frac{289}{16}$, by extracting the square root in the arithmetical sense, we get $x + \frac{225}{16x} = \frac{15 \times 17}{16}$; consequently, taking the half sum of this and the preceding equation, we have $x = \frac{15 \times 15}{32} + \frac{15 \times 17}{32} = 15$, as required.

Remark.—This method of solving a complete quadratic is almost as simple as the method of solving a simple quadratic; since it is suggested by the nature of the case, and does not require the formality of completing the square, etc.

14. If $2a$ and b equal the sum and the sum of the fourth powers of two numbers or quantities, then it is required to find them.

Let $a + x$ and $a - x$ stand for the numbers or quantities, then, since the first condition of the question is satisfied, we must have $(a + x)^4 + (a - x)^4 = b$, or we shall have $2a^4 + 12a^2x^2 + 2x^4 = b$; which gives $x^4 + 6a^2x^2 = \frac{b - 2a^4}{2}$. Solving the equation, we get $x = \sqrt{\left(\sqrt{\frac{b + 16a^4}{2}} - 3a^2\right)}$, and thence $a + x$ and $a - x$ will become $a + \sqrt{\left(\sqrt{\frac{b + 16a^4}{2}} - 3a^2\right)}$ and $a - \sqrt{\left(\sqrt{\frac{b + 16a^4}{2}} - 3a^2\right)}$, as required. Thus, if

$2a = 7$ and $b = 641$, we shall get 5 and 2 for the sought numbers.

15. If $2a$ and b are the sum and the sum of the fifth powers of two numbers or quantities, it is required to find them.

Representing them by $a + x$ and $a - x$, we must have $(a + x)^5 + (a - x)^5 = b$, or $2a^5 + 20a^3x^2 + 10ax^4 = b$; which gives $x^4 + 2a^2x^2 = \frac{b - 2a^5}{10a}$.

Solving the equation, we have $x = \sqrt{\left(\sqrt{\frac{b + 8a^5}{10a}} - a^2\right)}$; consequently, $a + x$ and $a - x$ become $a + \sqrt{\left(\sqrt{\frac{b + 8a^5}{10a}} - a^2\right)}$ and $a - \sqrt{\left(\sqrt{\frac{b + 8a^5}{10a}} - a^2\right)}$, as required. If $2a = 6$ and $b = 1056$, we get 4 and 2 for the sought numbers.

16. A person on a journey goes the distance a the first day, $a + d$ the second day, $a + 2d$ the third day, and so on, in arithmetical progression; then how long will it take him to go the distance b?

Ans. $-\frac{2a - d}{2d} + \frac{\sqrt{(4a^2 - 4ad + d^2 + 8bd)}}{2d}$ days.

17. To find two positive members, such that if the square of the first is multiplied by 5 and that of the second by 7, the sum of the products shall equal 157; and that if the square of the first is multiplied by 7 and that of the second by 5, the sum of the products shall equal 143.

Ans. 3 and 4.

18. Having the sum, $3a$, and the sum of the squares, b, of three numbers or quantities, in arithmetical progression, to find the numbers or quantities.

Representing the progression by $x - y$, x, and $x + y$, the conditions of the question give the equations $x - y + x + x + y = 3a$ or $x = a$, and $(x - y)^2 + x^2 + (x + y)^2 = 3x^2 + 2y^2 = 3a^2 + 2y^2 = b$, or we have $y = \sqrt{\frac{b - 3a^2}{2}}$. Hence the terms become $a - \sqrt{\frac{b - 3a^2}{2}}$, a, and $a + \sqrt{\frac{b - 3a^2}{2}}$; thus, if $3a = 15$ and $b = 83$, we have $a = 5$ and $\sqrt{\frac{b - 3a^2}{2}} = 2$; and the progression becomes 3, 5, 7.

19. Given, the sum, $4a$, and the sum of the squares, b, of four numbers or quantities, in arithmetical progression, to find the numbers or quantities.

Representing the progression by $x-3y$, $x-y$, $x+y$, and $x+3y$, the conditions give the equations $4x=4a$ or $x=a$, and $(x-3y)^2+(x-y)^2+(x+y)^2+(x+3y)^2=4x^2+20y^2=4a^2+20y^2=b$, or $y=\sqrt{\frac{b-4a^2}{20}}$; consequently, the terms of the progression are easily found.

Thus, if $4a=24$ and $b=164$, we have $a=6$ and $y=\sqrt{\frac{164-144}{20}}=1$; consequently, the progression is 3, 5, 7, 9.

Remark.—If we have the sum, na, and sum of the squares or cubes or fourth powers b, of n numbers or quantities in arithmetical progression, then we can, by methods similar to those used in this and the preceding question, find the terms of the progression from the solution of quadratic equations.

20. Given, the sum, $3a$, and the product, b, of three numbers or quantities, in arithmetical progression, to find the numbers or quantities.

Representing the progression by $x-y$, x, and $x+y$, the conditions give $x=a$ and $(x-y)x(x+y)=x(x^2-y^2)=a(a^2-y^2)=b$, or $y=\sqrt{\frac{a^3-b}{a}}$; consequently, the progression is easily found.

Thus, if $3a=15$ and $b=105$, we have $a=5$ and $y=\sqrt{\frac{125-105}{5}}=2$; and the progression is 3, 5, 7.

Remark.—If we have the sum and product of four or five numbers or quantities, in arithmetical progression, then it is easy to perceive (from what has been done) how the progression may be found by quadratics.

21. "Having the sum a, and the sum of the squares b, of four numbers or quantities, in geometrical progression, to find the progression."

Let x and y denote the second and third terms; then, by the nature of such a progression, $\frac{x^2}{y}$ and $\frac{y^2}{x}$ must be the first and fourth terms; consequently, from the conditions, we have the equations $\frac{x^2}{y} + x + y + \frac{y^2}{x} = a$ and $\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b$.

Since $\frac{x^2}{y} + \frac{y^2}{x} = \frac{x^3 + y^3}{xy} = (x + y)\left(\frac{x^2 - xy + y^2}{xy}\right) = (x + y)\left(\frac{x^2 + y^2}{xy} - 1\right)$, it is clear that the first of the preceding equations is equivalent to $\frac{x^2 + y^2}{xy} = \frac{a}{x + y}$; and it is easy, in like manner, to show that the second is equivalent to $\frac{x^4 + y^4}{x^2y^2} = \frac{b}{x^2 + y^2}$.

From the first of these equations we have $\frac{x^4 + y^4}{x^2y^2} = \left(\frac{x^2 + y^2}{xy}\right)^2 - 2 = \left(\frac{a}{x + y}\right)^2 - 2$; consequently, by substitution, the equations are reduced to $\frac{x^2 + y^2}{xy} = \frac{a}{x + y}$ and $\left(\frac{a}{x + y}\right)^2 - 2 = \frac{b}{x^2 + y^2}$.

Eliminating $x^2 + y^2$ from the preceding equations, and adding 2 to each member of the first, we shall get $\left(\frac{a}{x + y}\right)^3 - \frac{2a}{x + y} = \frac{b}{xy}$ and $\frac{(x + y)^2}{xy} = \frac{a}{x + y} + 2$, or $\frac{1}{xy} = \frac{a}{(x + y)^3} + \frac{2}{(x + y)^2}$.

Eliminating $\frac{1}{xy}$ from the first and third of these equations, and rejecting the common factor $\frac{1}{x + y}$ from the result, we get $\frac{a^3}{(x + y)^2} - 2a = \frac{ab}{(x + y)^2} + \frac{2b}{x + y}$, or (freeing from fractions, etc.) we have $(x + y)^2 + \frac{b}{a}(x + y) = \frac{a^2 - b}{2}$.

Solving this quadratic, we get $x + y = \sqrt{\frac{a^2 - b}{2} + \frac{b^2}{4a^2}} -$

$\frac{b}{2a}$; consequently, if we substitute the value of $x + y$ in $xy = \frac{(x + y)^3}{a + 2(x + y)}$, we shall find the value of xy.

Hence, from the values of $x + y$ and xy we can easily find the values of x and y, and thence the terms of the progression become known; thus, if $a = 15$, and $b = 85$, we get $x + y = 6$ and $xy = 8$, and solving these equations, we have $x = 2$ and $y = 4$, and thence the progression $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$ becomes 1, 2, 4, and 8.

22. Having the sum, a, and the sum of the squares, b, of five numbers or quantities, in geometrical progression, to find the progression.

Let x and y denote the second and third terms; then, by the nature of the progression, $\frac{x^2}{y}$, $\frac{y^2}{x}$, and $\frac{y^3}{x^2}$ will represent the first, fourth, and fifth terms.

Hence, by the question, we must have the equations $\frac{x^2}{y} + x + y + \frac{y^2}{x} + \frac{y^3}{x^2} = a$ and $\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} + \frac{y^6}{x^4} = b$; and dividing the second of these equations by the first, we get the equation $\frac{x^2}{y} - x + y - \frac{y^2}{x} + \frac{y^3}{x^2} = \frac{b}{a}$, so that the solution of the question is reduced to that of the equations $\frac{x^2}{y} + x + y + \frac{y^2}{x} + \frac{y^3}{x^2} = a$ and $\frac{x^2}{y} - x + y - \frac{y^2}{x} + \frac{y^3}{x^2} = \frac{b}{a}$.

By taking the half sum and half difference of these equations, we get $\frac{x^2}{y} + y + \frac{y^3}{x^2} = \frac{a^2 + b}{2a}$ and $x + \frac{y^2}{x} = \frac{a^2 - b}{2a}$; and multiplying the terms of the second of these by $\frac{y}{x}$, and subtracting the result from the first, we shall have $\frac{x^2}{y} = \frac{a^2 + b}{2a} - \left(\frac{a^2 - b}{2a}\right)\frac{y}{x}$ and $x + \frac{y^2}{x} = \frac{a^2 - b}{2a}$ for the equations, to whose solution the question is reduced.

Multiplying the terms of the second of these equations by $\frac{x}{y}$, we easily get $\frac{x^2}{y} = \left(\frac{a^2 - b}{2a}\right) \times \frac{x}{y} - y$; consequently, substituting this value of $\frac{x^2}{y}$ in the first of the preceding equations, we get $\left(\frac{a^2-b}{2a}\right)\times\frac{x}{y}-y=\frac{a^2+b}{2a}-\left(\frac{a^2-b}{2a}\right)\times\frac{y}{x}$, which is easily reduced to the form $\frac{a^2-b}{2a}\times\frac{x^2+y^2}{xy}=\frac{a^2+b}{2a}+y$.

Since the equation $x+\frac{y^2}{x}=\frac{a^2-b}{2a}$ gives $\frac{x^2+y^2}{x}=\frac{a^2-b}{2a}$, the preceding equation, by substitution, becomes $\left(\frac{a^2-b}{2a}\right)^2\div y=\frac{a^2+b}{2a}+y$, which is easily reduced to the quadratic $y^2+\frac{a^2+b}{2a}y=\left(\frac{a^2-b}{2a}\right)^2$.

Solving the quadratic, and taking the positive root, we have $y=-\frac{a^2+b}{4a}+\frac{\sqrt{(5a^4-6a^2b+5b^2)}}{4a}$; consequently, having found y, we easily get x from the equation $x+\frac{y^2}{x}=\frac{a^2-b}{2a}$; thus, if $a=31$ and $b=341$, we have $y=4$ and $x=2$, and thence the progression is 1, 2, 4, 8, and 16.

23. Having a, the sum of the means, and b, the sum of the extremes, of four numbers or quantities, in geometrical progression, to find the progression.

Ans. Supposing the terms of the progression to increase from left to right, then $a\sqrt{\frac{b-a}{b+3a}}$ equals the difference of the means, and of course the progression is easily found.

Thus, if $a=24$ and $b=56$, the progression is 2, 6, 18, and 54.

24. Having a, the sum of the extremes, and b, the product, of three numbers or quantities, in harmonical progression, to find the progression.

Let x, y, and z denote the progression; then, since their reciprocals are in arithmetical progression, we have the equation $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$, or we have $\frac{x+z}{xz}=\frac{2}{y}$, or (by the conditions) $\frac{a}{b}=\frac{2}{y^2}$; consequently, $y=\sqrt{\frac{2b}{a}}$ and the progression is easily found; thus, if $a=98$, and $b=99225$, we get $y=45$, and from $\frac{b}{y}=\frac{99225}{45}=xz$, or $xz=2205$ and $x+z=98$, we get $x=35$ and $z=63$.

25. Having a, the sum, and b, the product of the extremes of three numbers or quantities, in harmonical progression, to find the progression.

Ans. The mean term equals $\frac{a-\sqrt{a^2-8b}}{2}$, and thence $\frac{a+\sqrt{a^2-8b}}{2}$ equals the sum of the extreme terms; consequently, the progression is easily found; thus, if $a=143$, and $b=2205$, we get 45 for the mean, and 35 and 63 for the extremes.

26. "What number is that which, when divided by the product of its two digits, the quotient is 3, and if 18 be added to it, the digits will be inverted?"

If y is the unit's, and x the ten's digit, then $10x+y$ will express the number, and $\frac{10x+y}{xy}=3$ by the first condition; also, $10x+y+18=10y+x$ by the second condition. From the elimination of y we get the quadratic $x^2-\frac{5}{3}x=\frac{2}{3}$, whose solution gives $x=2$, and 24 will be the sought number.

27. There is a number consisting of three digits, such that the middle digit is a geometrical mean between the other two, and that the sum of the extreme digits, when diminished by 1, equals twice the middle digit. Moreover, supposing the sum of the digits to equal 19, and that the hundredth's digit is greater than the unit's digit, it is proposed to find the number. *Ans.* 964.

28. If a body moves 1 yard the first minute, 3 yards the second minute, 5 yards the third, 7 yards the fourth, and so on, then how long will the body be in moving 400 yards? *Ans.* 20 minutes.

29. Having the product of the sum of two numbers by the greater equal to 153, and the product of the sum by the less equal to 136, to find the numbers. *Ans.* 8 and 9.

30. The product of two numbers is 28, and after the first is increased by 6 and the second diminished by 4, the product of the resulting numbers is 30; then what are the numbers? *Ans.* 4 and 7.

31. Supposing the sum of a certain principal and its amount at compound interest for two years to be $2102.50, and that the amount of the same principal, at the same compound rate, in four years, is $1215.50625; then what is the principal? and the rate per cent. per annum?

Ans. The principal is $1000, and the rate 5 per cent.

32. The sum of the squares of two numbers equals 61, and their sum is to their product as 11 to 30; then what are the numbers? *Ans.* 6 and 5.

33. The sum of the reciprocals of two numbers is 5, and the sum of the squares of their reciprocals is 13; then what are the numbers? *Ans.* $\frac{1}{2}$ and $\frac{1}{3}$.

34. Having the sum of the n^{th} powers of a number or quantity, and of its reciprocal, equal to a, to find it.

Ans. $\sqrt[n]{\dfrac{a \pm \sqrt{a^2-4}}{2}}$.

35. Having the sum of the n^{th} powers of two numbers or quantities equal to a, and the sum of the n^{th} powers of their reciprocals equal to b, to find them.

Ans. $\sqrt[n]{\dfrac{a \pm \sqrt{a^2-\frac{4a}{b}}}{2}}$ and $\sqrt[n]{\dfrac{a \mp \sqrt{a^2-\frac{4a}{b}}}{2}}$.

36. Having the sum of the squares of two numbers or quantities equal to a, and the sum of the reciprocals of the numbers or quantities equal to b, to find them.

If x and y stand for the numbers or quantities, the conditions give the equations $x^2+y^2=a$ and $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=b$.

From the second of these equations we get $2xy - \frac{2}{b}(x+y) = 0$, which, added to the first equation, reduces it to the quadratic $(x+y)^2 - \frac{2}{b}(x+y) = a$, whose solution (regarding x and y as positive) gives $x + y = \frac{1 + \sqrt{ab^2 + 1}}{b}$.

Because $2xy = \frac{2}{b}(x+y) = \frac{2 + 2\sqrt{ab^2+1}}{b^2}$, we get $x^2 + y^2 - 2xy = a - 2xy = \frac{ab^2 - 2 - 2\sqrt{ab^2+1}}{b^2}$, whose square root gives $x - y = \pm\frac{\sqrt{(ab^2 - 2 - 2\sqrt{ab^2+1})}}{b}$. Hence, since $x + y$ and $x - y$ are known, it is easy to find the values of x and y.

SECTION XV.

METHOD OF UNDETERMINED COEFFICIENTS.

(**1.**) If an equation of the form $A + Bx + Cx^2 +$ etc., $= A' + B'x + C'x^2 +$ etc., (1), in which A, B, C, etc., A′, B′, C′, etc., do not contain x, must be satisfied by any value which may be given to x, then shall $A = A'$, $B = B'$, $C = C'$, and so on, or (regarding A and A′ as coefficients of $x^0 = 1$) *the coefficient of any power of x in one member of the equation equals the coefficient of the same power of x in the other member.*

For, by transposition, etc., (1) gives

$$x = \frac{A' - A}{B - B' + (C - C')x + \text{etc.}} \quad (2),$$

which clearly can not be satisfied for all the values that may be given to x (or so as to leave x arbitrary), except by putting the numerator and denominator of the right member separately equal to 0; consequently, we must have $A = A'$, and $B - B' + (C - C')x + (D - D')x^2 + \text{etc.} = 0$, (3).

It may in like manner be shown from (3) that we must have $B = B'$, and then that $C = C'$, and so on, as required.

Hence, if for A', B', C', etc., we put their equals, A, B, C, etc., (1) will become the identical equation $A + Bx + Cx^2 +$ etc. $= A + Bx + Cx^2 +$ etc.

Universally, if we have the equation $A + B\phi(x) + C\phi'(x)$ + etc. $= A' + B'\phi(x) + C'\phi'(x) +$ etc., (4), such that A, B, C, etc., A', B', C', etc., are independent of any values which $\phi(x)$, $\phi'(x)$, $\phi''(x)$ etc., $\frac{\phi'(x)}{\phi(x)}$, $\frac{\phi''x}{\phi x}$, etc., $\frac{\phi''(x)}{\phi'(x)}$, $\frac{\phi'''(x)}{\phi'(x)}$, etc., and so on (which are functions of x, or dependent on it), can have, then it is clear that, by a like reasoning to that used above, we can show that we must have $A = A'$, $B = B'$, $C = C'$, and so on; consequently, by the substitution of the values of A', B', C', etc., (4) becomes the identical equation $A + B\phi(x) + C\phi'(x) +$ etc. $= A + B\phi(x) + C\phi'(x) +$ etc., in which x is clearly arbitrary.

(2.) Since any function of x, when developed into a series of ascending or descending powers of x, or of any other functions of x (which the nature of the case may require), ought to be considered and treated as being identical with the series which represents it, it follows (from what has been done) that, after we have determined a suitable form for the development, we may represent the coefficients of its terms by any undetermined letters, as A, B, C, etc., and then determine the values of the letters by equating them to the coefficients of the corresponding powers (or functions) of x in the given function, so that the given function and its development shall constitute an identical equation; *and it is in this process, that the Method of Undetermined Coefficients essentially consists.*

(3.) APPLICATIONS OF THE METHOD TO EXAMPLES.

1. To convert the fraction $\frac{a}{b + cx}$ into a series, arranged according to the ascending powers of x.

If we divide the numerator and denominator of the fraction by b, and put $\frac{a}{b} = a'$ and $\frac{c}{b} = b'$, the fraction will become

$\frac{a'}{1 + b'x}$; and it is clear (from actual division) that we may assume $\frac{a'}{1 + b'x} = a' + Ax + Bx^2 + Cx^3 +$ etc., for the development of the fraction.

Freeing the equation from the fraction, we get $a' = a' + (A + a'b')x + (B + Ab')x^2 + (C + Bb')x^3 +$ etc., which must be an identical equation.

Consequently, if we equate the terms which do not contain x, and put the coefficients of x, x^2, x^3, etc., equal to 0 (since the coefficients of x, x^2, x^3, etc., in the first member of the equation equal 0), we get the identical equation $a' = a'$, and the equations $A + a'b' = 0$, $B + Ab' = 0$, $C + Bb' = 0$, etc., or $A = a' \times (-b')$, $B = A \times (-b')$, $C = B \times (-b')$, etc.

Hence, we have $\frac{a}{b + cx} = \frac{a'}{1 + b'x} = a' + a' \times (-b'x) + a' \times (-b'x) \times (-b'x) + a' \times (-b'x) \times (-b'x) \times (-b'x) +$ etc. $= \frac{a}{b} - \frac{acx}{b^2} + \frac{ac^2x^2}{b^3} - \frac{ac^3x^3}{b^4} +$ etc.; which is such that if we multiply any one of its terms by $-\frac{cx}{b}$, the product gives the next successive term.

Remark.—If we put $x = \frac{1}{y}$, we shall have $\frac{a}{b + cx} = \frac{a}{b + \frac{c}{y}} = \frac{ay}{c + by} = y \times \frac{a}{c + by}$; consequently, developing $\frac{a}{c + by}$ into a series arranged according to the ascending powers of y, we shall have $\frac{a}{c + by} = \frac{a}{c} - \frac{aby}{c^2} + \frac{ab^2y^2}{c^3} -$ etc.; and restoring the value of y, we easily get $\frac{a}{b + cx} = \frac{a}{cx + b} = \frac{a}{cx} - \frac{ab}{c^2x^2} + \frac{ab^2}{c^3x^3}$ etc. $= \frac{ax^{-1}}{c} - \frac{abx^{-2}}{c^2} + \frac{ab^2x^{-3}}{c^3} -$ etc.; a series arranged according to the descending powers of x.

2. To convert the fraction $\frac{1}{1 - 3x}$ into a series, arranged according to the ascending powers of x.

Ans. $1 + 3x + 9x^2 + 27x^3 + 81x^4 +$ etc.

3. To convert $\frac{1}{5 - 2x}$ into a series, arranged according to the ascending powers of x.

$$\textit{Ans.}\ \frac{1}{5} + \frac{2x}{25} + \frac{4x^2}{125} + \frac{8x^3}{625} + \text{etc.}$$

4. To convert $\frac{3 - 5x}{2 + 7x}$ into a series.

$$\textit{Ans.}\ \frac{3}{2} - \frac{31}{4}x + \frac{217}{8}x^2 - \frac{1519}{16}x^3 + \frac{10633}{32}x^4 - \text{etc.}$$

5. To convert $\sqrt{1 - z^2}$ into a series, arranged according to the ascending powers of z.

We may clearly assume $\sqrt{1 - z^2} = 1 + Az^2 + Bz^4 + Cz^6 + \text{etc.}$, or $1 - z^2 = (1 + Az^2 + Bz^4 + \text{etc.})^2 = 1 + 2Az^2 + (A^2 + 2B)z^4 + 2(AB + C)z^6 + \text{etc.}$; since it is clear that the odd powers of z can not enter into the series. Comparing the coefficients of like powers of z, we get $A = -\frac{1}{2}$, $B = -A^2 \div 2 = -\frac{1}{8} = -\frac{1}{2.4}$, $C = -AB = -\frac{1}{2} \times \frac{1}{2.4} = -\frac{3}{2.4.6}$, etc. Hence, by substitution, we shall have $\sqrt{1 - z^2} = 1 - \frac{z^2}{2} - \frac{z^4}{2.4} - \frac{3z^6}{2.4.6} - \frac{3.5z^8}{2.4.6.8} - \text{etc.}$

6. To find two fractions whose sum equals the fraction $\frac{1}{1 - 7x + 12x^2} = \frac{1}{(1 - 3x).(1 - 4x)}$.

It is clear that we may assume $\frac{A}{1 - 3x}$ and $\frac{B}{1 - 4x}$ for the fractions, and suppose A and B to be independent of x. Hence, we shall have the equation $\frac{1}{(1 - 3x).(1 - 4x)} = \frac{A}{1 - 3x} + \frac{B}{1 - 4x} = \frac{A + B - (4A + 3B)x}{(1 - 3x).(1 - 4x)}$, or by rejecting the common denominators we shall have $1 = A + B - (4A + 3B)x$, which must clearly be an identical equation; consequently, we must have the equations $A + B = 1$ and $4A + 3B = 0$, which give $A = -3$ and $B = 4$, and hence $\frac{1}{1 - 7x + 12x^2} = \frac{-3}{1 - 3x} + \frac{4}{1 - 4x}$.

7. To find two fractions, whose sum equals the fraction $\frac{1}{(1-3x).(1+x^2)}$.

Because x^2+1 has $x+\sqrt{-1}$ and $x-\sqrt{-1}$ for factors, we may clearly put $\frac{1}{(1-3x).(1+x^2)}=\frac{A}{1-3x}+\frac{m-n\sqrt{-1}}{x-\sqrt{-1}}+\frac{m+n\sqrt{-1}}{x+\sqrt{-1}}=\frac{A}{1-3x}+\frac{2n+2mx}{1+x^2}$, or putting B for $2n$ and C for $2m$, we may assume $\frac{1}{(1-3x).(1+x^2)}=\frac{A}{1-3x}+\frac{B+Cx}{1+x^2}$; which, freed from fractions, gives the identical equation $1=A+B+(C-3B)x+(A-3C)x^2$. Hence, we have the equations $A+B=1$, $C=3B$, and $A=3C$, which give $A=\frac{9}{10}$, $B=\frac{1}{10}$, and $C=\frac{3}{10}$; consequently, we shall have $\frac{1}{(1-3x).(1+x^2)}=\frac{9}{10(1-3x)}+\frac{1+3x}{10(1+x^2)}$.

8. To decompose the fraction $\frac{3-x}{(x-2)^3.(x-5)}$ into others (the most simple), whose sum shall equal the given fraction.

Assuming $\frac{3-x}{(x-2).(x-5)}=\frac{A}{x-2}+\frac{B}{x-5}$, we get $A=-\frac{1}{3}$ and $B=-\frac{2}{3}$, and thence we shall have $\frac{3-x}{(x-2)^2.(x-5)}=-\frac{1}{3(x-2)^2}-\frac{2}{3(x-2).(x-5)}$; and since $\frac{1}{(x-2).(x-5)}=-\frac{1}{3(x-2)}+\frac{1}{3(x-5)}$, we shall get $\frac{3-x}{(x-2)^2.(x-5)}=-\frac{1}{3(x-2)^2}+\frac{2}{9(x-2)}-\frac{2}{9(x-5)}$.

Hence, we shall have $\frac{3-x}{(x-2)^3.(x-5)}=-\frac{1}{3(x-2)^3}+\frac{2}{9(x-2)^2}-\frac{2}{9(x-2).(x-5)}=-\frac{1}{3(x-2)^3}+\frac{2}{9(x-2)^2}+\frac{2}{27(x-2)}-\frac{2}{27(x-5)}$, as required.

9. To find two fractions, whose sum equals the fraction $\frac{2x-5}{x^2-5x+6}$. *Ans.* $\frac{1}{x-3}$ and $\frac{1}{x-2}$.

10. To find three fractions whose sum shall equal $\frac{7x-11}{x^3-2x^2-x+2}$. *Ans.* $\frac{2}{x-1}$, $-\frac{3}{x+1}$, and $\frac{1}{x-2}$.

11. To find three fractions whose sum equals the fraction $\frac{1}{x^3+x^2}$. *Ans.* $\frac{1}{x^2}$, $-\frac{1}{x}$, and $\frac{1}{x+1}$.

12. To find three fractions whose sum equals the fraction $\frac{1}{x^4-1}$.

Ans. $\frac{1}{4(x-1)}$, $-\frac{1}{4(x+1)}$, and $-\frac{1}{2(x^2+1)}$.

13. To convert $\frac{a+bx}{c+mx+nx^2}$ into a series, arranged according to the ascending powers of x.

Dividing each term of the numerator and denominator by c, and putting $\frac{a}{c}=a'$, $\frac{b}{c}=b'$, $\frac{m}{c}=m'$, $\frac{n}{c}=n'$, we shall have reduced the given fraction to $\frac{a'+b'x}{1+m'x+n'x^2}$; consequently, if we convert this into a series, we shall readily obtain the required development.

It is easy to perceive that we may assume $\frac{a'+b'x}{1+m'x+n'x^2}$ $= a' + Ax + Bx^2 + Cx^3 + \text{etc.}$; which, freed from the fraction, gives the identical equation $a' + b'x = a' + (A + a'm')x + (B + Am' + a'n')x^2 + (C + Bm' + An')x^3 + (D + Cm' + Bn')x^4 + \text{etc.}$ Hence we get the equations $a' = a'$, $A = -a'm' + b'$, $B = a' \times (-n') + A \times (-m')$, $C = A \times (-n') + B \times (-m')$, $D = B \times (-n') + C \times (-m')$, and so on; which give $Ax = (-a'm' + b')x$, $Bx^2 = a' \times (-n'x^2) + Ax \times (-m'x)$, $Cx^3 = Ax \times (-n'x^2) + Bx^2 \times (-m'x)$, $Dx^4 = Bx^2 \times (-n'x^2) + Cx^3 \times (-m'x)$, and so on.

Hence, the development is easily found; *since any term after the second is found from the two preceding terms by multiplying the first of the two by* $-n'x^2$, *and the second by* $-m'x$, *and then adding the products according to their signs.*

Remark.—It is clear that fractions of the forms $\dfrac{a'+b'x+c'x^2}{1+m'x+n'x^2+p'x^3}$, $\dfrac{a'+b'x+c'x^2+d'x^3}{1+m'x+n'x^2+p'x^3+q'x^4}$, etc., can be developed in a similar way, when their first three, four, etc., terms are known.

14. To develop $\dfrac{1}{1-x-x^2}$ into a series, arranged according to the ascending powers of x.

By actual division, we find $1+x$ for the first two terms of the development; consequently, since $-n'x^2$ and $-m'x$ are here represented by x^2 and x, by proceeding as directed in the preceding example, we have $\dfrac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 +$ etc.

15. To convert $\dfrac{1}{1+2x+x^2}$ into a series.

Ans. $\dfrac{1}{1+2x+x^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 -$ etc.

16. To convert $\dfrac{1-x}{1+2x+x^2}$ into a series.

Ans. $\dfrac{1-x}{1+2x+x^2} = 1 - 3x + 5x^2 - 7x^3 + 9x^4 -$ etc.

17. To convert $\dfrac{1+2x}{1-3x+2x^2-x^3}$ into a series.

By division we get $1 + 5x + 13x^2$ for the first three terms of the development; and since $-p'x^3$, $-n'x^2$, $-m'x$ are here expressed by x^3, $-2x^2$, and $3x$, we easily get $\dfrac{1+2x}{1-3x+2x^2-x^3} = 1 + 5x + 13x^2 + 30x^3 + 69x^4 + 160x^5 + 372x^6 +$ etc.

18. To expand $\dfrac{1}{1-7x+12x^2} = \dfrac{-3}{1-3x} + \dfrac{4}{1-4x}$ into a series.

Since $\dfrac{1}{1-7x+12x^2} = 1 + 7x + 37x^2 + 175x^3 + 781x^4 + 3367x^5 +$ etc., and that $\dfrac{-3}{1-3x} = -3 - 9x - 27x^2 - 81x^3 -$

etc., $\frac{4}{1-4x} = 4 + 16x + 64x^2 + 256x^3 +$ etc., it follows that by adding the last two series, according to their signs, the sum must equal the first series. If we use n to stand for the number of any term in either of the two last series, it is clear that $-3^n x^{n-1}$ will represent the n^{th} term in the first of them, and that $4^n x^{n-1}$ will represent the n^{th} term of the second. Hence, if we add these terms, we shall get $(4^n - 3^n)x^{n-1}$ for their sum, which will equal the n^{th} term (or what is called the *general term*) of the development of $\frac{1}{1-7x+12x^2}$; thus, if $n = 5$, $(4^n - 3^n)x^{n-1}$ becomes $(1024 - 243)x^4 = 781x^4$, which is the fifth term of the series $1 + 7x + 37x^2 + 175x^3 + 781x^4 +$ etc.

Remark.—From this example we perceive how we may proceed in order to find the general term of any series of the kind given above.

REVERSION OF SERIES.

1. To revert a series is to express the unknown letter contained in it by another series, which may be regarded as the development of the value of the unknown letter.

2. We shall suppose that we have $y = ax + bx^2 + cx^3 + dx^4 +$ etc., (1), and that it is proposed to find x.

Then it is clear that we may assume $x = Ay + By^2 + Cy^3 + Dy^4 +$ etc., (2), and regard A, B, C, etc., as being independent of y. If we substitute the values of y, y^2, y^3, etc., from (1) in (2), it will become $x = A(ax + bx^2 + \text{etc.}) + B(ax + bx^2 + \text{etc.})^2 + C(ax + bx^2 + \text{etc.})^3 +$ etc., (3), which must be an identical equation.

It is easy to perceive that the developments required in the right member of (3) can be obtained from the expression $Aax + Ba^2x^2 + Ca^3x^3 +$ etc., by changing a into $a + bx$, b into $b + cx$, c into $c + dx$, and so on successively, and developing by the rule given in Involution for the expansion of a function of a polynomial.

Thus, by using the vertical bar to express the coefficients of the powers of x, (3) becomes

$$x = Aax + \left.\begin{array}{c} Ba^2 \\ Ab \end{array}\right| x^2 + \left.\begin{array}{c} Ca^3 \\ 2Bab \\ Ac \end{array}\right| x^3 + \left.\begin{array}{c} Da^4 \\ 3Ca^2b \\ 2Bac \\ Bb^2 \\ Ad \end{array}\right| x^4 + \left.\begin{array}{c} Ea^5 \\ 4Da^3b \\ 3Cab^2 \\ 3Ca^2c \\ 2Bad \\ 2Bbc \\ Ae \end{array}\right| x^5 + \text{etc., (4).}$$

Because (4) is an identical equation, we must have the equations $Aa = 1$, $Ba^2 + Ab = 0$, $Ca^3 + 2Bab + Ac = 0$, and so on, which give $A = \frac{1}{a}$, $B = -\frac{b}{a^3}$, $C = \frac{2b^2 - ac}{a^5}$, $D = \frac{-5b^3 + 5abc - a^2d}{a^7}$, and so on.

Substituting the values of A, B, C, etc., in (2), we get $x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2b^2 - ac}{a^5}y^3 - \frac{5b^3 - 5abc + a^2d}{a^7}y^4 +$ etc., (5), which agrees with the formula given by Cagnoli, at p. 46 of his Trigonometry.

If the coefficients of the even powers of x in (1) equal 0, then it will be reduced to the form $y = ax + bx^3 + cx^5 + dx^7 +$ etc., (1′); and instead of (5) we shall have $x = \frac{y}{a} - \frac{b}{a^4}y^3 + \frac{3b^2 - ac}{a^7}y^5 - \frac{12b^3 - 8abc + a^2d}{a^{10}}y^7 +$ etc., (5′).

Remarks.—1. If we have $y' = m + ax + bx^2 + cx^3 +$ etc., then, by transposition, and putting $y = y' - m$, we shall have $y = ax + bx^2 +$ etc.; consequently, the series can be reversed by (5) or (5′).

2. If we have $a'z + b'z^2 + c'z^3 +$ etc. $= ax + bx^2 + cx^3 +$ etc.; then, by putting $y = a'z + b'z^2 + c'z^3 +$ etc., we shall have $y = ax + bx^2 + cx^3 +$ etc., and thence $x = \frac{y}{a} - \frac{by^2}{a^3} +$ etc.; consequently, putting $a'z + b'z^2 +$ etc., for y, we shall get $x = \frac{a'z}{a} + \frac{a^2b' - a'^2b}{a^3}z^2 +$ etc.

To illustrate what has been done, take the following common

EXAMPLES.

1st. Given, $y = x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} +$ etc., to find x.

If we put $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{4}$, $d = \frac{1}{8}$, etc., then (5) gives $x = y - \frac{y^2}{2} + \frac{y^3}{4} - \frac{y^4}{4} +$ etc.

2d. Given, $y' = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\,.\,3} + \frac{x^4}{2\,.\,3\,.\,4} +$ etc., to find x.

Putting $y = y' - 1$ $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{2.3} = \frac{1}{6}$, $d = \frac{1}{2.3.4} = \frac{1}{24}$, etc., then (5) gives $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} +$ etc. $=$ the hyperbolic logarithm of $1 + y = y'$.

3d. Given, $z + \frac{z^2}{2} + \frac{z^3}{4} + \frac{z^4}{8} +$ etc. $= x + x^2 + x^3 +$ etc., to find x.

Putting $a' = 1$, $b' = \frac{1}{2}$, $c' = \frac{1}{4}$, etc., $a = 1$, $b = 1$, $c = 1$, etc., then we shall have $x = z - \frac{z^2}{2} + \frac{z^3}{4} - \frac{z^4}{8} +$ etc.

4th. Given, $y = x - \frac{x^3}{2\,.\,3} + \frac{x^5}{2\,.\,3\,.\,4\,.\,5} - \frac{x^7}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}$ $+$ etc., to find x.

Putting $a = 1$, $b = -\frac{1}{2\,.\,3} = -\frac{1}{6}$, $c = \frac{1}{2\,.\,3\,.\,4\,.\,5} = \frac{1}{120}$, etc.; then, by (5′), we shall have $x = y + \frac{y^3}{2\,.\,3} + \frac{3y^5}{2\,.\,4\,.\,5} + \frac{3\,.\,5y^7}{2\,.\,4\,.\,6\,.\,7} +$ etc.

3. We shall suppose that we have the equation $x^n + ax^{n+m} + bx^{n+2m} + cx^{n+3m} + dx^{n+4m} +$ etc. $= \pm\, y^n$, (1), and that it is hence proposed to develop the value of x.

It is easy to perceive that we may assume $x = y[A + By^m + Cy^{2m} + Dy^{3m} +$ etc.], (2), in which A, B, C, etc., are independent of y; and it is clear that we may regard By^m

as being an increment of A, Cy^m as an increment of B, Dy^m as an increment of C, and so on.

From the substitution of the value of x from (2) in (1), we get $y^n[A + By^m + \text{etc.}]^n + ay^{n+m}[A + By^m + \text{etc.}]^{n+m} + by^{n+2m}[A + By^m + \text{etc.}]^{n+2m} + \text{etc.} = \pm y^n$, or omitting the factor y^n, which is common to the two members of the equation, we have $[A + By^m + \text{etc.}]^n + ay^m[A + By^m + \text{etc.}]^{n+m} + by^{2m}[A + By^m + \text{etc.}]^{n+2m} + \text{etc.} = \pm 1$, (3), which must be an identical equation.

It is easy to perceive that the developments required in the first member of (3) can be obtained from $A^n + aA^{n+m}y^m + bA^{n+2m}y^{2m} + cA^{n+3m}y^{3m} + \text{etc.}$, by changing A into $A + By^m$, B into $B+Cy^m$, C into $C+Dy^m$, and so on, successively, and developing according to the rule given in Involution. Thus, we get

$$A^n + \left.\begin{array}{l} aA^{n+m} \\ nA^{n-1}B \end{array}\right| y^m + \left.\begin{array}{l} bA^{n+2m} \\ (n+m)aA^{n+m-1}B \\ \frac{n(n-1)}{1\,.\,2}A^{n-2}B^2 \\ \qquad nA^{n-1}C \end{array}\right| y^{2m} + \text{etc.} = \pm 1, \quad (4).$$

Because (4) is an identical equation, we must have the equations $A^n = \pm 1$, (5), (noticing that + must be used in this for ±, when the right member of the given equation is +, and *vice versa*), and $aA^{n+m} + nA^{n-1}B = 0$, $bA^{n+2m} + (n+m)aA^{n+m-1}B + \frac{n(n-1)}{1\,.\,2}A^{n-2}B^2 + nA^{n-1}C = 0$, and so on.

From these equations we get $B = -\frac{a}{n}A^{m+1}$, $C = \frac{(n+2m+1)a^2 - 2nb}{2n^2} \times A^{2m+1}$, and so on.

If y represents the arithmetical root of y^n, and A the algebraic root of equation (5), then, if we put $Ay = y'$, (6), and substitute the values of B, C, D, etc., in (2), we shall get

$$x = y' - \frac{ay'^{m+1}}{n} + \left[\frac{n+2m+1}{2n^2}a^2 - \frac{b}{n}\right]y'^{2m+1} + \left[-\frac{(n+3m+1).(2n+3m+1)}{2\,.\,3\,.\,n^3}a^3 + \frac{n+3m+1}{n^2}ab - \frac{c}{n}\right]y'^{3m+1}$$
$$+ \left[\frac{(n+4m+1).(2n+4m+1).(3n+4m+1)}{2\,.\,3\,.\,4n^4}a^4 - \frac{(n+4m+1).(2n+4m+1)}{2\,.\,n^3}a^2b + \frac{n+4m+1}{2n^2}(b^2+2ac) - \frac{d}{n}\right]y'^{4m+1}$$

$$+\left[-\frac{(n+5m+1).(2n+5m+1).(3n+5m+1).(4n+5m+1)}{2\ .\ 3\ .\ 4\ .\ 5n^5}a^5 + \frac{(n+5m+1).(2n+5m+1).(3n+5m+1)}{1\ .\ 2\ .\ 3n^4}\times a^3b - \frac{(n+5m+1).(2n+5m+1)}{1\ .\ 2n^3}(a^2c+ab^2) + \frac{n+5m+1}{n^2}(ad+bc) - \frac{e}{n}\right]y'^{5m+1} + \text{ etc., } (a),$$

whose law of continuation is manifest.

If n is a positive whole number, and $m = -1$, then (1) becomes $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \text{etc.} = \pm y'^n$, (6′); and ($a$) is reduced to

$$x = y' - \frac{a}{n} + \left(\frac{n-1}{2n^2}a^2 - \frac{b}{n}\right)\frac{1}{y'} + \left[-\frac{(n-2).(n-1)}{3n^3}a^3 + \frac{n-2}{n^2}ab - \frac{c}{n}\right]\frac{1}{y'^2} + \left[\frac{(n-3)(2n-3)(n-1)}{2\ .\ 4n^4}a^4 - \frac{(n-3)(2n-3)}{2n^3}a^2b + \frac{n-3}{2n^2}(b^2+2ac) - \frac{d}{n}\right]\frac{1}{y'^3} +$$

$$\left[-\frac{(n-4)(n-2)(3n-4)(n-1)}{3\ .\ 5n^5}a^5 + \frac{(n-4)(n-2)(3n-4)}{3n^4}a^3b - \frac{(n-4)(n-2)}{n^3}(a^2c+ab^2) + \frac{n-4}{n^2}(ad+bc) - \frac{e}{n}\right]\frac{1}{y'^4} +$$

$$\left[\frac{(n-5)(2n-5)(3n-5)(4n-5)(n-1)}{2\ .\ 3\ .\ 4\ .\ 6\ n^6}a^6 - \frac{(n-5)(2n-5)(3n-5)(4n-5)}{2\ .\ 3\ .\ 4\ .\ n^5}a^4b + \frac{(n-5)(2n-5)(3n-5)}{2\ .\ 3\ .\ n^4}\times\left(\frac{3a^2b+2a^3c}{2}\right) - \frac{(n-5)(2n-5)}{2\ .\ 3\ .\ n^3}(3a^2d+6abc+b^3) + \frac{n-5}{2n^2}(2ae+2bd+c^2) - \frac{f}{n}\right]\frac{1}{y'^5} + \text{ etc., } (b).$$

If in (6′) we put $x - \frac{a}{n}$ instead of x, the resulting equation will be of the same form as (6′), with the exception that the term which corresponds to ax^{n-1} will be wanting, because its coefficient will equal 0; consequently, (6′) may represent the resulting equation, if we put $a = 0$.

Hence, putting $a = 0$, (b) is reduced to

$$x = y' - \frac{b}{ny'} - \frac{c}{ny'^2} + \frac{(n-3)b^2 - 2nd}{2n^2y'^3} + \frac{(n-4)bc - ne}{n^2y'^4} + \left[-\frac{(n-5)(2n-5)}{6n^3}b^3\right.$$

$+ \frac{n-5}{2n^2}(c^2 + 2bd) - \frac{f}{n}\Big]\frac{1}{y'^5} + \Big[-\frac{(n-3)(n-6)}{n^3}b^2c + \frac{n-6}{n^2}$ $(be + cd) - \frac{g}{n}\Big]\frac{1}{y'^6} + \Big[\frac{(n-7)\,(2n-7)\,(3n-7)}{2\,.\,3\,.\,4n^4} \times b^4 -$ $\frac{(n-7)(2n-7)}{2n^3}(b^2d + bc^2) + \frac{n-7}{2n^2}(d^2 + 2bf + 2ce) - \frac{h}{n}\Big]\frac{1}{y'^7}$ $+ \Big[\frac{(n-4)(n-8)(3n-8)}{3n^4}b^3c - \frac{(n-4)(n-8)}{3n^3}(c^3 + 6bcd +$ $3b^2e) + \frac{n-8}{n^2}(de + bg + cf) - \frac{i}{n}\Big]\frac{1}{y'^8}$ + etc., (*c*), which is of a much simpler form than (*b*), while it has the same generality.

Supposing n to be a positive whole number, and a, b, c, d, etc., together with $\pm y^n$, to be real numbers or quantities, it is clear that (6′) may represent any algebraic equation of the n^{th} degree, observing that one or more of the letters a, b, c, etc., may be naught or negative, if required.

Because $\pm y^n = \pm 1 \times y^n = (\mathrm{A}y)^n = y'^n$, it results that the right member of (6′) equals y'^n. To get $y' = \mathrm{A}y$, we must (according to what has been shown) multiply the positive root of y^n by the value of A, as found from the solution of the *binomial equation* (5).

Since it is proved in the Treatises on Analytical Trigonometry, that (5) has n different roots, it follows that y' must have n different values. Also, because the imaginary roots of (5) occur in pairs of the forms $p+q\sqrt{-1}$ and $p-q\sqrt{-1}$ (in which p may equal 0, if required), it follows that there must be similar corresponding values of y'. Hence, it is clear, from (*b*) or (*c*), that the equation of the n^{th} degree represented by (6′) has n roots, whose developments are given by (*b*) or (*c*), by putting the different values of y' for y' in them; it is also clear, if (6′) has imaginary roots, that they must occur in pairs of the forms $\mathrm{P} + \mathrm{Q}\sqrt{-1}$ and $\mathrm{P} - \mathrm{Q}\sqrt{-1}$, observing that P may equal 0, if required.

It may be added, if a, b, c, etc., and y'^n are one or more of them imaginary, then it may be shown, in much the same way as before, that (6′) has n roots.

To show the use of what has been done, take the following

EXAMPLES.

1. To develop the roots of $x^2 + 2x = 9 = 3^2 \times 1$.

Comparing the equation to (6′), we have $n = 2$, $a = 2$, $b = 0$, $c = 0$, etc., $y'^n = y'^2 = 3^2 \times 1$, or, since $y'^2 = y^2 \times A^2$, we put $y^2 = 3^2$ and $A^2 = 1$; consequently, the positive root of y^2 is 3, and the algebraic roots of $A^2 = 1$ give $A = \pm 1$, and we shall have $y' = \pm 3$.

Substituting the preceding values in (b), we get $x = y' - 1 + \frac{1}{2y'} - \frac{1}{8y'^3} + \frac{1}{16y'^5} -$ etc., and putting ± 3 for y' in this, we have $x = \pm 3 - 1 \pm \frac{1}{6} \mp \frac{1}{216} \pm \frac{1}{3888} +$ etc. $= \pm 3 - 1 \pm 0.1666\dot{6} \mp 0.004629$, $\pm 0.000257 \mp$ etc. Hence, using the upper signs in the ambiguous signs ($\pm$, $\mp$), in the preceding series, we get $x = 2.1622$; and using the lower signs, we have $x = -4.1622$. These roots, which are correct to four decimal places, have been obtained by using only five terms of the preceding series.

2. To find z, from the equation $z^2 - 700z = 59829$.

To simplify the equation, we put $z = 100x$, and thence get $x^2 - 7x = 5.9829$, or, multiplying by x^{-1}, and transposing, we have $x - 5.9829x^{-1} = 7$.

In this equation we have $n = 1$, $a = 0$, $b = -5.9829$, $c = 0$, $d = 0$, etc., and $y' = y = 7$. Hence, by substituting the preceding values in (c), we get $x = 7 + \frac{5.9829}{7} - \frac{(5.9829)^2}{7^3} + \frac{2(5.9829)^3}{7^5} - \frac{5(5.9829)^4}{7^7} +$ etc. $= 7 + 0.8547 - 0.10435887 + 0.0254844 - 0.0074875 +$ etc. $= 7.768308 +$, when we use only five terms of the series; consequently, we have $z = 100x = 776.8308 +$. The preceding value of z is correct to two places of figures, and nearly to three places, for the correct value is $z = 777$. (See Young's Theory of Equations, p. 103.) It may be added, that the remaining root of the equation is easily found to be -77.

3. To find x from the equation $x^3 - 3x^2 = 5$.

Multiplying the terms by x^{-2} and transposing, the equation becomes $x - 5x^{-2} = 3$; which gives $a = 0$, $b = 0$, $c = -5$, $d = 0$, $e = 0$, etc., $y' = y = 3$ and $n = 1$.

Substituting the preceding values in (*c*), we get $x = 3 + \frac{5}{9} - \frac{50}{243} + \frac{875}{6561} - \text{etc.} = 3 + 0.555\dot{5}. - 0.2058 + 0.1333. - \text{etc.} = 3.42.$, correct to two decimal places nearly.

4. To find x from the equation $x^5 - 10x^3 + 6x + 1 = 0$.

Multiplying by x^{-3} and transposing, the equation becomes $x^2 + 6x^{-2} + x^{-3} = 10$; which gives $n = 2$, $a = 0$, $b = 0$, $c = 0$, $d = 6$, $e = 1$, $f = 0$, $g = 0$, etc., $y^2 = 10$ and $A^2 = 1$, or $y = \sqrt{10}$ and $A = \pm 1$ and $y' = \pm \sqrt{10}$.

Hence, $x = \pm \sqrt{10} \mp \frac{3}{10\sqrt{10}} - \frac{1}{200} \mp \frac{5}{8} \times \frac{36}{10^3\sqrt{10}} - \frac{3}{2} \times \frac{6}{10^4} + \text{etc.} = \pm 3.162277. \mp 0.094868. - 0.005 \mp 0.0071151. - 0.0009 + \text{etc.}$

Taking the upper signs in the ambiguous signs, we get $x = 3.05$, correct to two decimal places; and using the lower signs we have $x = -3.06$, which is also correct to two decimal places.

5. To find x from the equation $x^4 - 3x^2 + 75x = 10000$.

Here we have $n = 4$, $a = 0$, $b = -3$, $c = 75$, $d = 0$, $e = 0$, etc., also, $y'^4 = 10000 = 10^4 \times 1 = y^4 \times A^4$, or $y^4 = 10^4$ and $A^4 = 1$, so that $y = 10$, $A = \pm 1$, and $A = \pm \sqrt{-1}$; consequently, y' has the four values expressed by $y' = \pm 10$ and $y' = \pm 10 \sqrt{-1}$.

By the substitution of the preceding values in (*c*), we get $x = y' + \frac{3}{4y'} - \frac{75}{4y'^2} + \frac{9}{32y'^3} - \frac{22473}{128y'^5} + \frac{675}{32y'^6} - \frac{270405}{2048y'^7} + \text{etc.}$ Putting ± 10 for y', we get $x = \pm 10 \pm 0.075 - 0.1875 \pm 0.00028125 \mp 0.00175570 + 0.00002109 \mp 0.00001320 \pm \text{etc.}$

Hence, we get $x = 9.8860$ and $x = -10.26099$ for the approximate values of two of the roots; the first being correct to four, and the second to three decimal places.

Again, by putting $\pm 10 \sqrt{-1}$ for y', we get $x = \pm 10 \sqrt{-1} \pm \frac{0.075}{\sqrt{-1}} - \frac{0.1875}{(\sqrt{-1})^2} \pm \frac{0.00028125}{(\sqrt{-1})^3} \mp \frac{0.00175570}{(\sqrt{-1})^5} + \frac{0.00002109}{(\sqrt{-1})^6} \mp \frac{0.00001320}{(\sqrt{-1})^7} \pm \text{etc.} = \pm 10 \sqrt{-1} \mp 0.075 \sqrt{-1} + 0.1875 \pm 0.00028125 \sqrt{-1} \pm 0.0017557 \sqrt{-1} - 0.00002109 \mp$

$0.0000132\sqrt{-1} \pm$ etc. $= 0.18747 \pm 9.9270\sqrt{-1}$, which is correct to four decimal places, both in the real and imaginary parts; consequently, we have $x = 0.1874 + 9.9270\sqrt{-1}$ and $x = 0.1874 - 9.9270\sqrt{-1}$ for the approximate values of the imaginary roots of the given equation.

Remarks.—We have taken this question from p. 106 of Young's Theory of Equations, where one value of x is found to be 9.8860027. The question appears to have been originally proposed by Dr. Halley, who found 9.8862603936495 for the approximate value of one of the roots. (See a small tract on the Solution of Equations, at the end of Newton's Universal Arithmetic, Example 1.)

6. To find x from the equation $x^3 - 5x^2 + 8x - 1 = 0$.

By putting $x = \frac{1}{x'} = x'^{-1}$, the equation is easily changed to $x'^{-1} - \frac{5}{8}x'^{-2} + \frac{1}{8}x'^{-3} = \frac{1}{8} = y'^{-1}$; which gives $y' = 8$. Comparing this equation to (6′), we have $n = -1$, $a = -\frac{5}{8}$, $b = \frac{1}{8}$; consequently, from (b), we have $x' = 8 - \frac{5}{8} - \frac{17}{512} - \frac{65}{16384} -$ etc. $= 7.3379$; correct to three decimal places.

Hence, we have $x = \frac{1}{x'} = \frac{1}{7.3379} = 0.136$ for the approximate value of x, which is correct in all its figures.

METHOD OF APPROXIMATING TO THE RATIO OF TWO NUMBERS OR QUANTITIES.

1. Let A and A′ represent the numbers or quantities; then if we divide A by A′, and divide A′ by the remainder, and so on, as in finding the greatest common divisor, by using a, b, c, etc., for the successive quotients, we shall have the following process:

$$\begin{array}{l}
\underline{\mathrm{A}'}\,|\,\mathrm{A}\,\underline{|\,a} \\
\qquad a\mathrm{A}' \\
\hline
\underline{\mathrm{A} - a\mathrm{A}'}\,|\,\mathrm{A}'\,\underline{|\,b} \qquad\qquad\qquad (1). \\
\qquad\quad b\mathrm{A} - ab\mathrm{A}' \\
\underline{(1 + ab)\mathrm{A}' - b\mathrm{A}}\,|\,\mathrm{A} - a\mathrm{A}'\,\underline{|\,c} \\
\qquad\qquad\qquad (c + abc)\mathrm{A}' - bc\mathrm{A} \\
\qquad\qquad \underline{(1 + bc)\mathrm{A} - (a + c + abc)\mathrm{A}'}\,|\,(1+ab)\mathrm{A}' - b\mathrm{A}\,\underline{|\,d}
\end{array}$$

and so on, to any extent that may be required.

It is clear, from (1), that any dividend and its corresponding divisor (neglecting their signs) may be expressed by the forms $P_0A' - Q_0A$ and $QA - PA'$, (2); consequently, if m stands for the quotient resulting from the division, the remainder will evidently be of the form $P_1A' - Q_1A = (P_0 + mP)A' - (Q_0 + mQ)A$; which gives $P_1 = mP + P_0$ and $Q_1 = mQ + Q_0$, (3).

2. Supposing A and A′ to be positive, and a, b, c, etc., the greatest positive integers in the successive quotients, it is clear that the remainders in (1) will all be positive. Because the remainder $A - aA'$ is positive, it is plain that $\frac{A}{A'}$ is greater than $a = \frac{a}{1}$; and in like manner from the remainder, $(1 + ab)A' - bA$, we shall have $\frac{A}{A'}$ less than $\frac{1 + ab}{b}$; and from the remainder, $(1 + bc)A - (a + c + abc)A'$, we shall have $\frac{A}{A'}$ greater than $\frac{a + c + abc}{1 + bc}$, and so on; consequently, *the ratio* $\frac{A}{A'}$ *lies between any two of the successive ratios* $\frac{a}{1}$, $\frac{1 + ab}{b}$, $\frac{a + c + abc}{1 + bc}$, etc., *or they are limits to it;* $\frac{A}{A'}$ *being greater than the first of the preceding ratios, less than the second, greater than the third, and so on.* It is hence clear that we may call $\frac{a}{1}$, $\frac{1 + ab}{b}$, $\frac{a + c + abc}{1 + bc}$, etc., *the first, second, third, etc., approximate ratios of* $\frac{A}{A'}$; observing that $\frac{A}{A'}$, which expresses the exact geometrical ratio of A to A′, *is sometimes called the exact or complete quotient,* while the approximate ratios *are called inexact or incomplete quotients.*

3. If we represent any successive three approximate ratios by $\frac{P_0}{Q_0}$, $\frac{P}{Q}$, and $\frac{P_1}{Q_1}$, then from (3) these will become $\frac{P_0}{Q_0}$, $\frac{P}{Q}$, and $\frac{mP + P_0}{mQ + Q_0}$, (4).

Since m represents the quotient in (1), which corresponds to $\frac{P_1}{Q_1}$ (the last of the three ratios), it follows, from (4), *that* *if we multiply the numerator of the second by m, and increase the product by the numerator of the first, the result will give the numerator of the third; and in the very same way the denominator of the third can be found from the denominators of the second and first fractions or ratios.* Thus, if we take the first two approximate ratios or fractions, $\frac{a}{1}, \frac{1+ab}{b}$, and represent the third by $\frac{P_1}{Q_1}$; then, since c is the third quotient, we must put c for m, and we shall have $\frac{P_1}{Q_1} = \frac{(1+ab)c+a}{bc+1} = \frac{a+c+abc}{1+bc}$, as it ought to be.

4. If we multiply the first of (3) by Q and the second by P, we shall have $P_0Q + mPQ = QP_1$ and $Q_0P + mPQ = PQ_1$, which, by subtraction, give $P_0Q - Q_0P = -(PQ_1 - QP_1)$; and if $\frac{P_2}{Q_2}, \frac{P_3}{Q_3}$, etc., are the ratios which succeed $\frac{P_1}{Q_1}$, we shall have $PQ_1 - QP_1 = -(P_1Q_2 - Q_1P_2)$, and so on. Hence, if for P_0 and Q_0 we put a and 1, and for P and Q we put $1+ab$ and b, we shall have $P_0Q - Q_0P = ab - 1 - ab = -1$; consequently (from what has been done), if $\frac{P_0}{Q_0}, \frac{P}{Q}, \frac{P_1}{Q_1}, \frac{P_2}{Q_2}$, etc., represent the first, second, third, etc., approximate ratios, we shall have the equations $P_0Q - Q_0P = -1$, $PQ_1 - QP_1 = 1$, $P_1Q_2 - Q_1P_2 = -1$, etc., (5); *observing that the right members of* (5) *are negative when their first members are obtained from the last two of an even number of approximating ratios, and that the right members are positive when their first members are obtained from the last two of an odd number of approximating ratios.*

5. If the first of (5) is divided by Q_0Q, the second by QQ_1, and so on, they will be reduced to $\frac{P_0}{Q_0} - \frac{P}{Q} = -\frac{1}{Q_0Q}$, $\frac{P}{Q} - \frac{P_1}{Q_1} = \frac{1}{QQ_1}$, etc., (6).

Because a, b, c, etc., are positive integers, and that $Q_0 = 1$, $Q = b$, $Q_1 = 1 + bc$, etc., it is clear that Q_0Q is less than QQ_1, and that QQ_1 is less than Q_1Q_2, and so on; consequently, it follows, from (6), that the ratios $\frac{P_0}{Q_0}, \frac{P}{Q}, \frac{P_1}{Q_1}$, etc., will continually approach to a state of equality, as their number is increased.

Since the ratio $\frac{A}{A'}$ is between any two of the ratios $\frac{P_0}{Q_0}$, $\frac{P}{Q}$, $\frac{P_1}{Q_1}$, etc., or limited by them, it results that as the number of ratios is increased they will continually approach to $\frac{A}{A'}$ as their limit; *consequently, we may call* $\frac{P_0}{Q_0}$, $\frac{P}{Q}$, $\frac{P_1}{Q_1}$, etc., *converging ratios or fractions.* If the ratio $\frac{A}{A'}$ is rational, it is clear that the number of converging ratios will be finite; but if $\frac{A}{A'}$ is irrational, the number of converging ratios must be infinite, and such that they will ultimately differ from $\frac{A}{A'}$ by a difference which will be less than any given difference.

6. If A and A′ have a common divisor, since the quotients a, b, c, etc., are independent of the divisor, it follows that the converging fractions will also be independent of it; consequently, if the number of fractions is finite (or $\frac{A}{A'}$ rational), the last of them must be the ratio $\frac{A}{A'}$, when reduced to its lowest terms. Hence, if $\frac{A}{A'}$, when reduced to its lowest terms, is expressed by $\frac{U}{V}$, and that $\frac{S}{T}$ is the fraction which immediately precedes it, it results from (5) that we shall have the equation $SV - TU = \pm 1$, (7); in which $-$ must be used for $\pm$ when there is an even number of converging fractions, and $+$ must be used for $\pm$ when there is an odd number of converging fractions.

If C represents the greatest common divisor of A and A′, then if we multiply (7) by C (since $CV = A'$ and $CU = A$) we get $A'S - AT = \pm C$, (8); consequently, if $C = 1$, the last of the converging fractions will be $\frac{A}{A'}$, and $\frac{S}{T}$ will be the fraction (or ratio) which immediately precedes it.

7. It is impossible to insert a rational fraction, such as $\frac{R}{S}$, between two successive converging fractions, such as $\frac{P}{Q}$ and $\frac{P_1}{Q_1}$, unless its denominator S is greater than Q_1 (the greater denominator of the fractions, between which the fraction is inserted).

For, by the supposition, we must have $\frac{P}{Q} - \frac{P_1}{Q_1} = \pm \frac{1}{QQ_1}$ greater than $\frac{P}{Q} - \frac{R}{S} = \frac{PS - QR}{QS}$, or (rejecting the signs) we must have $\frac{1}{Q_1} > \frac{PS - QR}{S}$, (9). Because P, Q, R, and S are integers, it is clear that PS — QR can not (numerically) be less than 1; consequently, it is evident that the inequality (9) can not be satisfied unless S is greater than Q_1.

Hence, the fractions $\frac{P_0}{Q_0}$, $\frac{P}{Q}$, $\frac{P_1}{Q_1}$, etc., *are the simplest fractions which converge toward* $\frac{A}{A'}$; *or to the same degree of approximation, they express the approximate values of* $\frac{A}{A'}$ *in the simplest manner.*

8. Since, from (1), we have $\frac{A}{A'} = a + \frac{A - aA'}{A'}$, and $\frac{A'}{A - aA'} = b + \frac{(1 + ab)A' - bA}{A - aA'}$, and so on, it is clear that we shall have $\frac{A}{A'} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$, *which is called a continued fraction;* observing that an expression which is partly integral and partly fractional, or wholly fractional, and is such that the denominator of the fractional part is a mixed number, and the denominator of the fractional part of the denominator is also a mixed number, and so on, to any required extent, *is called a continued fraction.*

If we take only the integral part of the continued fraction, we have $\frac{a}{1}$ for the first approximation to $\frac{A}{A'}$; and if we take the part $a + \frac{1}{b}$, we have $\frac{1 + ab}{b}$ for the second approximation to $\frac{A}{A'}$, and so on. Hence, our method of approximating to the ratio $\frac{A}{A'}$ is substantially the same as the ordinary method, by using continued fractions.

9. To perceive the use of what has been done, take the following

EXAMPLES.

1st. To find the approximate ratios of $\frac{29}{24}$.

Here we have

$$\begin{array}{llllll}
24) & 29 & (1 & & & \\
 & \underline{24} & & & & \\
 & 5) & 24 & (4 & & \\
 & & \underline{20} & & & \\
 & & 4) & 5 & (1 & \\
 & & & \underline{4} & & \\
 & & & 1) & 4 & (4 \\
 & & & & \underline{4} & \\
 & & & & 0; &
\end{array}$$

consequently, the quotients a, b, c, d are severally represented by 1, 4, 1, 4. Hence, the first two converging fractions become $\frac{a}{1} = \frac{1}{1}$ and $\frac{1 + ab}{b} = \frac{1 + 1 \times 4}{4} = \frac{5}{4}$; consequently, $\frac{1 + 5 \times 1}{1 + 4 \times 1} = \frac{6}{5}$ and $\frac{5 + 6 \times 4}{4 + 5 \times 4} = \frac{29}{24}$ are the remaining fractions. Because the last fraction is equal to the given fraction, it follows that the given fraction can not be reduced to lower terms; and since there are an even number of fractions, we have $6 \times 24 - 5 \times 29 = -1$.

2d. To find the approximate values of $\frac{372}{246}$.

Here a, b, c, and d are represented by 1, 1, 1, and 20, and thence $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{62}{41} = \frac{372}{246}$ are the sought fractions. Because there are an even number of fractions, we have $3 \times 41 -$

$2 \times 62 = -1$; and because 6 is the greatest common divisor of 246 and 372, we shall have $3 \times 246 - 2 \times 372 = -6$.

3d. To find the approximate values of $\frac{1728}{1892}$.

Here a, b, c, etc., are represented by 0, 1, 10, 1, 1, 6, and 3, and we have $\frac{0}{1}$, $\frac{1}{1}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{21}{23}$, $\frac{137}{150}$, $\frac{432}{473} = \frac{1728}{1892}$.

Because there are an odd number of fractions, we have $473 \times 137 - 432 \times 150 = 1$; and because 4 is the greatest common divisor of 1728 and 1892, we also have $1892 \times 137 - 1728 \times 150 = 4$.

4th. Given, $\frac{3.1415926535, \text{etc.}}{1} =$ the ratio of the circumference of a circle to its diameter, to find the approximate ratios.

Here the numbers corresponding to a, b, c, d, etc., are 3, 7, 15, 1, 292, 1, 1, etc.; consequently, $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{103993}{33102}$, etc., are the sought ratios. The most celebrated of these ratios are $\frac{22}{7}$ and $\frac{355}{113}$, which are the ratios of Archimedes and Metius.

5th. Given, $25x - 31y = 1$, to find positive integral values of x and y which will satisfy the equation.

If we find the approximate ratios of $\frac{31}{25}$, we get 1, 4, and 6 for the numbers which correspond to a, b, c; consequently, we have $\frac{1}{1}$, $\frac{5}{4}$, and $\frac{31}{25}$ for the ratios. Since the number of ratios is odd, we have $25 \times 5 - 31 \times 4 = 1$; or, using m to stand for any positive integer, we have $25 \times (5 + 31m) - 31 \times (4 + 25m) = 1$; consequently, we may put $x = 5 + 31m$, and $y = 4 + 25m$. If we put 0, 1, 2, etc., successively, for m, we get $x = 5$, $y = 4$; $x = 36$, $y = 29$; $x = 67$, $y = 54$, and so on, for the corresponding values of x and y; consequently, the question admits of an unlimited number of answers.

6th. Given, $41x - 56y = 1$, to find positive integral values of x and y which will satisfy the equation.

Here, since $\frac{1}{1}, \frac{3}{2}, \frac{4}{3}, \frac{11}{8}, \frac{15}{11}$, and $\frac{56}{41}$ are the converging fractions, we have $41 \times 15 - 56 \times 11 = -1$, or $41 \times (56m - 15) - 56 \times (41m - 11) = 1$. Putting 1, 2, etc., for m, we get $56m - 15 = 41$ and $41m - 11 = 30$, and so on.

Remark.—If A and B are positive integers prime to each other, it follows that we may, in like manner, find an unlimited number of positive integers for x and y which will satisfy the equation $Ax - By = 1$.

7th. To find the sum or (more properly) the generating function of the series $1 + 2x + 3x^2 + 4x^3 +$ etc.

If we can find the generating function of the fraction $\frac{1}{1 + 2x + 3x^2 + 4x^3 + \text{etc.}}$, it is clear that, by taking its reciprocal, we shall get the generating function of the proposed series.

If we divide the numerator of the fraction by its denominator, we shall get 1 for the quotient, and $-2x - 3x^2 - 4x^3 -$ etc., for the remainder.

Then, dividing $1 + 2x + 3x^2 + 4x^3 +$ etc., by the preceding remainder, we get $-\frac{1}{2x}$ for the quotient, and $\frac{x}{2} + x^2 + \frac{3x^3}{2} + 2x^4 + \frac{5x^5}{2} +$ etc., for the remainder; and dividing the last divisor by the last remainder, we have -4 for the quotient, and $x^2 + 2x^3 + 3x^4 +$ etc., for the remainder; also, dividing the last divisor by the last remainder, we have $\frac{1}{2x}$ for the quotient, and 0 for the remainder; consequently, the divisions terminate.

Hence, the quotients a, b, c, d are here represented by 1, $-\frac{1}{2x}, -4$, and $\frac{1}{2x}$; consequently, the fractions $\frac{a}{1}, \frac{ab+1}{b}$,

$\frac{(ab+1)c+a}{bc+1}$, and $\frac{[(ab+1)c+a]\times d+ab+1}{(bc+1)\times d+b}$ will become $\frac{1}{1}$, $\frac{-\frac{1}{2x}+1}{-\frac{1}{2x}}$, $\frac{\frac{2}{x}-3}{\frac{2}{x}+1}$, and $\frac{\frac{1}{x^2}-\frac{2}{x}+1}{\frac{1}{x^2}}=1-2x+x^2=$ $(1-x)^2$, which is the generating function of $\frac{1}{1+2x+3x^2+\text{etc.}}$. Hence, we shall clearly have $1+2x+3x^2+4x^3+\text{etc.}=$ $(1-x)^{-2}=$ the sought generating function.

8th. To find the generating function of the series $1+2x+3x^2+5x^3+8x^4+13x^5+$ etc., or of the fraction $\frac{1}{1+2x+3x^2+5x^3+8x^4+\text{etc.}}$.

Proceeding as in the last equation, we have a, b, c, d expressed by 1, $-\frac{1}{2x}$, -4, and $-\frac{1}{2x}$; consequently, we easily get $\frac{1-x-x^2}{1+x}$ for the generating function of $\frac{1}{1+2x+3x^2+\text{etc.}}$; by taking the reciprocals, we shall have $1+2x+3x^2+5x^3+\text{etc.}=\frac{1+x}{1-x-x^2}$.

Remark.—We have taken this question from p. 265 of the New York edition of Young's Algebra, where $\frac{1}{(1-x)^2}$ is given for the generating function of $1+2x+3x^2+5x^3+$ etc.

9th. To find the generating function of the series $3+5x+7x^2+13x^3+23x^4+45x^5+87x^6+$ etc.

Proceeding as in the last two questions to find the generating function of the fraction $\frac{1}{3+5x+7x^2+\text{etc.}}$, we get $\frac{1}{3}$, $-\frac{9}{5x}$, $-\frac{25}{12}$, $-\frac{1}{5x}$, 2, and $-\frac{1}{x}$ for the representatives of a, b, c, d, e, and f; consequently, we get $\frac{1-2x-x^2+2x^3}{3-x-6x^2}$ for the generating function of the fraction; and taking the reciprocals, we shall have $3 + 5x + 7x^2 + 13x^3 + 23x^4 + \text{etc.}$ $= \dfrac{3 - x - 6x^2}{1 - 2x - x^2 + 2x^3}$, as required.

Remarks.—1. The answer which we have given to this question agrees with that given by Mr. Young, at the place cited above.

2. The method used in solving the three preceding questions is substantially the same as that given by Mr. Young at p. 174, etc., of his Theory of Algebraical Equations. Mr. Young ascribes the method to M. Le Barbier.

3. It is clear that the method is applicable to any recurring series, when the terms are rational, and the scale of relation consists of a finite number of terms.

10th. To find the approximate fractions of $\dfrac{x^3 - 3x - 1}{1}$.

Here, dividing the numerator (arranged according to the descending powers of x) by the denominator, 1, and 1 by the remainder, and so on, we get x^3, $-\dfrac{1}{3x}$, $9x^2$, and $\dfrac{1}{3x}$ for the representatives of a, b, c, and d.

Hence, $\dfrac{x^3}{1}$, $\dfrac{1-\dfrac{x^2}{3}}{-\dfrac{1}{3x}}$, $\dfrac{9x^2\left(1-\dfrac{x^2}{3}\right)+x^3}{9x^2\left(-\dfrac{1}{3x}\right)+1}$, and $\dfrac{3x\left(1-\dfrac{x^2}{3}\right)+\dfrac{x^2}{3}+1-\dfrac{x^2}{3}}{3x\left(-\dfrac{1}{3x}\right)+\dfrac{1}{3x}-\dfrac{1}{3x}}$ $= \dfrac{x^3 - 3x - 1}{1}$ are the converging fractions.

Again, putting the given expression in the form $\dfrac{-1 - 3x + x^3}{1}$, and proceeding as before, we get -1, $-\dfrac{1}{3x}$, $-\dfrac{9}{x}$, and $\dfrac{1}{3x}$ for the representatives of a, b, c, and d; consequently, the converging fractions become $\dfrac{-1}{1}$, $\dfrac{1+\dfrac{1}{3x}}{-\dfrac{1}{3x}}$,

$\dfrac{\dfrac{-9}{x}-\dfrac{3}{x^2}-1}{\dfrac{3}{x^2}+1}$, and $\dfrac{-1 - 3x + x^3}{1}$.

Remarks.—If the fractions converge, with sufficient rapidity, to $\frac{x^3-3x-1}{1}$, it is clear that those values of x which satisfy the equation $x^3-3x-1=0$ ought to reduce the converging fractions to 0 very nearly. Hence, if we put the numerators of the fractions which contain x, separately equal to 0, the roots of the resulting equations ought to be the approximate values of the roots of $x^3-3x-1=0$.

Thus, if we put the numerators of the second and third fractions (in the first set) equal to 0, we get the equations $x^2=3$ and $3x^2-x=9$. Solving these equations, the first gives $x=1.732$ and $x=-1.732$, and the second gives $x=1.904$ and $x=-1.57$ for approximate values of the roots 1.879 and -1.532 of the equation $x^3-3x-1=0$.

In like manner, if we take the second set of converging fractions, and put the numerators of the second and third equal to 0, we get the equations $3x+1=0$ and $x^2+9x+3=0$. Solving these equations, we get $x=-\frac{1}{3}=-0.333$ and $x=-0.347$ for approximate values of the remaining root -0.3472 of the equation $x^3-3x-1=0$.

Hence, we perceive in what manner we may (often) find the first figures of one or more of the roots of an equation. Thus, to find the approximate root of the equation $x^3+18x-6=0$, we convert $\frac{-6+18x+x^3}{1}$ into the converging fractions

$$\frac{-6}{1},\ \frac{1-\frac{1}{3x}}{\frac{1}{18x}},\ \frac{-6-\frac{324}{x}\left(1-\frac{1}{3x}\right)}{1-\frac{18}{x^2}},\ \text{and}\ \frac{-6+18x+x^3}{1}.$$

Then, by putting the numerators of the second and third fractions equal to 0, we get the equations $3x-1=0$ and $x^2+54x=18$. Solving these equations, we get $x=\frac{1}{3}=0.333$ and $x=0.33130$; the second of these values being correct to four decimal places. (See Vol. I. of Bonnycastle's Algebra, pp. 147 and 148.)

SECTION XVI.

SOLUTION OF RECIPROCAL OR RECURRING EQUATIONS.

(**1.**) LET $A_0z^m + A_1z^{m-1} + A_2z^{m-2} + \dots + A_{m-2}z^2 + A_{m-1}z + A_m = 0$, (1), stand for any equation of the m^{th} degree; then, if $A_0 = A_m$, $A_1 = A_{m-1}$, $A_2 = A_{m-2}$, and so on, the equation is called a *reciprocal or recurring equation.*

(**2.**) Supposing (1) to be a reciprocal or recurring equation, by putting A_0, A_1, A_2, etc., for A_m, A_{m-1}, A_{m-2}, etc., it is clear that the equation may be written in the form $A_0(z^m+1) + A_1z(z^{m-2} + 1) + A_2z^2(z^{m-4} + 1) +$ etc. $= 0$, (2).

If m is an odd number, since $m - 2$, $m - 4$, and so on, are odd numbers, it is clear that $z^m + 1$, $z^{m-2} + 1$, $z^{m-4} + 1$, and so on, will each be exactly divisible by $z + 1$; consequently, $z = -1$ must be a root of (2) or (1). To find the remaining roots of (1), we divide it by $z + 1$, and put the quotient equal to 0, and we shall get the equation $A_0z^{m-1} + (A_1 - A_0)z^{m-2} + (A_2 - A_1 + A_0) \times z^{m-3} + \dots + (A_{m-2} - A_{m-1} + A_m)z^2 + (A_{m-1} - A_m)z + A_m = 0$, (3); and since $A_0 = A_m$, $A_1 = A_{m-1}$, $A_2 = A_{m-2}$, and so on, it is clear that (3) is a reciprocal or recurring equation, of the degree represented by the even number $m - 1$.

(**3.**) If m is an odd number, and $A_0 = -A_m$, $A_1 = -A_{m-1}$, $A_2 = -A_{m-2}$, and so on, then it is clear that (1) may be written in the form $A_0(z^m - 1) + A_1z(z^{m-2} - 1) + A_2z^2(z^{m-4} - 1) +$ etc. $= 0$; consequently, since m, $m - 2$, $m - 4$, and so on, are odd numbers, $z^m - 1$, $z^{m-2} - 1$, $z^{m-4} - 1$, and so on, must each be exactly divisible by $z - 1$, and of course $z = 1$ must be a root of (1). Dividing (1) by $z - 1$, and putting the quotient equal to 0, we shall get the reciprocal or recurring equation $A_0z^{m-1} + (A_1 + A_0)z^{m-2} + (A_2 + A_1 + A_0)z^{m-3} + \dots + (A_2 + A_1 + A_0)z^2 + (A_1 + A_0)z + A_0 = 0$, of the even degree $m - 1$; and solving this equation, we shall get the remaining roots of (1).

(4.) If m is an even number, and $A_0 = -Am$, $A_1 = -Am - 1$, and so on, and if the middle term of (1) is wanting, then by writing (1) in the form $A_0(z^m - 1) + A_1z(z^{m-2} - 1) + A_2z^2(z^{m-4} - 1) + \text{etc.} = 0$, since $z^m - 1$, $z^{m-2} - 1$, and so on, are exactly divisible by $z^2 - 1$, it follows that $z = 1$ and $z = -1$ must be roots of (1). Dividing (1) by $z^2 - 1$, and putting the quotient equal to 0, we get the reciprocal or recurring equation $A_0z^{m-2} + A_1z^{m-3} + (A_0 + A_2)z^{m-4} + (A_1 + A_3)z^{m-5} + (A_0 + A_2 + A_4)z^{m-6} + \ldots\ldots + (A_0 + A_2 + A_4)z^4 + (A_1 + A_3)z^3 + (A_0 + A_2)z^2 + A_1z + A_0 = 0$; consequently, from the solution of this equation, we shall get the remaining roots of (1).

(5.) Hence, supposing (1) to be a reciprocal or recurring equation of an even degrée, and that $m = 2n$, then (1) may be written in the form $A_0(z^{2n} + 1) + A_1z(z^{2n-2} + 1) + A_2z^2(z^{2n-4}+1) + A_3z^3(z^{2n-6}+1) + \ldots\ldots + A_{n-1}z^{n-1}(z^2 + 1) + A_nz^n = 0$; or dividing the terms of this by z^n, we shall have the equation $A_0\left(z^n + \frac{1}{z^n}\right) + A_1\left(z^{n-1} + \frac{1}{z^{n-1}}\right) + A_2\left(z^{n-2} + \frac{1}{z^{n-2}}\right) + \ldots\ldots + A_{n-1}\left(z + \frac{1}{z}\right) + A_n = 0$, (*a*). Putting $z + \frac{1}{z} = x$, (*b*), we easily get $z^2 + \frac{1}{z^2} = x^2 - 2$, $z^3 + \frac{1}{z^3} = x^3 - 3x$, $z^4 + \frac{1}{z^4} = (x^3 - 3x) \times x - (x^2 - 2) = x^4 - 4x^2 + 2$, $z^5 + \frac{1}{z^5} = (x^4 - 4x^2 + 2)x - (x^3 - 3x) = x^4 - 5x^3 + 5x$, and so on. Substituting the preceding values of $z + \frac{1}{z}$, $z^2 + \frac{1}{z^2}$, $z^3 + \frac{1}{z^3}$, and so on, in (*a*), it will be reduced to an equation of the form $A_0x^n + Bx^{n-1} + Cx^{n-2} + \text{etc.} = 0$, (*c*); which being of the n^{th} degree, its degree is half that of (1).

Solving (*c*) we shall get the values of x, which will of course be the n roots of the equation; then putting these roots separately, for x in (*b*), the values of z and $\frac{1}{z}$ corresponding to each root, will be two of the roots of (1); consequently, the $2n$ roots of (1) will become known.

Because the values of z and $\frac{1}{z}$ are the roots of (1) which correspond to each root of (*c*), it is plain that the roots of (1) may be considered as consisting of corresponding pairs, of the forms $a', \frac{1}{a'}$; $a'', \frac{1}{a''}$, and so on.

(6.) To illustrate what has been done, take the following

EXAMPLES.

1. To find the roots of $2z^3 - 3z^2 - 3z + 2 = 0$.

Since -1 is clearly one of the roots, by dividing the equation by $z + 1$ and putting the quotient equal to 0, we get the equation $2z^2 - 5z + 2 = 0$; which is a reciprocal or recurring equation, whose solution gives 2 and $\frac{1}{2}$ for the remaining roots of the given equation.

2. To find the roots of $5z^5 + 4z^4 + 3z^3 - 3z^2 - 4z - 5 = 0$.

Since 1 is one of the roots, if we divide the equation by $z-1$, and put the quotient equal to 0, we shall get $5z^4 + 9z^3 + 12z^2 + 9z + 5 = 0$, or $5\left(z^2 + \frac{1}{z^2}\right) + 9\left(z + \frac{1}{z}\right) + 12 = 0$. Putting $z + \frac{1}{z} = x$, the preceding equation is easily reduced to $5x^2 + 9x = -2$; whose solution gives $x = -\frac{9 - \sqrt{41}}{10}$ and $x = -\frac{9 + \sqrt{41}}{10}$. Hence, we shall have $z + \frac{1}{z} = -\frac{9 - \sqrt{41}}{10}$ and $z + \frac{1}{z} = -\frac{9 + \sqrt{41}}{10}$; whose solution will give the remaining roots of the given equation, which are all imaginary.

3. To find the roots of $z^4 + 2z^3 - 2z - 1 = 0$.

Since 1 and -1 are clearly two of the roots, if we divide the equation by $z^2 - 1$ and put the quotient equal to 0, we shall get the reciprocal or recurring equation $z^2 + 2z + 1 = 0$; whose solution gives -1 and -1 for the remaining roots of the given equation.

4. To find the roots of $z^3 + 1 = 0$.

Ans. -1 and $\frac{1 \pm \sqrt{-3}}{2}$.

5. To find the roots of $z^3 - 1 = 0$.

Ans. 1 and $\frac{-1 \pm \sqrt{-3}}{2}$

6. To find the roots of $z^4 + 1 = 0$. *Ans.* $\pm\dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}$.

7. To find the roots of $z^4 - 1 = 0$.

Ans. 1, -1, and $\pm\sqrt{-1}$.

SOLUTION OF BINOMIAL EQUATIONS.

(1.) Agreeably to what has been said in Evolution, at p. 282, etc., we may clearly represent any binomial equation by $y^n \pm a^n = 0$, (1). Putting $y = ax$, (1) becomes $a^n x^n \pm a^n = 0$, which, by rejecting the factor a^n, is reduced to $x^n \pm 1 = 0$; for which we shall take the equivalent separate equations $x^n = -1$, (2), and $x^n = 1$, (3).

(2.) Because $\left(\dfrac{1+\sqrt{-1}}{\sqrt{2}}\right)^4 = -1$, and $\left(\dfrac{1+\sqrt{-1}}{\sqrt{2}}\right)^8 = 1$, if we represent any positive or negative odd integer by m, and any positive or negative integer (including 0) by p, it is clear that for (2) and (3) we may write $x^n = \left(\dfrac{1+\sqrt{-1}}{\sqrt{2}}\right)^{4m}$, and $x^n = \left(\dfrac{1+\sqrt{-1}}{\sqrt{2}}\right)^{8p}$; whose n^{th} roots give $x = \left(\dfrac{1+\sqrt{-1}}{\sqrt{2}}\right)^{\frac{4m}{n}}$ and $x = \left(\dfrac{1+\sqrt{-1}}{\sqrt{2}}\right)^{\frac{8p}{n}}$. Because $\left(\dfrac{1+\sqrt{-1}}{\sqrt{2}}\right)^{-1} = \dfrac{1}{\dfrac{1+\sqrt{-1}}{\sqrt{2}}}$ $= \dfrac{1-\sqrt{-1}}{\sqrt{2}}$, it is clear that for the preceding equations we may write $x = \left(\dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4m}{n}}$, (4), and $x = \left(\dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8p}{n}}$, (5), and suppose m and p to be positive integers. It is clear from (4) and (5), that the values of x (generally) occur in pairs; since by using + for ± in $1 \pm \sqrt{-1}$ we get one root, and by taking − for ± we get another corresponding root.

(3.) Putting $m = n \pm q$, multiplying by 4 and dividing by n, we get $\dfrac{4m}{n} = 4 \pm \dfrac{4q}{n}$, which being put for the exponent in (4), reduces it to $x = \left(\dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{4+\frac{4q}{n}} = \left(\dfrac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{4} \times$

$\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\pm\frac{4q}{n}}$, or $\left(\text{since } \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{4} = -1\right)$, $x = -\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\mp\frac{4q}{n}}$; which is equivalent to the separate equations $x = -\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4q}{n}}$, and $x = -\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{-\frac{4q}{n}} = -\left(\frac{1 \mp \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4q}{n}}$. Because the last two of these values of x are the same as the first two taken in a reverse order, it follows that the values of x which correspond to m greater than n, will be merely a repetition of the values of x which correspond to m less than n.

Hence, supposing n to be an odd integer, by putting 1, 3, 5, ... n successively, for m in (4), we shall get the n different values of x expressed by $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{n}}$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{12}{n}}$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{20}{n}}$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{28}{n}}$, ... $\frac{n-1}{2}$ pairs of values of x, together with the value of x expressed by $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4n}{n}} = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{4} = -1$; which will evidently be the n roots of (4), since it is clear that (4) can not have equal roots.

And if n is an even number, by putting 1, 3, 5, ... $n-1$ successively, for m in (4), we shall in like manner get the n different values of x, expressed by $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{n}}$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{12}{n}}$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{20}{n}}$, ... $\frac{n}{2}$ pairs of values of x, which, as before, will be the n roots of (2).

(4.) If n in (5) is an odd number, and we put $p = \frac{n-1}{2}$

$\pm r$, we get $\frac{8p}{n} = 4 - \frac{4}{n} \pm \frac{8r}{n}$, which being put for the exponent in (5), reduces it to $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{4 - \frac{4}{n} \pm \frac{8r}{n}} = - \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{-\frac{4}{n} \pm \frac{8r}{n}} = - \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{-\frac{4}{n}} \times \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\pm\frac{8r}{n}}$, which, being equivalent to the separate equations $x = - \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{-4}{n}} \times \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8r}{n}}$ and $x = - \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{-\frac{4}{n}} \times \left(\frac{1 \mp \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8r}{n}}$, it is clear that the values of x which result from taking p greater than $\frac{n-1}{2}$ will be merely a repetition of the values of x resulting from taking p not greater than $\frac{n-1}{2}$.

Hence, if we put 0, 1, 2, 3, ... $\frac{n-1}{2}$ successively, for p in (5), we shall get the n values of x expressed by $x=1$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8}{n}}$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{16}{n}}$, ... $\frac{n-1}{2}$ pairs of values, which will be the n roots of (3).

Again, if n in (5) is an even number, and we put $p = \frac{n}{2} \pm s$ we get $\frac{8p}{n} = 4 \pm \frac{8s}{n}$, which, being put for the exponent in (5), reduces it to $x = -\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\pm\frac{8s}{n}}$; consequently, as before, the values of x resulting from taking p greater than $\frac{n}{2}$ will be merely a repetition of the values of x which result from taking p not greater than $\frac{n}{2}$.

Hence, if we put 0, 1, 2, 3, ... $\frac{n}{2}$ successively, for p in (5), we shall get the n values of x expressed by $x = 1$, $x =$

$\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8}{n}}$, $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{16}{n}}$, ... $x = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4n}{n}} =$ $\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{4} = -1$, which will be the n roots of (3).

(6.) It is evident that (4) and (5) may be written in the forms $x = \left[\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{n}}\right]^{m}$, (6), and $x = \left[\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8}{n}}\right]^{p}$, (7); consequently, having found the values of $\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{n}}$ and $\left(\frac{1 \pm \sqrt{-1}}{\sqrt{1}}\right)^{\frac{8}{n}}$, the n roots of (2) will be found by putting 1, 3, 5, etc., successively, for m in (6), and those of (3) will be found by putting 0, 1, 2, 3, etc., successively, for p in (7), and then developing the powers.

To find $\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{n}}$ and $\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8}{n}}$, it will clearly be sufficient to find $\left(\frac{1 + \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{n}}$ and $\left(\frac{1 + \sqrt{-1}}{\sqrt{2}}\right)^{\frac{8}{n}}$, and then to change $\sqrt{-1}$ (in them) into $\pm \sqrt{-1}$.

If we convert the binomial $(1 + a\sqrt{-1})^{r}$ into a series arranged according to the ascending powers of $a\sqrt{-1}$, we shall (by the binomial theorem) get $(1 + a\sqrt{-1})^{r} = 1 + ra\sqrt{-1} - \frac{r(r-1)}{1\,.\,2}a^{2} - \frac{r(r-1)(r-2)}{1\,.\,2\,.\,3}a^{3}\sqrt{-1} + \frac{r(r-1)}{1\,.\,2}\,.\,\frac{(r-2)(r-3)}{3\,.\,4}a^{4} - \text{etc.} = (1 - \frac{r(r-1)}{1\,.\,2}a^{2} + \frac{r(r-1)(r-2)(r-3)}{1\,.\,2\,.\,3\,.\,4}a^{4} - \text{etc.}) + (ra - \frac{r(r-1)(r-2)}{1\,.\,2\,.\,3}a^{3} + \frac{r(r-1)(r-2)(r-3)(r-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^{5} - \text{etc.}) \times \sqrt{-1}$, (8); consequently, if r is a positive integer, the power will be exact, and if r is fractional, and a sensibly less than 1, the value of $(1 + a\sqrt{-1})^{r}$ can be found to any required degree of exactness by converging series.

Because $\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{n}} = \left(\frac{3\sqrt{2}}{5}\right)^{\frac{4}{n}} \times \left(1+\frac{\sqrt{-1}}{2}\right)^{\frac{4}{n}} \times \left(1+\frac{\sqrt{-1}}{3}\right)^{\frac{4}{n}}$, (9), and $\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{\frac{8}{n}} = \left(\frac{3\sqrt{2}}{5}\right)^{\frac{8}{n}} \times \left(1+\frac{\sqrt{-1}}{2}\right)^{\frac{8}{n}} \times \left(1+\frac{\sqrt{-1}}{3}\right)^{\frac{8}{n}}$, (10), it is clear, if we develop the binomials in the right members of these equations, that we shall get the values of the first members of the equations by converging series when $\frac{4}{n}$ and $\frac{8}{n}$ are fractional.

Thus, if $n = 3$, and we put $\frac{4}{3}$ for r, and $\frac{1}{2}$ for a in (8), we shall get $\left(1+\frac{\sqrt{-1}}{2}\right)^{\frac{4}{3}} = 0.94563651 + 0.67252751\sqrt{-1}$; and by putting $\frac{1}{3}$ for a in (8), we shall also have $\left(1+\frac{\sqrt{-1}}{3}\right)^{\frac{4}{3}} = 0.97555394 + 0.44623019\sqrt{-1}$; consequently, from (9) we shall have $\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{3}} = \left(\frac{3\sqrt{2}}{5}\right)^{\frac{4}{3}} \times \left(1+\frac{\sqrt{-1}}{2}\right)^{\frac{4}{3}} \times \left(1+\frac{\sqrt{-1}}{3}\right)^{\frac{4}{3}} = 0.499999 + 0.866025\sqrt{-1}$.

Changing $\sqrt{-1}$ into $\pm\sqrt{-1}$, this result clearly indicates that two of the roots of the equation $x^3 + 1 = 0$ are $0.5 \pm \frac{\sqrt{3}}{2}\sqrt{-1} = \frac{1 \pm \sqrt{-3}}{2}$, agreeably to what has been shown in Ex. 4 of Recurring Equations.

To find the remaining root of $x^3 + 1 = 0$, we shall put $n = 3$ and $m = 3$ in (6), and shall get $x = \left[\left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^{\frac{4}{3}}\right]^3 = \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right)^4 = -1$, which also agrees with the answer given in the example previously cited.

It is clear from (10), that if we square the preceding roots, we shall get $\frac{-1 \pm \sqrt{-3}}{2}$ and 1 for the roots of the equation $x^3 - 1 = 0$.

Hence the method of proceeding in order to find the roots of any binomial equation becomes too evident to require any further comment.

(6.) *Finally, we have shown, from the principles of pure algebra, that any binomial equation must have as many different roots as there are units in the index of the unknown letter.*

We have also given a simple method for finding the roots of any such equation to any required degree of exactness, so that we may consider the solution of binomial equations as being completed.

SECTION XVII.

SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

(1.) In conformity to what has been shown, we may clearly take $y^3 + A_1y^2 + A_2y = A_3$, (1) for the general representative of any cubic equation.

If for y we put $x - \frac{A_1}{3}$, it is easy to perceive that (1) will be reduced to the form $x^3 + ax = b$, (2), which is of a much more simple form than (1).

Putting $x = p + q$, and substituting $p + q$ for x in (2), it becomes $(p + q)^3 + a(p + q) = b$, or $p^3 + q^3 + (3pq + a)(p + q) = b$; consequently, by assuming $3pq + a = 0$ or $pq = -\frac{a}{3}$, (3), we get $p^3 + q^3 = b$, (4).

From (4) and (3) we easily get $(p^3 + q^3)^2 - 4p^3q^3 = b^2 + \frac{4a^3}{27}$, or $(p^3 - q^3)^2 = b^2 + \frac{4a^3}{27}$, whose square root gives $p^3 - q^3 = \sqrt{b^2 + \frac{4a^3}{27}}$, (5).

The half sum and half difference of (4) and (5) give $p^3 = \frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$ and $q^3 = \frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$; consequently, by taking the sum of the cube roots of these expressions (since $x = p + q$), we get $x = \sqrt[3]{\left(\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)} + \sqrt[3]{\left(\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)}$, (A), or $\left(\text{since } q = -\frac{a}{3p}\right)$, $x = \sqrt[3]{\left(\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)} - \frac{\frac{a}{3}}{\sqrt[3]{\left(\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)}}$, (A′); which are called Cardan's Formulæ.

If a is positive, while b is positive or negative, it is clear that x can be found from (A) or (A′); also, if a is negative $\left(\text{since } \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \text{ will become } \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}\right)$, it is evident, if $\frac{a^3}{27}$ is not greater than $\frac{b^2}{4}$, that x can also be found from (A), or (A′); and if $\frac{b^2}{4} - \frac{a^3}{27} = 0$, we shall from (A) get $x = 2\sqrt[3]{\frac{b}{2}} = \sqrt[3]{4b}$.

If a is negative and $\frac{a^3}{27}$ greater than $\frac{b^2}{4}$, or if $\frac{b^2}{a^3}$ is less than $\frac{4}{27}$, it is clear that $\frac{b^2}{4} - \frac{a^3}{27}$ will be negative, and of course $\sqrt{\frac{b^2}{4} - \frac{a^3}{27}}$ will be imaginary; consequently, x can not be found from (A) or (A′), unless the cube roots can be extracted in such a way that the imaginary parts shall destroy each other; and any example in which the cube roots can not be thus extracted is said to fall under the *irreducible case of cubic equations.*

To illustrate (A) and (A′), take the following

EXAMPLES.

1. To find x from the equation $x^3 + 3x = 2$.

Since a and b are here expressed by 3 and 2, we easily get from (A) and (A′), $x = \sqrt[3]{(1 + \sqrt{2})} + \sqrt[3]{(1 - \sqrt{2})}$ and $x = \sqrt[3]{(1 + \sqrt{2})} - \frac{1}{\sqrt[3]{(1 + \sqrt{2})}}$.

2. To find x from the equation $x^3 - 6x = -6$.

Since a and b are expressed by -6 and -6, we get $x = -\sqrt[3]{2} - \sqrt[3]{4}$, or $x = -\sqrt[3]{2} - \frac{2}{\sqrt[3]{2}}$.

3. To find y from the equation $y^3 - 3y^2 = 5$.

By putting $x + 1$ for y, the equation is easily reduced to $x^3 - 3x = 7$. Solving this equation by (A′), we get $x = \sqrt[3]{\left(\frac{7 + 3\sqrt{5}}{2}\right)} + \frac{1}{\sqrt[3]{\left(\frac{7 + 3\sqrt{5}}{2}\right)}}$; consequently, by adding 1 to the value of x, we get the value of y.

Thus we perceive how to apply (A) or (A′) to the solution of any cubic equation which does not fall under the irreducible case.

(2.) SOLUTION OF CUBIC EQUATIONS WHICH FALL UNDER THE IRREDUCIBLE CASE.

Putting $-a$ for a in (2), it becomes $x^3 - ax = b$, (1′); in which b may be positive or negative, while a must be positive, and at the same time $\frac{b^2}{a^3}$ must not be greater than $\frac{4}{27}$.

Adding $-max + max = 0$ to the first member of (1′), it becomes $x^3 - max + max - ax = b$; consequently, if we assume $x^3 - max = 0$, or $x^2 = ma$, we shall get $max - ax = b$, or $x = \frac{b}{a(m-1)}$, which gives $x^2 = \frac{b^2}{a^2(m-1)^2}$; and equating these values of x^2, we shall get $m(m-1)^2 = \frac{b^2}{a^3}$.

Putting $m = v + \frac{2}{3}$, we have $m - 1 = v - \frac{1}{3}$, and thence

$x=\frac{3b}{a(3v-1)}$, (2′), and $m(m-1)^2=\frac{b^2}{a^3}$ becomes $\left(v+\frac{2}{3}\right)\left(v-\frac{1}{3}\right)^2=v^3-\frac{v}{3}+\frac{2}{27}=\frac{b^2}{a^3}$, or $v^3-\frac{v}{3}=\frac{b^2}{a^3}-\frac{2}{27}$, (3′).

If $\frac{b^2}{a^3}=\frac{4}{27}$, (3′) reduces to $v^3-\frac{v}{3}=\frac{4}{27}-\frac{2}{27}=\frac{2}{27}$, or to $v^3-\frac{v}{3}-\frac{2}{27}=\left(v-\frac{2}{3}\right)\times\left(v+\frac{1}{3}\right)^2=0$, the roots of this equation being $v=\frac{2}{3}$, $v=-\frac{1}{3}$, $v=-\frac{1}{3}$. Substituting $\frac{2}{3}$, $-\frac{1}{3}$, $-\frac{1}{3}$, for v in (2′), we get $x=\frac{3b}{a}$, $x=-\frac{3b}{2a}$, and $x=-\frac{3b}{2a}$ for the roots of (1′), when $\frac{b^2}{a^3}=\frac{4}{27}$.

Putting $\sqrt{\frac{27b^2}{a^3}}=\mathrm{N}$, and assuming $\mathrm{V}^3-\frac{\mathrm{V}}{3}=\frac{\mathrm{N}}{27}$, (1″), and $\mathrm{V}=\frac{\mathrm{N}}{3(3v-1)}$ (2″); then it is clear that we may consider (3′) as being derived from (1″), by the same process that (3′) has been derived from (1′); and it is evident, if we can find V from (1″), that (2″) will give the value of v, and then (2′) will give the value of x.

In like manner (1″) may be supposed to be derived from $\overset{\prime}{\mathrm{V}}{}^3-\frac{\overset{\prime}{\mathrm{V}}}{3}=\frac{\sqrt{\mathrm{N}+2}}{27}$, (3″), by putting $\mathrm{V}=\frac{\sqrt{\mathrm{N}+2}}{3(3\overset{\prime}{\mathrm{V}}-1)}$, (4″); and (3″) may be supposed to be derived from $\overset{\prime\prime}{\mathrm{V}}{}^3-\frac{\overset{\prime\prime}{\mathrm{V}}}{3}=\frac{\sqrt{(\sqrt{\mathrm{N}+2}+2)}}{27}$, (5″), by putting $\overset{\prime\prime}{\mathrm{V}}=\frac{\sqrt{(\sqrt{\mathrm{N}+2}+2)}}{3(3\overset{\prime}{\mathrm{V}}-1)}$, (6″), and so on indefinitely.

Since $\frac{b^2}{a^3}$ is supposed not to be greater than $\frac{4}{27}$, it is plain that N is not greater than 2; and if N is less than 2, it is clear that $\sqrt{\mathrm{N}+2}$ is less than 2, and nearer to it than N; and $\sqrt{(\sqrt{\mathrm{N}+2}+2)}$ is also less than 2, and nearer to it than $\sqrt{\mathrm{N}+2}$; and so on to infinity.

Hence, we may manifestly consider the irreducible case of cubic equations as being reduced, by the preceding process, to the solution of an equation of the form $w^3 - \frac{w}{3} = \frac{2}{27}$, (a), since the right members of $(1'')$, $(3'')$, $(5'')$, etc., as their number is increased, continually approximate to $\frac{2}{27}$ as their limit; and it is clear, when their number is infinitely great, that their right members will differ from $\frac{2}{27}$ by a difference which will be less than any given difference. Now, although $(1'')$, $(3'')$, $(5'')$, etc., can not in practice be continued to infinity, yet they can easily be continued so far as to enable us to find x to any required degree of exactness; since, if they are continued until an equation is found whose right member agrees with $\frac{2}{27}$ to n decimal places, it is clear that we may put $\frac{2}{3}$ the greatest root of (a) for the greatest root of the equation thus obtained, and that we can thence find x correctly to at least $n - 1$ decimal places; and it is clear that the value of x thus found will (generally) be the numerically greatest root of $(1')$.

If r stands for the root of $(1')$, thus found, then, reducing $(1')$ to the form $x^3 - ax - b = 0$, we shall, by division, get $\frac{x^3 - ax - b}{x - r} = x^2 + rx + r^2 - a + \frac{r^3 - ar - b}{x - r}$; consequently, since $r^3 - ar - b = 0$, to n decimal places, if we put $x^2 + rx + r^2 - a$ equal to 0, we shall get the quadratic equation $x^2 + rx + r^2 - a = 0$, whose solution gives $x = -\frac{r}{2} \pm \frac{\sqrt{4a - 3r^2}}{2}$, (b), which gives the two remaining roots of $(1')$ when the values of a and r are substituted for them in the equation, these roots being thus correctly found to at least $n - 1$ decimal places.

To illustrate what has been done, take the following

EXAMPLES.

1. To find the roots of $x^3 - 6x = 2$ to five decimal places.

Here $a = 6$, $b = 2$, $x = \frac{3b}{a(3v - 1)} = \frac{1}{3v - 1}$, $\mathrm{N} = \sqrt{\frac{27b^2}{a^3}}$

$=\sqrt{\frac{1}{2}}=0.70710678$, and (1″) and (2″) give $V^3-\frac{V}{3}=\frac{0.70710678}{27}$, and $V=\frac{0.70710678}{3(3v-1)}$; then $\overset{\prime}{V}{}^3-\frac{\overset{\prime}{V}}{3}=\frac{\sqrt{2.70710678}}{27}=\frac{1.64532876}{27}$ and $\overset{\prime}{V}=\frac{1.64532876}{3(3V-1)}$, and $\overset{\prime\prime}{V}{}^3-\frac{\overset{\prime\prime}{V}}{3}=\frac{\sqrt{3.64532876}}{27}=\frac{1.90927440}{27}$, $\overset{\prime\prime}{V}=\frac{1.90927440}{3(3\overset{\prime}{V}-1)}$; $\overset{\prime\prime\prime}{V}{}^3-\frac{\overset{\prime\prime\prime}{V}}{3}=\frac{\sqrt{3.90927440}}{27}=\frac{1.97718843}{27}$, $\overset{\prime\prime\prime}{V}=\frac{1.97718843}{3(3\overset{\prime\prime}{V}-1)}$; $\overset{\text{IV}}{V}{}^3-\frac{\overset{\text{IV}}{V}}{3}=\frac{1.99428895}{27}$, $\overset{\text{IV}}{V}=\frac{1.99428895}{3(3\overset{\prime\prime\prime}{V}-1)}$; $\overset{\text{V}}{V}{}^3-\frac{\overset{\text{V}}{V}}{3}=\frac{1.99857172}{27}$, $\overset{\text{V}}{V}=\frac{1.99857172}{3(3\overset{\text{IV}}{V}-1)}$; $\overset{\text{VI}}{V}{}^3-\frac{\overset{\text{VI}}{V}}{3}=\frac{1.99964289}{27}$, $\overset{\text{VI}}{V}=\frac{1.99964289}{3(3\overset{\text{V}}{V}-1)}$; $\overset{\text{VII}}{V}{}^3-\frac{\overset{\text{VII}}{V}}{3}=\frac{1.99991072}{27}$, $\overset{\text{VII}}{V}=\frac{1.99991072}{3(3\overset{\text{VI}}{V}-1)}$; $\overset{\text{VIII}}{V}{}^3-\frac{\overset{\text{VIII}}{V}}{3}=\frac{1.99997768}{27}$, $\overset{\text{VIII}}{V}=\frac{1.99997768}{3(3\overset{\text{VII}}{V}-1)}$; $\overset{\text{IX}}{V}{}^3-\frac{\overset{\text{IX}}{V}}{3}=\frac{1.99999442}{27}$, $\overset{\text{IX}}{V}=\frac{1.99999442}{3(3\overset{\text{VIII}}{V}-1)}$, and so on.

Because $\frac{1.99999442}{27}$ differs from $\frac{2}{27}$ only by $\frac{0.00000558}{27}$, it is clear that we may put $\overset{\text{IX}}{V}=\frac{2}{3}$; consequently, from $\overset{\text{IX}}{V}=\frac{1.99999442}{3(3\overset{\text{VIII}}{V}-1)}$, by putting $\frac{2}{3}$ for $\overset{\text{IX}}{V}$, we get $3\overset{\text{VIII}}{V}=1.99999721$.

Similarly $3\overset{\text{VIII}}{V}=\frac{1.9999768}{3\overset{\text{VII}}{V}-1}$, gives $3\overset{\text{VII}}{V}=1.99999023$, and so on, until we finally get $x=\frac{1}{3v-1}=\frac{3V}{0.70710678}=2.601679$; which is one of the roots of the given equation, correctly found to five (and very nearly to six) decimal places.

Substituting 6 for a and 2.601679 for r in (b), we easily get $x=-2.261802$ and $x=-0.339876$ for the remaining roots of the given equation; these roots being correctly found to five decimal places. (See Vol. I., p. 155 of Bonnycastle's Algebra, Question 4.)

2. To find the roots of $x^3 - 3x = 1$ to three places of figures.

Since $a = 3$, $b = 1$, we have $x = \frac{3b}{a(3v-1)} = \frac{1}{3v-1}$ and $N = \sqrt{\frac{27b^2}{a^3}} = 1$; and thence $V^3 - \frac{V}{3} = \frac{1}{27}$, $V = \frac{1}{3(3v-1)}$; $\overset{\prime}{V}{}^3 - \frac{\overset{\prime}{V}}{3} = \frac{\sqrt{3}}{27} = \frac{1.7320}{27}$, $\overset{\prime}{V} = \frac{1.7320}{3(3V-1)}$; $\overset{\prime\prime}{V}{}^3 - \frac{\overset{\prime\prime}{V}}{3} = \frac{1.9318}{27}$, $\overset{\prime\prime}{V} = \frac{1.9318}{3(3\overset{\prime}{V}-1)}$; $\overset{\prime\prime\prime}{V}{}^3 - \frac{\overset{\prime\prime\prime}{V}}{3} = \frac{1.9828}{27}$, $\overset{\prime\prime\prime}{V} = \frac{1.9828}{3(3\overset{\prime\prime}{V}-1)}$; $\overset{\text{IV}}{V}{}^3 - \frac{\overset{\text{IV}}{V}}{3} = \frac{1.9956}{27}$, $\overset{\text{IV}}{V} = \frac{1.9956}{3(3\overset{\prime\prime\prime}{V}-1)}$, and so on. Hence, putting $\frac{2}{3}$ for $\overset{\text{IV}}{V}$, we get $3\overset{\prime\prime\prime}{V} = 1.9978$, $3\overset{\prime\prime}{V} = 1.9924$, $3\overset{\prime}{V} = 1.9695$, $3V = 1.879$; consequently, since $x = \frac{1}{3v-1} = 3V$, we shall have $x = 1.879$ for one of the roots of the given equation; correctly found to three decimal places. Putting 3 for a and 1.879 for r in (b), we easily get $x = -1.532$ and $x = -0.347$ for the first figures of the two remaining roots of the given equation.

3. To find the roots of $x^3 - 7x = -7$ to four places of figures.

Here we have $a = 7$, $b = -7$, and $x = -\frac{3}{3v-1}$, $N = \sqrt{\frac{27}{7}} = 1.96396049$; and hence $V^3 - \frac{V}{3} = \frac{1.96396049}{27}$, $V = \frac{1.96396049}{3(3v-1)}$, $\overset{\prime}{V}{}^3 - \frac{\overset{\prime}{V}}{3} = \frac{1.990969}{27}$, $\overset{\prime}{V} = \frac{1.990969}{3(3V-1)}$.

Hence, putting $\overset{\prime}{V} = \frac{2}{3}$, we get $3V = 1.99548$; consequently, since $x = -\frac{3}{3v-1} = -\frac{3V}{0.65465}$, we shall get $x = -3.048$ for one of the roots of the given equation, which is correct in all its figures. It is hence easy, from (b), to find 1.692 and 1.356 for the first figures of the two remaining roots of the given equation.

(**3.**) *Hence it follows that we have completed the solution of cubic equations.* For those which do not fall under the irreducible case can be solved by the formulæ of *Cardan*, and those which fall under the irreducible case can be solved to any degree of exactness by the preceding method.

SOLUTION OF BIQUADRATIC EQUATIONS.

(**1.**) In accordance with the definitions that have been given, we may clearly take $y^4 + Ay^3 + By^2 + Cy = D$, (1), to represent a biquadratic equation.

If we put $y = x - \frac{A}{4}$, it is easy to perceive that (1) will be reduced to the more simple form $x^4 + ax^2 + bx + c = 0$, (2).

(**2.**) Putting $x^2 = mx + n$, (3), and substituting this value of x^2 in (2), it will become $(mx + n)^2 + a(mx + n) + bx + c = m^2x^2 + 2mnx + n^2 + amx + an + bx + c = m^2x^2 + (2mn + am + b)x + n^2 + an + c = 0$; and if in this we put $mx + n$ for x^2, it will become $m^2(mx + n) + (2mn + am + b)x + n^2 + an + c = (m^3 + 2mn + am + b)x + n^2 + m^2n + an + c = 0$; in which m and n are undetermined.

If we determine m and n on the supposition that the coefficient of x is 0 (so as to leave x arbitrary), we shall get the equations $m^3 + 2mn + am + b = 0$ and $n^2 + m^2n + an + c = 0$.

From the first of these equations we easily get $n = -\frac{m^3 + am + b}{2m}$, which being put for n in the second equation, reduces it to $(m^2 + a)\left(-\frac{m^3 + am + b}{2m}\right) + \left(\frac{m^3 + am + b}{2m}\right)^2 + c = \left[\frac{m^3 + am + b}{2m} - (m^2 + a)\right] \times \left[\frac{m^3 + am + b}{2m}\right] + c = \left(\frac{b + ma + m^3}{2m}\right) \times \left(\frac{b - (ma + m^3)}{2m}\right) + c = 0$, or we shall have $b^2 - (ma + m^3)^2 + 4cm^2 = 0$, which is easily reduced to the equation $m^6 + 2am^4 + (a^2 - 4c)m^2 = b^2$.

If we put $m^2 = z$, (4), the preceding equation becomes $z^3 + 2az^2 + (a^2 - 4c)z = b^2$, (5), which agrees with the resul obtained in a very different manner by Euler. (See Bonnycastle's Algebra, Vol. I., p. 161.)

(**3.**) If we substitute the value of n in (3), it will become $x^2 = mx - \frac{m^3+am+b}{2m}$; which is easily reduced to $\left(x-\frac{m}{2}\right)^2 = -\frac{m^3+2am+2b}{4m}$, which gives $x = \frac{m}{2} \pm \frac{1}{2}\sqrt{-\frac{m^3+2am+2b}{m}}$, (6).

Hence, the solution of the biquadratic equation (2) is reduced to that of the cubic equation (5); for having found z from (5), we get m from (4), and then x can be found from (6); consequently, since a cubic equation can either be exactly solved, or solved to any required degree of exactness, *it follows that we have completed the solution of biquadratic equations.*

(**4.**) To illustrate what has been done, take the following

EXAMPLES.

1. To find the roots of $x^4 - 27x^2 + 14x + 120 = 0$.

Since a, b, and c are represented by -27, 14, and 120, (5) becomes $z^3 - 54z^2 + 249z = 196$, one of whose roots is easily seen to be 1.

Substituting 1 for z in (4), we get $m^2 = 1$, whose roots give $m = 1$ and $m = -1$.

Putting 1 and -1 separately for m in (6), we easily get 3, -2, 4, and -5 for the roots of the proposed equation.

2. To find the roots of $x^4 - 17x^2 - 20x - 6 = 0$.

Here a, b, and c being expressed by -17, -20, and -6, (5) becomes $z^3 - 34z^2 + 313z = 400$; and it is easy to see that 16 is one of the roots of this equation.

Putting 16 for z in (4), we get $m = 4$ and $m = -4$ for the corresponding values of m.

Substituting 4 and -4 separately for m in (6), we get $2+\sqrt{7}$, $2-\sqrt{7}$, $-2+\sqrt{2}$, and $-2-\sqrt{2}$ for the roots of the proposed equation. (See Question 4, p. 162, Vol I., of Bonnycastle's Algebra.

3. To find the roots of $y^4 - 8y^3 + 20y^2 - 19y + 12 = 0$.

Putting $y = x + 2$ in the given equation, it becomes $x^4 - 4x^2 - 3x + 6 = 0$, which wants its second term.

Because a, b, and c are represented by -4, -3, and 6, (5) becomes $z^3 - 8z^2 - 8z = 9$, one of whose roots equals 9. Since 3 and -3 are the values of m, which correspond to $z = 9$, we (from (6)), get $2, 1, \frac{-3 + \sqrt{-3}}{2}, \frac{-3 - \sqrt{-3}}{2}$ for the roots of the equation in x; consequently, from $y = x + 2$ we shall have $4, 3, \frac{1 + \sqrt{-3}}{2}, \frac{1 - \sqrt{-3}}{2}$ for the roots of the proposed equation.

4. To find the roots of $y^4 - 8y^3 + 24y^2 - 36y + 27 = 0$.

Putting $y = x + 2$, the equation becomes $x^4 - 4x + 3 = 0$; consequently, (5) gives $z^3 - 12z = 16$, which has 4 for one of its roots.

Because 2 and -2 are the values of m which correspond to $z = 4$, we readily get $1, 1, -1 + \sqrt{-2}, -1 - \sqrt{-2}$ for the roots of the equation in x; consequently, $3, 3, 1 + \sqrt{-2}, 1 - \sqrt{-2}$ are the roots of the given equation.

SECTION XVIII.

LIMITS OF THE REAL ROOTS OF AN EQUATION OF THE nth DEGREE.

(1.) USING $x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots . + N = 0$, (1) to stand for an equation of the n^{th} degree, which is supposed to have real roots; then, if some of the coefficients are positive (regarding N as being the coefficient of x^0), while others are negative, if we use $-b$ to stand for the greatest negative coefficient, $b + 1$ shall be greater than the greatest positive root of (1), or $b + 1$ shall be what is called *a superior limit* to the positive roots of (1).

For it is clear that $x^n - bx^{n-1} - bx^{n-2} - \ldots . - b$ can not be greater than the first member of (1), and since $(b + 1)^n - b(b + 1)^{n-1} - b(b + 1)^{n-2} -$ etc. $= (b + 1)^{n-1} - b(b + 1)^{n-2} -$ etc. $= (b + 1)^{n-2} - b(b + 1)^{n-3}$ etc. = etc. = etc. $= 1$, it follows if we put $b + 1$, or any greater number for x in (1), that the result can not be less than 1; consequently, since the greatest positive root, when put for x, reduces the first member of (1) to 0, it results that $b+1$ must be greater than the greatest positive root.

Thus, 5 is a superior limit to the positive roots of the equation $x^4-4x^3+3x^2-x-3=0$.

(**2.**) On the same supposition, if M is the greatest positive coefficient which precedes the first negative term, and if $-N'$ is the greatest negative coefficient, then $\frac{N'}{M}+1$ shall be a superior limit to the positive roots of (1).

For if we divide the coefficients by M, and disregard the terms which precede the term whose coefficient is M, then, since $-\frac{N'}{M}$ is the greatest negative coefficient in the result, it follows from what has been shown that $\frac{N'}{M}+1$ is a superior limit to the positive roots.

Thus, 3 is a superior limit to the positive roots of $x^5+2x^4+6x^3-7x^2+9x-12=0$.

(**3.**) Using $-b$ to represent the greatest negative coefficient in (1), and x^{n-m} to express the power of x in the first negative term, then shall $\sqrt[m]{b}+1$ be a superior limit to the positive roots of (1).

For since $x^n-b(x^{n-m}+x^{n-m-1}+x^{n-m-2}+\ldots\ldots+x+1)$ $=x^n-b\left(\frac{x^{n-(m-1)}-1}{x-1}\right)$ is manifestly not greater than the first member of (1), if we determine x so as to make this expression positive, it is manifest that the value of x so found, and all greater values, will be superior limits to the positive roots of (1).

Because the preceding condition is equivalent to the inequality $x^n>\frac{b}{x^{m-1}}\times\frac{x^n-x^{m-1}}{x-1}$, if we put $b=r^m$ or $r+1=\sqrt[m]{b}+1$, and substitute $r+1$ for x, we shall reduce the inequality to $(r+1)^n>\left(\frac{r}{r+1}\right)^{m-1}\times[(r+1)^n-(r+1)^{m-1}]$; which being evidently true, it results that $\sqrt[m]{b}+1$ is a superior limit to the positive roots of (1).

Thus, 3 is a superior limit to the positive roots of the equation $x^4 + 2x^3 - 3x^2 - 4x + 6 = 0$; since $-b = -4$, $n = 4$, and $n - m = 2$.

(4.) If M is the greatest positive coefficient which precedes the first negative term in (1), and if m' is the difference of the indices of the corresponding powers of x; then, if $-N'$ is the greatest negative coefficient in the equation, it is evident, from what has been shown in (3), that $\sqrt[m']{\frac{N'}{M}} + 1$ will be a superior limit to the positive roots of (1).

Thus, 3 is a superior limit to the positive roots of the equation $x^6 + 8x^5 + 4x^4 - 5x^3 - 7x^2 - 9x - 32 = 0$, because $M = 8$, $-N' = -32$, and that -5 is the first negative coefficient, and that $5 - 3 = 2 = m'$.

(5.) If for x we put $-x$ in (1), (and change the signs of all the terms in the result when n is an odd number), we shall clearly obtain an equation such, that its positive roots, when their signs are changed, will be the negative roots of (1).

Hence, having found a superior limit to the positive roots of (1), and then put $-x$ for x in (1), and obtained a superior limit to the positive roots in the resulting equation, by changing the sign we shall get what is called an *inferior limit* to the negative roots of (1); consequently, in this way we shall (generally) find a positive and negative number, between which all the real roots of (1) must lie, the numbers thus found being called *limits* to the real roots of the equation.

EXAMPLES.

1. To find the limits of the roots of $x^3 + 2x^2 - 3x - 16 = 0$.

Here $\sqrt{16} + 1 = 5$ is a superior limit to the positive roots; and changing x into $-x$, or (which is the same) changing the signs of the second and fourth terms, we get $x^3 - 2x^2 - 3x + 16 = 0$; and since $3 + 1 = 4$ is a superior limit to the positive roots of this, it follows that -4 is an inferior limit to the negative roots of the given equation, and its real roots lie between -4 and 5.

2. To find the limits of the roots of $x^3 + 2x^2 + x + 5 = 0$.

Because the equation has no negative terms, it clearly has no positive roots; and putting $-x$ for x, we get -6 for the limit of the negative roots, and the real roots lie between -6 and 0.

3. To find the limits of the roots of $x^3 - 2x^2 + 4x - 7 = 0$.

The number $7 + 1 = 8$ is a limit to the positive roots; and putting $-x$ for x, the equation becomes $x^3 + 2x^2 + 4x + 7 = 0$, which has no positive roots, and of course the given equation has no negative roots, and the roots lie between 0 and 8.

(6.) Putting $x = \frac{1}{y}$, and substituting $\frac{1}{y}$ for x in (1), it is easily reduced to an equation of the form $y^n + A'_{n-1}y^{n-1} + A'_{n-2}y^{n-2} + \dots + N' = 0$, (2), whose roots are clearly the reciprocals of the roots of (1).

Representing the limits of the roots of (2) by a and $-b$, it is evident that $\frac{1}{a}$ can not be greater than the least positive root of (1), and $\frac{1}{a}$ will be what is called an *inferior limit* to the positive roots of (1); and in like manner $-\frac{1}{b}$, not being numerically greater than the numerically least negative root of (1), will be what is called a *superior limit* to the negative roots of (1).

Hence, limits can be assigned to the real positive and negative roots of any equation which has such roots.

EXAMPLES.

1. To find limits to the roots of $x^3 - 4x - 1 = 0$.

Since -4 is the greatest negative coefficient, (3) gives $\sqrt{4} + 1 = 3 =$ a superior limit to the positive roots, and putting $-x$ for x, we get (in like manner) -3 for an inferior limit of the negative roots; consequently, the roots lie between -3 and 3.

Putting $\frac{1}{y}$ for x in the equation, it is easily reduced to $y^3 + 4y^2 - 1 = 0$, and 2 is easily seen to be a superior limit to the positive roots; and in like manner (by putting $-y$ for y) we get -5 for the inferior limit of the negative roots;

consequently, $\frac{1}{2}$ and $-\frac{1}{5}$ are inferior and superior limits to the positive and negative roots of the given equation.

2. To find limits to the roots of $3x^4 + 6x^3 - 12x^2 + 4x - 5 = 0$.

Dividing the terms by 3, the equation is reduced to the proper form, $x^4 + 2x^3 - 4x^2 + \frac{4}{3}x - \frac{5}{3} = 0$; then, from (2), (3), or (4) we get 3 for a superior limit to the positive roots; and putting $-x$ for x, or changing the signs of the second and fourth terms, we get -5 for an inferior limit to the negative roots; consequently, the real roots lie between -5 and 3.

Putting $\frac{1}{y}$ for x, we easily reduce the equation to $y^4 - \frac{4}{5}y^3 + \frac{12}{5}y^2 - \frac{6}{5}y - \frac{3}{5} = 0$, whose real roots have $\frac{6}{5} + 1 = 2.2$ and $-\left(1 + \sqrt[4]{\frac{3}{5}}\right)$ for limits; and of course $1 \div 2.2$ and $-1 \div \left(1 + \sqrt[4]{\frac{3}{5}}\right)$ are inferior and superior limits to the positive and negative roots of the given equation.

Hence, we easily find that the equation has two real roots, one of which lies between 1 and 2, and the other between -3 and -4.

(7.) Representing an equation of the n^{th} degree by $Ax^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + An = f(x) = 0$, (3), supposing A to be positive, while the other coefficients may be positive or negative, according to the nature of the case; then, if we denote the first, second, third, etc., derived functions of $f(x)$ by $f'(x)$, $f''(x)$, $f'''(x)$, etc., we shall get the equations $nAx^{n-1} + (n-1)A_1x^{n-2} + (n-2)A_2x^{n-3} + \ldots + 2A_{n-2}x + A_{n-1} = f'(x) = 0$, $n(n-1)Ax^{n-2} + (n-1)(n-2)A_1x^{n-3} + (n-2)(n-3)A_2x^{n-4} + \ldots + 3 \times 2A_{n-3}x + 2A_{n-2} = f''(x) = 0$, and so on to $n(n-1)(n-2) \times \ldots \times 2 \times 1A = N$, a term which does not contain x.

Now, if x is so assumed that all the terms of the series $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$, ... N, (4), shall be positive, it is clear, from what has been shown, that the assumed value of x will be a superior limit to the positive roots of (3).

If we put $-x$ for x, or change the signs of the second, fourth, sixth, etc., terms in the first member of (3), then if we find a superior limit to the positive roots of the resulting equation (as before), and change the sign of the limit, it will become an inferior limit to the negative roots of (3); consequently, we shall thus get limits to the real roots of (3).

EXAMPLES.

1. To find limits to the real roots of $x^3 - 3x^2 + 2x - 7 = 0$.

Here $f(x) = 0$, $f'(x) = 0$, etc., become $x^3 - 3x^2 + 2x - 7 = 0$, $3x^2 - 6x + 2 = 0$, $x - 1 = 0$, and the constant N is reduced to 1.

Hence, putting 3.1 for x, the series (4) becomes 0.161, 12.23, 2.1, 1; consequently, 3.1 is a superior limit to the positive roots, and it is easy to perceive that the real roots lie between 0 and 3.1.

2. To find limits to the real roots of $4x^3 - 8x - 17 = 0$.

Here $f(x) = 0$ and $f'(x) = 0$ become $4x^3 - 8x - 17 = 0$, and $12x^2 - 8 = 0$, or $3x^2 - 2 = 0$; consequently, since the roots of $3x^2 - 2 = 0$ have -0.9 and 0.9 for their limits, it is clear (from what has been shown), if the roots of $4x^3 - 8x - 17 = 0$ are real, that one of them must lie between -0.9 and 0.9. Because the substitution of -0.9 and 0.9 for x in $4x^3 - 8x - 17 = 0$ gives -12.716 and -21.284, which have the same sign, the equation has no real root which lies between -0.9 and 0.9, and of course it has a pair of imaginary roots.

And since the substitution of 2 and 2.1 in $4x^3 - 8x - 17 = 0$ gives -1 and 3.244, which have contrary signs, it follows that the equation has one root which lies between 2 and 2.1.

3. To find limits to the real roots of $x^4 - 19x^3 + 132x^2 - 302x + 200 = 0$.

Here $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$ become $x^4 - 19x^3 + 132x^2 - 302x + 200 = 0$, $4x^3 - 57x^2 + 264x - 302 = 0$, and $2x^2 - 19x + 44 = 0$.

Because 4 and $\frac{11}{2}$ are the roots of the last of the preceding equations, if the roots of the first are all real, it follows that one of the roots of the second must lie between the preceding numbers. Hence, substituting 4 and $\frac{11}{2}$ separately for x in the second equation, we get 98 and 91.25 for the results, which being positive, or having the same sign, it follows that the second or cubic equation has no real root which lies between 4 and $\frac{11}{2}$; consequently, it has a pair of imaginary roots, and thence (according to what has heretofore been shown) it results that the first or proposed equation has a pair of imaginary roots.

Since the substitution of 1 and 2 separately for x, in the cubic equation gives -91 and 30 for the results, which have unlike signs, it follows, from what has been shown, that the equation has a real root which lies between 1 and 2.

Since the preceding root must lie between the two remaining roots of the first or proposed equation (if they are real), we put 1.2 and 0 separately for x in the proposed equation, and get -3.0784 and 200 for the results; which, having contrary signs, it follows that the given equation has one real root which lies between 0 and 1.2, and it is clear that the root can not be much less than 1.2. Putting 2 and $\infty =$ infinity separately for x, in the given equation, we get -12 and $+\infty$ for the results; consequently, the equation has another real root, which is a little greater than 2.

Remark.—This question has been taken from p. 162, Vol. I., of Bonnycastle's Algebra, where 1.02804, 4, 6.57653, 7.39543 are given for the roots; and the same answer is given by P. Nicholson, at p. 84 of his Analytical and Arithmetical Essays.

4. To find the number of real roots of $x^4 - 8x^3 + 3x^2 - 7x - 18 = 0$, which may lie between $x = 0$ and $x = 8$.

Here, $f(x)$, $f'(x)$, etc., become $x^4 - 8x^3 + 3x^2 - 7x - 18 = 0$, $4x^3 - 24x^2 + 6x - 7 = 0$, $6x^2 - 24x + 3 = 0$, $x - 2 = 0$, and $N = 1$. Putting 0 for x in the preceding equations, and writing the signs of the results in their proper order, together with the sign of N, we get the row of signs expressed by $-$, $-$, $+$, $-$, $+$, (1); also, putting 8 for x, and writing the signs of the results and N, as before, we get the row of signs $+$, $+$, $+$, $+$, $+$, (2).

In accordance with what has heretofore been shown, it is evident from (2) that 8 is a superior limit to the positive roots of the proposed equation.

Also, since the signs in (1) and (2) show that the given equation changes its sign between $x = 0$ and $x = 8$, it follows, from principles heretofore established, that it must have one or three real roots between 0 and 8.

Now the given equation can not have three real roots between 0 and 8; for if it can, then the first derived equation must have two real roots between 0 and 8, since it must have a real root between each successive two of the real roots of the given equation; but from the signs in (1) and (2), it appears that the first derived equation changes its sign between $x = 0$ and $x = 8$; consequently, the proposed equation has but one real positive root; and since the absolute term is negative, the equation has one negative root; and it is easy to show that 7 and -0.9 are the principal parts of these roots.

(**s.**) Let $Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + \dots + Tx + U = 0$, (1), stand for a complete equation of the n^{th} degree, and it will clearly consist of $n + 1$ terms; then if we multiply it by $x - a$, it will be changed to $Ax^{n+1} + (B - aA)x^n + (C - aB)x^{n-1} + (D - aC)x^{n-2} + (E - aD)x^{n-3} + \dots + (U - aT)x - aU = 0$, (2), which is an equation of the $(n + 1)^{th}$ degree, consisting of $n + 2$ terms, and having $x = a$ for one of its roots.

Calling any two successive unlike signs (in (1) and (2)) a *variation*, and any two successive signs which are alike a *permanence*, we propose to show that (2) has at least one more variation than (1), when a is positive, and that (2) has at least one more permanence than (1), when a is negative.

1. When a is positive. Since (2) has one more sign than (1), it will clearly be sufficient to show that (2) has no more permanences than (1), because (2) must then necessarily have at least one more variation than (1).

If (1) has no permanences, or if the signs of $B - aA$, $C - aB$, etc., are the same as the signs of B, C, etc., it is evident that the signs of the $n + 1$ first terms of (2) will (severally) be the same as the signs of the terms in (1); consequently, since the sign of $U - aT$, the coefficient of x, is unlike that of $-aU$, the absolute term in (2), it follows that (2) has one more variation than (1).

If a permanence exists between the first and second terms of (1), and a variation between the second and third terms; if the sign of $B - aA$ is the same as the sign of $-A$, it is clear that a variation will exist between the first and second terms of (2), and a permanence will exist between the second and third terms (of (2)), which will represent the permanence between the first two terms of (1); but the permanence in (1) can not introduce any other permanence into (2); indeed, there may be a permanence between the third and fourth terms of (2), which results from a permanence between the third and fourth terms of (1), when D is greater than aC; but it is clear that this permanence does not, in the least, depend on the permanence between the first two terms of (1).

If there is a permanence between the third and fourth terms of (1), and $D - aC$ has the same sign as $-C$, then (as before) there will be a permanence between the fourth and fifth terms of (2), when the sign of E is unlike the signs of C and D; and there may be a permanence between the second and third terms of (2), resulting (as before shown) from a permanence between the first and second terms of (1).

Because the preceding reasonings and conclusions are applicable to any number of successive permanences, taken anywhere in (1), and to the corresponding permanences introduced by them into (2), it follows that when a is positive, (2) must have at least one more variation of signs than (1); agreeably to what was to be proved.

2. When a is negative, or when $-a$ is put for a.

If we change the signs of the second, fourth, sixth, etc., terms of (1), the negative roots of (1) will be changed into positive roots, and the positive roots into negative roots; consequently, multiplying the result by $x - a$, it follows that the *resulting equation* must have at least one more variation of signs than the equation into which (1) has been changed. Now if we change the signs of the second, fourth, sixth, etc., terms of the *resulting equation*, the signs of its roots will be changed, and its variations will be changed into permanences, and its permanences into variations; consequently, since the equation thus obtained agrees with (2), when the sign of a in it is changed, it follows, if a is negative, that (2) must at least have one more permanence of signs than (1); and at the same time (2) will have the negative root, which is expressed by $x = -a$.

From what has been shown, we deduce the following conclusions, which constitute what is called the Rule or

THEOREM OF DESCARTES.

A complete equation can not have a greater number of positive roots than there are variations among the signs of its terms; nor can the equation have a greater number of negative roots than there are permanences among the signs of its terms.

EXAMPLES.

1. To find the number of positive and negative roots that may exist in the equation $x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$.

Because there is one permanence and three variations, the equation may have one negative and three positive roots.

2. To find the number of positive and negative roots that may be contained in $6x^4 + 35x^3 + 25x^2 - 60x + 24 = 0$.

Since the equation has two variations and two permanences, it may have two positive and two negative roots.

3. To find the number of positive and negative roots that may be contained in $x^3 - 22x - 24 = 0$.

Completing the equation by the addition of the infinitesimal $\pm 0.x^2$, it becomes $x^3 \pm 0.x^2 - 22x - 24 = 0$.

Since the equation has one variation and two permanences (whether we use + or − for ±), it results that one of the roots may be positive and the remaining two negative.

4. To find the number of positive and negative roots that may be contained in $4x^3 - 8x - 17 = 0$.

According to the solution of the last question, the equation may have one positive and two negative roots; nevertheless, it has one positive and two imaginary roots.

5. To find the number of positive and negative roots that may be contained in $3x^3 - 5x^2 - 8 = 0$.

Completing the equation by the addition of the infinitesimal $0.x$, it becomes $3x^3 - 5x^2 + 0\,x - 8 = 0$; also completing the equation by the addition of $-0.x$, it becomes $3x^3 - 5x^2 - 0.x - 8 = 0$.

Now, according to the first of these complete equations, the roots of the proposed equation may all be positive; while, according to the second complete equation, two of them must be negative; consequently, on account of the incompatibility of the preceding conclusions, two of the roots must be imaginary.

6. To find how many positive and negative roots may be in $x^5 - 7x^4 - 50 = 0$.

Because the equation may be completed in the forms $x^5 - 7x^4 + 0\,x^3 - 0.x^2 + 0.x - 50 = 0$ and $x^5 - 7x^4 - 0.x^4 - 0.x^2 - 0.x - 50 = 0$, it follows that it must have four imaginary roots; and it is easy to perceive that it has one real positive root.

It is hence clear that any equation which wants $2m - 1$ terms between terms having like signs, must have at least $2m$ imaginary roots.

7. To find how many imaginary roots there are in the equation $7x^6 - 9x^5 - 80 = 0$.

Because the equation wants four terms between terms which have like signs, it is evident, as in the last question, that it has four imaginary roots; and it is easy to perceive that the equation has two real roots, one of which is positive and the other negative.

Hence, any equation which wants $2m$ terms between terms which have like signs, must have at least $2m$ imaginary roots.

8. To find how many imaginary roots there are in $8x^5 - 20x^4 + 92 = 0$.

Writing $8x^5 - 20x^4 - 0.x + 92$ for the equation, then, since two terms are wanting between $-20x^4$ and $-0.x$, terms which have like signs, it follows that the equation has (at least) two imaginary roots.

Hence any equation which wants $2m + 1$ terms between terms having unlike signs, must have at least $2m$ imaginary roots.

9. To find how many imaginary roots there are in $10x^6 - 15x^5 + 72 = 0$.

Writing $10x^6 - 15x^5 - 0.x + 72 = 0$ for the equation, then according to what has been shown, since the equation wants three terms between $-15x^5$ and $-0.x$, terms which have like signs, it results that the equation must at least have four imaginary roots.

Hence, any equation which wants $2m$ terms between terms which have unlike signs, must at least have $2m$ imaginary roots.

10. To show that $y^4 - 19y^3 + 132y^2 - 302y + 200 = 0$ has a pair of imaginary roots.

Putting $y = x + \frac{19}{4}$, we get an equation of the form $x^4 + Ax^2 + \text{etc.} = 0$, in which A is positive; consequently, this equation has a pair of imaginary roots, since it wants a term between terms having like signs; and of course the proposed equation must have a pair of imaginary roots.

11. To show that $y^4 - 5y^3 + 13y^2 - 17y + 10 = 0$ has a pair of imaginary roots.

Putting $y = x + \frac{5}{4}$, and proceeding as in the last question, it is easy to show that the proposed equation has at least a pair of imaginary roots.

Indeed, since the equation is equivalent to $\left(y^2 - \frac{5}{2}y\right)^2 + \frac{27y^2}{4} - 17y + 12 = 0$, and that $\frac{27y^2}{4} - 17y + 12$ is positive for all real values of y, it results that all the roots of the equation are imaginary.

(9.) STERN'S THEOREM.

Resuming (4), given in (7), we shall have the series $f(x)$, $f'(x), f''(x), f'''(x), \ldots$ N, (1), in which $f(x) = 0$ denotes an algebraic equation, $f'(x)$, $f''(x)$, etc., being the first, second, etc., derived functions of $f(x)$, and N is the last derived function, which is evidently independent of x.

If m represents the number of variations of signs among *the terms of the series* (1) *which correspond to the* (*real*) *value* p *of* x, *and* m' *the number of variations corresponding to the value* q *of* x, *then the number of real roots of the equation* $f(x) = 0$, *which lie between* p *and* q, *can not exceed the number of units in the difference of the numbers* m *and* m'.

In the demonstration of the theorem we shall suppose that p is greater than q, in such a sense that $p = q + r$, r being positive; and that x increases by very small increments from the value q to p, and as it increases passes through the real roots of $f(x) = 0$, which lie between q and p.

According to what has heretofore been shown, it follows that the signs of $f(x)$ and $f'(x)$ will form a variation just before x passes through any real root of $f(x) = 0$, which becomes a permanence just after the passage; consequently, a variation of signs between the first two terms of the series (1) will be lost by the passage of x through any real root of $f(x) = 0$.

Because $f''(x)$ is the first derived function of $f'(x)$, it follows, as above, that as x increases and passes through a root of $f'(x) = 0$, a variation between the signs of $f'(x)$ and $f''(x)$ will be lost, which can not be recovered, unless a permanence exists between the signs of $f(x)$ and $f'(x)$ before the passage, which is changed into a variation, in consequence of the change of the sign of $f'(x)$; consequently, the variations among the signs of the terms of (1) can not be increased by the passage of x through a real root of $f'(x) = 0$.

Because a similar reasoning and conclusion is applicable to each of the terms of (1) which lie between $f(x)$ and the constant N, it follows that any changes of signs among these terms, resulting from the supposed increase of x, can not increase the number of variations in (1).

Because a variation of signs between $f(x)$ *and* $f'(x)$ *in* (1) *is lost by the passage of* x, *during its increase, through each real root of* $f(x) = 0$, *it results that the theorem is true; and that whether* $f(x) = 0$ *has or has not any equal roots.*

To perceive the use of the theorem, take the following

EXAMPLES.

1. To find the number of real roots of $x^3 - 3x^2 + 3x - 1 = 0$ which may lie between $x = 0$ and $x = 2$.

Here the successive terms of (1) become $x^3 - 3x^2 + 3x - 1$, $3x^2 - 6x + 3$, $6x - 6$, and 6; and $q = 0$ and $p = 2$.

Putting 0 for x in the preceding series, we get the signs $-$, $+$, $-$, $+$, three variations; and putting 2 for x, the signs become $+$, $+$, $+$, $+$, no variation; consequently, m and m' are here represented by 0 and 3.

Consequently, since $m' - m = 3$, it follows that the proposed equation may have three real roots, which may lie between 0 and 2.

Indeed, because the equation is equivalent to $(x-1).(x-1).(x-1) = 0$, it has three equal real roots, expressed by $x = 1$, $x = 1$, $x = 1$, which lie between 0 and 2.

2. To find how many real roots of $x^3 - 2x - 1 = 0$ may lie between -2 and 2.

Here (1) become $x^3 - 2x - 1$, $3x^2 - 2$, $6x$, and 6; and putting -2 for x, their signs are $-$, $+$, $-$, $+$, three variations; also, putting 2 for x, the signs are $+$, $+$, $+$, $+$, no variation; consequently, the roots of the equation must all lie between -2 and 2. Indeed, since the (proposed) equation is equivalent to $(x+1).(x^2 - x - 1) = (x+1).\left(x - \frac{1+\sqrt{5}}{2}\right).\left(x - \frac{1-\sqrt{5}}{2}\right) = 0$, it is easy to perceive that -2 and 2 are inferior and superior limits to the roots.

3. To find how many roots of $x^3 - 4x^2 - 9x + 36 = 0$ lie between -4 and 5.

Here (1) become $x^3 - 4x^2 - 9x + 36$, $3x^2 - 8x - 9$, $6x - 8$, and 6; and putting -4 for x, the signs are $-$, $+$, $-$, $+$, three variations; also, putting 5 for x, the signs are $+$, $+$, $+$, $+$, no variation; consequently, -4 and 5 are inferior and superior limits to the roots. Indeed, the roots are -3, 3, and 4.

Remarks.—If we suppose the equation $x^3 - 4x^2 - 9x + 36 = 0$ has equal roots with contrary signs, then, putting x and $-x$ for a pair of these roots, it is clear that the equations $x^3 - 4x^2 - 9x + 36 = 0$ and $-x^3 - 4x^2 + 9x + 36 = 0$ must be coexistent; consequently, we must have $x^3 - 9x = 0$ or $x^2 = 9$ and $4x^2 = 36$, which concur in giving -3 and 3 for a pair of equal roots having opposite signs; and dividing the equation by $x^2 - 9$, the quotient is $x - 4$, and of course 4 is the remaining root.

Hence, *if we unite all the terms of an equation which contain the odd powers of the unknown letter, according to their signs, into one sum, then the greatest common divisor of the sum thus obtained and the remaining terms of the equation, when put equal to* 0, *will give the equal roots, with opposite signs, in the equation.*

Thus, in the equation $x^7 - 9x^5 + 6x^4 + 15x^3 - 12x^2 - 7x + 6 = 0$, we find $x^4 - 2x^2 + 1 = (x^2 - 1)^2$ to be the greatest common divisor of $x^7 - 9x^5 + 15x^3 - 7x$, and $6x^4 - 12x^2 + 6$; consequently, from $x^4 - 2x^2 + 1 = 0$ we get $x = 1$, $x = -1$, and $x = 1$, $x = -1$, for the equal roots having opposite signs.

4. To find how many of the roots of $x^4 - x^3 - 13x^2 + 16x - 48 = 0$ lie between -5 and 5.

Here (1) become $x^4 - x^3 - 13x^2 + 16x - 48$, $4x^3 - 3x^2 - 26x + 16$, $12x^2 - 6x - 26$, $24x - 6$, and 6; and putting -5 for x in these, the signs are $+, -, +, -, +$, four variations; also, putting 5 for x, the signs are $+, +, +, +, +$, no variation; consequently, -5 and 5 are inferior and superior limits to the roots.

Because $x^3 - 16x$ and $x^4 - 13x^2 - 48$ have $x^2 - 16$ for a factor, it follows that -4 and 4 are roots of the equation; and it is hence easy to show that the remaining roots are imaginary.

5. Find the number of roots of $x^4 - 4x^3 - 29x^2 + 156x - 180 = 0$, which may lie between 1 and 6. *Ans.* Three.

6. Find the number of roots of $5x^3 - 6x + 2 = 0$, which may lie between 0 and -2. *Ans.* One.

(10.) STURM'S THEOREM.

1. Representing any algebraic equation by $Ax^n + A_1x^{n-1} + A_2x^{n-2} + \dots + A_{n-1}x + A_n = 0$, (1), and its first member by V; then multiplying each term of (1) by the corresponding index of x, and subtracting 1 from the index, we get $nAx^{n-1} + (n-1)A_1x^{n-2} + (n-2)A_2x^{n-3} + \dots + 2A_{n-2}x + A_{n-1}$, (2), for the first derived function, which we denote by V_1.

Proceed with V and V_1 as in finding their greatest common divisor, being careful to change each sign in the successive remainders before they are used as divisors, and to continue each division until the remainder is of less dimensions in x than the corresponding divisor; then if Q_1, Q_2, Q_3, etc., stand for the successive quotients, and $-V_2$, $-V_3$, $-V_4$, etc., for the corresponding remainders, the equations $V = V_1Q_1 - V_2$, $V_1 = V_2Q_2 - V_3$, $V_2 = V_3Q_3 - V_4$, etc., (3), will be obtained.

Calling the remainders in (3), after their signs are changed, the *modified remainders*, write V and V_1, together with the modified remainders, in their proper order, and the series $V, V_1, V_2, V_3 \ldots . N$, (4), will be obtained; noticing that N is used for the last modified remainder.

If the equations $V = 0$ and $V_1 = 0$ can be satisfied by the same value of x, or if $V = 0$ has equal roots, it is clear that N must be a function of x, which is the greatest common divisor of V and V_1; consequently, those values of x which satisfy the equation $N = 0$ will evidently reduce each term of (4) to 0. Also, those values of x which reduce any two or more adjacent terms in (4) to 0, will reduce all the terms to 0, and will result from the equal roots of $V = 0$. Thus, let the equations $V_2 = 0$ and $V_3 = 0$ be satisfied by the same value of x; then it is clear, from (3), that the same value of x reduces each term of (4) to 0.

Hence, there is no value of x that is not one of the equal roots of $V = 0$, which can reduce two successive terms of (4) to 0.

If $V = 0$ has no equal roots, it follows, from what has been done, that N is independent of x, and that there is no value of x which can reduce two successive terms of (4) to 0.

If a is such a value of x as reduces one of the terms of (4), between V and N, to 0; then, if a is not one of the equal roots of $V = 0$, the terms which are adjacent to the term reduced to 0 in (4) shall have contrary signs.

Thus, suppose that a, when put for x, gives the equation $V_4 = 0$; then the equation $V_3 = V_4Q_4 - V_5$ of (3) is reduced to $V_3 = - V_5$; consequently, V_3 and V_5 in (4) have unlike signs.

2. *Let p and q be two numbers, such that p is greater than*

q, or that $p = q + r$, r being positive; then substitute q and p successively for x in the terms of (4), *and represent the number of their variations of signs which correspond to q and p by m and m'; and the number of units in $m - m'$ will exactly equal the number of unequal real roots of* (1) *or* $V = 0$, *which lie between the numbers q and p.*

If $-\infty = -$ infin. and $\infty = +$ infin. are put for q and p, it is plain that the terms in (4) will be reduced to their first terms nearly, or to those which contain the highest powers of x in them; consequently, the number of units in $m - m'$ will exactly equal the number of unequal real roots of $V=0$.

Remark.—Any factor that is suppressed or introduced (as in finding the greatest common divisor), to give integral quotients in obtaining (3), must be positive, so that the signs of the terms of the remainder may not be changed.

PROOF OF THE THEOREM.

To simplify the reasoning, we shall conceive x to increase by very small increments from q to p, and to pass through the roots of $V = 0$, which may lie between q and p.

1. If x, during its increase, passes through one of the unequal roots of $V = 0$, then a variation, which will exist between the signs of V and V_1, just before x passes through the root, will be changed into a permanence just after the passage; consequently, a variation between the signs of V and V_1 in (4) will be lost.

2. If x passes through n' equal roots during its increase; then, if a denotes one of these roots, it is clear that $(x - a)^{n'}$ must be a factor of V; consequently, if V' stands for the remaining factor, we shall have $V = (x - a)^{n'} \times V'$. If we put $x + y$ for x in $(x - a)^{n'} \times V'$, and represent the coefficient of y resulting from the substitution of $x + y$ for x in V' by R; then we shall get $n'(x - a)^{n'-1}V' + (x - a)^{n'}R$ for the coefficient of y in the result, which will be the first derived function of V. Hence, V and V_1 will be represented by $(x-a)^{n'}V'$ and $n'(x-a)^{n'-1}V'+(x-a)^{n'}R$; consequently, by division, we shall get $\dfrac{V}{V_1} = \dfrac{(x - a)^{n'}V'}{n'(x-a)^{n'-1}V' + (x-a)^{n'}R} =$

$\frac{(x-a)V'}{n'V'+(x-a)R}=\frac{x-a}{n'}$ very nearly, when $x-a$ is very small.

Because $x-a$ is negative, when x is a very little less than a, and positive when x is a very little greater than a, it follows, that when x increases from a value, a very little less to one, a very little greater than a, the sign of $\frac{V}{V_1}$ will be changed from $-$ to $+$; consequently, a variation between the signs of V and V_1 in (4) will be lost by the passage of x through the n' equal roots of $V=0$.

Because $V=(x-a)(x-a)^{n'-1}V'$ and $V_1=[n'V'+(x-a)R]\times(x-a)^{n'-1}$, it follows that they have $(x-a)^{n'-1}$ for a factor, which is also a factor of all the remaining terms of (4); and it is evident that the loss of a variation between the signs of V and V_1 is due to the change of the sign of $x-a$ (and not to that of $(x-a)^{n'-1}$) in V, resulting from the passage of x through the n' equal roots of $V=0$; indeed, because the change of the sign of $(x-a)^{n'-1}$ changes the signs of all the terms of (4), it clearly can not affect the number of variations of their signs.

Since all those values of x which reduce N, or any two or more successive terms of (4) to 0, have been shown to result from equal roots of $V=0$; it plainly follows, from what has been done, that such values of x can not affect the number of variations in (4).

3. If x passes through a value which is not one of the equal roots of $V=0$, and reduces one of the terms between V and N, as V_4 in (4) to 0, then, according to what has been shown, the terms V_3 and V_5, adjacent to V_4, will have contrary signs; consequently, since the passage of x through the root of $V_4=0$ can not affect the signs of V_3 and V_5, it follows that there will be one variation of signs between V_3 and V_5, when $V_4=0$, and that one variation of signs between the signs of V_3, V_4, and V_5 will exist just before and after V_4 is reduced to 0, and of course the passage of x through the root of $V_4=0$ can not affect the number of variations of signs among the terms of (4).

Hence, if x passes through a value which is not one of the equal roots of $V=0$, and reduces two or more of the terms in (4) to 0; then, since the terms reduced to 0 can not be adjacent to each other, and that the signs of the terms adjacent to the terms thus reduced are unlike and not changed by the reduction of the terms to 0, it follows, from what has been shown, that the passage of x through any such value can not affect the number of variations among the signs of the terms in (4).

4. Neglecting the multiplicity of a root, or considering such roots as single roots; then, since a variation is lost among the signs of (4), when x during its supposed increase from q to p reduces V to 0, or passes through a real root of $V = 0$, and that those values of x which reduce one or more of the remaining terms of (4) to 0 can not affect the number of variations of signs in (4), it follows that the number of variations of signs lost in (4) is exactly equal to the number of unequal real roots of $V = 0$ which lie between q and p; consequently, the theorem is true.

To perceive the use of the theorem, take the following

EXAMPLES.

1. To find the number of real roots of $x^3 - 3x - 1 = 0$.

Here we have $3x^2 - 3$ for the first derived function, and omitting the useless factor 3 we may take $x^2 - 1$ for it; and dividing $x^3 - 3x - 1$ by $x^2 - 1$, we have

$$\begin{array}{r|l|l}
\underline{x^2 - 1} & x^3 - 3x - 1 & \underline{x} \\
 & \underline{x^3 - x \phantom{{}-1}} & \\
 & -2x - 1 = &
\end{array}$$

the remainder, whose signs being changed, give $2x + 1$ for the first modified remainder; also multiplying $x^2 - 1$ by 4, and dividing the product by $2x + 1$, we have

$$\begin{array}{r|l|l}
\underline{2x + 1} & 4x^2 - 4 & \underline{2x - 1} \\
 & \underline{4x^2 + 2x} & \\
 & -2x - 4 & \\
 & \underline{-2x - 1} & \\
 & -3 = &
\end{array}$$

the remainder, whose sign being changed, gives 3 for the second modified remainder; hence, collecting the results, we get $x^3 - 3x - 1$, $x^2 - 1$, $2x + 1$, 3, for the representatives of the terms in (4).

Putting -2 for x in the preceding functions, their signs are $-$, $+$, $-$, $+$, three variations, also putting 2 for x in the same functions, the signs are $+$, $+$, $+$, $+$, no variation; consequently, since $m = 3$ and $m' = 0$, we have $m - m' = 3 =$ the number of real roots of the equation which lie between -2 and $+2$. Hence, because the equation can have only three roots, it follows that the roots are all real and have -2 and 2 for inferior and superior limits.

Again, putting -1 for x in the functions, they become 1, 0, -1, and 3, whose signs, by omitting 0, become $+$, $-$, $+$, two variations; consequently, since $x = -2$ gives three variations, it follows that one of the roots lies between -2 and -1, and of course -1 must be the first figure of one of the roots.

Also, because $x = 0$ gives one variation, it follows that one of the roots lies between -1 and 0; and since $x = -0.4$ gives two variations and $x = -0.3$ gives one variation, it follows that -0.3 are the first figures of this root.

Similarly, because $x = 1$ gives one variation, and that $x = 2$ gives no variation, it follows that 1 is the first figure of the remaining root; thus, -1, -0.3, and 1 are the principal parts of the three roots.

Remarks.—If we put $x = \frac{1}{x'} = x'^{-1}$, the proposed equation is easily reduced to $x'^{-1} - \frac{x'^{-3}}{3} = -\frac{1}{3} = y^{-1}$; which, compared to (6) of Sec. XV., gives $n = -1$, $y = -3$, $a = 0$, $b = -\frac{1}{3}$, $c = 0$, $d = 0$, etc.

Hence, from (since y' is here $= y$) (c) of the same Sec., we get $x' = -3 + \frac{1}{9} + \frac{2}{243} + \frac{7}{6561} + \frac{10}{59049} + \text{etc.} = -\frac{170027}{59049}$ very nearly; consequently, we get $x = \frac{1}{x'} = -\frac{59049}{170027} = -0.347291$ nearly, whose first five decimal places are correct.

Again, if we multiply the terms of the given equation by x^{-1}, the product is easily reduced to the form $x^2 - x^{-1} = 3$; consequently, from (c) we (as before) get $x = \pm\sqrt{3} + \frac{1}{6} \mp \frac{3}{8 \times 9\sqrt{3}} + \frac{1}{162} + \text{etc.} = \pm 1.73205 + 0.166\dot{6}. \mp 0.02405$

$+ 0.0061 +$ etc.; and taking the upper and lower signs in the ambiguous signs separately, we have $1.880 +$ and -1.535 for the first figures of the two remaining roots, the first of these being correct to two and very nearly to three places of figures, and the second being correct to three places of figures. It is manifest that the preceding method in the present question is much more simple than to find the first figures by Sturm's theorem.

2. To find the number of real roots contained in $3x^4 - 12x^2 + 8x - 24 = 0$.

Here the first derived function may be expressed by $3x^3 - 6x + 2$, and thence we deduce $x^2 - x + 4$ for the first modified remainder; and since the modified remainder is clearly positive for all real values of x, it is evidently unnecessary to get any more of the derived functions. Hence (4) may be represented by $3x^4 - 12x^2 + 8x - 24$, $3x^3 - 6x + 2$, $x^2 - x + 4$; and if in these we put $-\infty$ and ∞ for x, they may be expressed by $3(\infty)^4$, $-3(\infty)^3$, $(\infty)^2$, two variations, and $3(\infty)^4$, $3(\infty)^3$, $(\infty)^2$, no variation; consequently, the given equation has but two real roots.

If -3 is put for x, the signs are $+$, $-$, $+$, two variations; and if -2 is put for x, the signs are $-$, $-$, $+$, one variation; consequently, one of the roots lies between -2 and -3, and -2 is the principal part of the root.

In like manner, we easily get 2 for the principal part of the remaining root.

3. To find the number of real roots of $x^4 - x^3 + 6x^2 + 24$.

Here the first derived function is $(4x^2 - 3x + 12)x$, or, since $4x^2 - 3x + 12$ is positive for all values of x, it may be rejected, and we may take x for the first derived function.

Dividing $x^4 - x^3 + 6x^2 + 24$ by x, we get 24 for the remainder, whose sign, being changed, gives -24 for the modified remainder; consequently, (4) become $x^4 - x^3 + 6x^2 + 24$, x, -24.

Putting $-\infty$ and ∞ for x in the preceding functions, they may be expressed by $(\infty)^4$, $-\infty$, -24, one variation, and $(\infty)^4$, ∞, -24, one variation; consequently, the proposed equation has no real roots.

4. To find the number of real roots of $x^4 + 8x^3 + 17x^2 - 6x - 36 = 0$.

Here $2x^3 + 12x^2 + 17x - 3$ may be taken for the first derived function, and thence we get $7x^2 + 43x + 66$ for the first modified remainder, and $x + 3$ is the second and last modified remainder; consequently, the given equation has equal roots.

Hence, (4) become $x^4 + 8x^3 + 17x^2 - 6x - 36$, $2x^3 + 12x^2 + 17x - 3$, $7x^2 + 43x + 66$, $x + 3$; and putting $-\infty$ and ∞ for x, these become $(\infty)^4$, $-2(\infty)^3$, $7(\infty)^2$, $-\infty$, three variations, and $(\infty)^4$, $2(\infty)^3$, $7(\infty)^2$, ∞, no variation; consequently, the equation has three unequal real roots; indeed, the equation has a pair of equal roots, whose common value is -3, and two other roots, one of which lies between -3.3 and -3.2, and the other between 1.2 and 1.3.

5. To find the number of real roots of $2x^4 - 20x + 19 = 0$.

Here the first derived function is $8x^3 - 20$, or, rejecting the useless factor 4, it may be represented by $2x^3 - 5$.

Dividing $2x^4 - 20x + 19$ by $2x^3 - 5$, we get $-15x + 19$ for the remainder, whose signs being changed, give $15x - 19$ for the first modified remainder.

Multiplying $2x^3 - 5$ by 15×15, we get $450x^3 - 1125$ for the product, which, divided by $15x - 19$, gives $-\frac{3157}{15}$ for the remainder, whose sign being changed, gives $\frac{3157}{15}$ for the second modified remainder, which may clearly be expressed by 3157; hence, (4) become $2x^4 - 20x + 19$, $2x^3 - 5$, $15x - 19$, 3157.

Because the proposed equation (clearly) has no negative roots, and that the signs of the preceding functions are the same for $x = 0$ that they are when $x = 1$, we put $x = 1$, and the signs are $+, -, -, +$, two variations; and putting $x = 2$, the signs are $+, +, +, +$, no variation; consequently, the given equation has only two real roots, and they lie between 1 and 2.

6. To find the number of real roots of $x^3 + 11x^2 - 102x + 181 = 0$.

Here $3x^2 + 22x - 102$ is the first derived function; and dividing 9 times the given equation by it, we get $854x - 2751$ for the first modified remainder; or, rejecting the factor 7, it may be expressed by $122x - 393$.

Multiplying the first derived function by $(122)^2$, and dividing the product by $122x - 393$, we get $-(122)^2 \times 102 + 3863 \times 393 = -9$ for the remainder, whose sign being changed, gives 9 for the second modified remainder.

Since the preceding remainder is so small, it is plain that $122x - 393$ is nearly a factor of the given equation; consequently, putting it equal to 0, we get $x = 3.221$ nearly for one of two roots of the given equation that are nearly equal to each other.

Collecting the functions, we shall get $x^3 + 11x^2 - 102x + 181$, $3x^2 + 22x - 102$, $122x - 393$, 9, for the representatives of (4).

To simplify the preceding expressions, we put $x = x' + 3$, and they become $x'^3 + 20x'^2 - 9x' + 1$, $3x'^2 + 40x' - 9$, $122x' - 27$, 9.

Putting 0.2 and 0.3 for x', the signs of the functions become $+, -, -, +$, two variations, and $+, +, +, +$, no variation; consequently, the equation $x'^3 + 20x'^2 - 9x' + 1 = 0$ has a pair of nearly equal roots, which lie between 0.2 and 0.3, whose first figures are easily found to be 0.213 and 0.229.

Since $x = x' + 3$, we get 3.213 and 3.229 for the nearly equal roots of the proposed equation; and since 3.221, before found, must clearly lie between these roots, and differ but little from them, it is clear that they can readily be found without using Sturm's theorem.

Putting the equation in x' under the form $x' - 9x'^{-1} + x'^{-2} = -20$, then developing by (*c*) of Sec. XV., we get $x' = -20 - \frac{9}{20} - \frac{1}{400} + \frac{81}{8000} + \frac{27}{160000} - \frac{1456}{3200000} - \frac{81}{6400000} + \frac{3267}{128000000} +$ etc., whose first eight terms give $x' = -20.44264839$, which is correct to six decimal places; consequently, since $x = x' + 3$, we get $x = -17.442648$ for one of the roots of the proposed equation, which is correct in all its figures.

Remark.—This example was given by Sturm, in his Memoir, to show the use of his theorem in separating the nearly equal roots of an equation from each other.

SECTION XIX.

DEVELOPMENT OF THE REAL AND IMAGINARY ROOTS OF AN EQUATION OF THE n^{th} DEGREE.

(1.) Let $Ay^n + A_1y^{n-1} + A_2y^{n-2} + A_3y^{n-3} + \dots + A_{n-1}y + A_n = 0$, (1), stand for the equation; then if a' represents a near value of one of the roots and $a' + x$ its exact value, by putting $a' + x$ for y, we change (1) to $A(a' + x)^n + A_1(a' + x)^{n-1} + A_2(a' + x)^{n-2} + \dots + A_{n-1}(a' + x) + A_n = 0$, (2).

If we put $Aa'^n + A_1a'^{n-1} + A_2a'^{n-2} + \text{etc.} = f(a') =$ a function of a', and the first derived function of $f(a')$, equal to $f'(a')$, and that of $f'(a')$ equal to $f''(a')$, and so on, to any required extent; then by expanding the powers in (2), and ordering the result according to the ascending powers of x, we shall get $f(a') + f'(a')x + \frac{f''(a')}{1.2}x^2 + \frac{f'''(a')}{1.2.3}x^3 + \text{etc.} = 0$, (3).

If we divide the terms of (3) by $f'(a')$, and put $-\frac{f(a')}{f'(a')} = u$, $\frac{f''(a')}{1.2} \div f'(a') = a$, $\frac{f'''(a')}{1.2.3} \div f'(a') = b$, and so on, then (3) can easily be reduced to $x + ax^2 + bx^3 + cx^4 + \text{etc.} = u$, ($a$).

If we revert the series (a) (see On the Reversion of Series, p. 435), we shall get $x = u - au^2 - (b - 2a^2)u^3 - (c + 5a^3 - 5ab)u^4 - \text{etc.}$, ($b$); which will enable us to find x to any degree of exactness after we have found a suitable value of a', and thence the value of the root $a' + x$ will be found as required.

To obtain a suitable value of a', we shall put $a' = a_1 + b_1\sqrt{-1}$, and a' will be expressed in the proper form (on the supposition that a_1 and b_1 are not imaginary), for if a' is real, by putting $b_1 = 0$ we shall have $a' = a_1$, and if a' is imaginary, then, as has (heretofore) been shown, $a_1 + b_1\sqrt{-1}$ is the proper form for a'; noticing that a_1 may be 0 if required.

Putting $a_1 + b_1\sqrt{-1}$ for a' in $Aa'^n + A_1a'^{n-1} + A_2a'^{n-2} +$

etc. $= f(a')$ and $-\frac{f(a')}{f'(a')} = u$, we get the forms $f(a_1 + b_1\sqrt{-1}) = P + Q\sqrt{-1}$, (c), and $u = \frac{S + T\sqrt{-1}}{R}$, (d).

Let $r, r_1, r_2, r_3, \ldots . r_{n-1}$ stand for the n roots of (1), whether they are positive or negative, real or imaginary; then, since (1) is equivalent to the product $A(y - r) . (y - r_1) . (y - r_2) . (y - r_3) \times \ldots . \times (y - r_{n-1}) = 0$, if we substitute $a' + x$ for y in this, we shall clearly get $f(a') = A(a' - r) . (a' - r_1) . (a' - r_2) \times \ldots . \times (a' - r_{n-1})$, (e), and $f'(a') = A[(a' - r_1) . (a' - r_2) . (a' - r_3) \times \ldots . \times (a' - r_{n-1}) + (a' - r) . (a' - r_2) . (a' - r_3) \times \ldots . \times (a' - r_{n-1}) + (a' - r) . (a' - r_1) . (a' - r_3) . (a' - r_4) \times \ldots . \times (a' - r_{n-1}) + \text{etc.}]$, (f).

If no two of the roots r, r_1, r_2, etc., are equal to each other, and we assume a' very nearly equal to any one of them, as to r, then since $a' - r$ is a factor of all the terms of (f) but the first, if we omit them on account of their comparative minuteness, it is clear that (f) will give $f'(a') = A(a' - r_1) . (a' - r_2) . (a' - r_3) \times \ldots . \times (a' - r_{n-1})$ nearly; consequently, we shall get $u = -\frac{f(a')}{f'(a')} = -(a' - r)$ very nearly, which is very small, on account of the supposed minuteness of $a' - r$.

Hence, *if the roots of* (1) *are unequal to each other, we can assume a real or imaginary value for* a', *such that it shall differ so little from any real or imaginary root of* (1) *that the corresponding value of* u *shall be very small; and it is clear that* $f(a')$ *will (at the same time) also be very small.*

If two or more of the roots r, r_1, r_2, r_3, etc., are equal to each other, it is clear that $f(a')$ and $f'(a')$ will have a common divisor; consequently, if we divide $f(a')$ and $f'(a')$ by their common divisor, and represent the quotients by $F(a')$ and $F'(a')$, we shall have $u = -\frac{f(a')}{f'(a')} = -\frac{F(a')}{F'(a')}$, from which we can find the unequal roots of (1), and that whether they are real or imaginary; it may be added, that if we put the common divisor equal to 0, the values of a', which satisfy the resulting equation, will clearly give the equal roots of (1), after the number of each kind is increased by unity.

(2.) If the root $a' + x$ of (1) is real, we can always find a' to any required degree of exactness by Sturm's theorem; and it is clear that a sufficiently near value of a' can often be found with great facility by one of the more simple methods of limits given at pp. 471, 472, etc.

Finally, since all the conditions required in the determination of $a' = a_1 + b_1\sqrt{-1}$ are embraced in the condition that u, as given by (d), shall be so small as to make the series (b) converge with sufficient rapidity, in order to find x to the required degree of exactness by using a few of the first terms of (b), it is clear that if we assume a_1 and b_1, such that P and Q in (c) are small, and thence find that $\frac{\text{S}}{\text{R}}$ and $\frac{\text{T}}{\text{R}}$ are also very small, we shall manifestly have found suitable values for a_1 and b_1, or such values as will enable us to find x from (b) to the required degree of exactness.

EXAMPLES.

1. To find all the roots of the equation $4y^3 - 8y - 17 = 0$.

Putting $y = a' + x$, the equation is easily reduced to $4x^3 + 12a'x^2 + (12a'^2 - 8)x = 17 + 8a' - 4a'^3$, which is easily reduced to $x + \frac{3a'}{3a'^2 - 2}x^2 + \frac{1}{3a'^2 - 2}x^3 = \frac{17 + 8a' - 4a'^5}{12a'^2 - 8}$, (1); and this, compared to (a), gives $u = \frac{17 + 8a' - 4a'^3}{12a'^2 - 8}$, $a = \frac{3a'}{3a'^2 - 2}$, $b = \frac{1}{3a'^2 - 2}$, $c = 0$, $d = 0$, etc.

After a few trials, we find that by putting $a' = 2$, we shall get $u = \frac{1}{40} = 0.025$, $a = \frac{6}{10} = 0.6$, $b = 0.1$; consequently, we have $au^2 = 0.000375$, $(b - 2a^2)u^3 = -0.0000096875$, $(c + 5a^3 - 5ab)u^4 = 0.00000030468$, etc.

Hence, using only four terms of (b), we shall get $x = 0.025 - 0.000375 + 0.0000096875 - 0.00000030468 = 0.024634 +$, which is correct to six decimal places; consequently, we shall have $y = a' + x = 2.024634 +$ for the real root of the proposed equation, which is correct to six decimal places.

Again, if we put $a' = a_1 + b_1\sqrt{-1} = -1 + \sqrt{-1}$, we shall get $17 + 8a' - 4a'^3 = 1$ and $12a'^2 - 8 = -8(1 + 3\sqrt{-1})$, which give $u = -\frac{1}{8(1 + 3\sqrt{-1})} = -\frac{1 - 3\sqrt{-1}}{8(1 + 3\sqrt{-1})} \div (1 - 3\sqrt{-1}) = -\frac{1 - 3\sqrt{-1}}{80}$; and we shall have $a =$

$-\frac{3}{10}(1+2\sqrt{-1})$, $b=-\frac{1-3\sqrt{-1}}{20}$, $c=0$, $d=0$, etc.; also, $b-2a^2=\frac{49-57\sqrt{-1}}{100}$, $au^2=-\frac{6-33\sqrt{-1}}{32000}$, $(b-2a^2)u^3=\frac{49-57\sqrt{-1}}{100}\times\frac{13-9\sqrt{-1}}{256000}=\frac{62-591\sqrt{-1}}{12800000}$.

Hence, using only three terms of (b), converting u, au^2, and $(b-2a^2)u^3$ into equivalent decimals, etc., we shall get $x=-0.0123173+0.03647\sqrt{-1}$; consequently, we shall have $a'+x=-1.0123173+1.03647\sqrt{-1}$ for one of the imaginary roots, and changing the sign of the imaginary part, we get $-1.0123173-1.03647\sqrt{-1}$ for the other imaginary root of the proposed equation; these roots being correct to about seven decimals in their real parts, and nearly to six places in their imaginary parts.

2. To find the imaginary roots of $y^4-19y^3+132y^2-302y+200=0$.

By putting $a'+x$ for y, the equation is easily reduced to
$$x+\frac{6a'^2-57a'+132}{4a'^3-57a'^2+264a'-302}x^2+\frac{4a'-19}{4a'^3-57a'^2+264a'-302}x^3+\frac{1}{4a'^3-57a'^2+264a'-302}x^4=-\frac{a'^4-19a'^3+132a'^2-302a'+200}{4a'^3-57a'^2+264a'-302}=u;$$
which, compared to (a), gives $a=\frac{6a'^2-57a'+132}{4a'^3-57a'^2+264a'-302}$, $b=\frac{4a'-19}{4a'^3-57a'^2+264a'-302}$, $c=\frac{1}{4a'^3-57a'^2+264a'-302}$, $d=0$, $e=0$, etc.

Also, if we put $a'=a_1+b_1\sqrt{-1}$, we have $a'^4-19a'^3+132a'^2-302a'+200=a_1^4-6a_1^2b_1^2+b_1^4-19a_1^3+57a_1b_1^2+132a_1^2-132b_1^2-302a_1+200+(4a_1^3b_1-4a_1b_1^3-57a_1^2b_1+19b_1^3+264a_1b_1-302b_1)\sqrt{-1}$, which will be found to give $a_1=8$ and $b_1=4$ for the proper values of a_1 and b_1; which being put for a_1 and b_1, the expression becomes $-104+8\sqrt{-1}$. Similarly, by putting $a'=8+4\sqrt{-1}$, we shall get $4a'^3-57a'^2+264a'-302=-414+224\sqrt{-1}$.

Hence, we shall have $u=\frac{-104+8\sqrt{-1}}{414-224\sqrt{-1}}=-$

$\frac{44848 + 19984 \sqrt{-1}}{221572} = -0.202 - 0.090 \sqrt{-1}$ nearly, for the first approximation to the value of x; and adding this to $a' = 8 + 4 \sqrt{-1}$, we shall get $a' = 7.8 + 3.9 \sqrt{-1}$ nearly, for a more correct value of a'.

By putting $a' = 7.8 + 3.9 \sqrt{-1}$, we shall get $u = \frac{-5.9707 + 7.6674 \sqrt{-1}}{369.158 - 171.756 \sqrt{-1}} = \frac{-3521.053625 + 1804.9785 \sqrt{-1}}{165777.7525}$ $= -0.0212396028. + 0.0108879410. \sqrt{-1}$, $a = \frac{38.82 - 142.74 \sqrt{-1}}{369.158 - 171.756 \sqrt{-1}} = \frac{38847.16500 - 46026.04500 \sqrt{-1}}{165777.7525}$ $= 0.234332800 - 0.277637043 \sqrt{-1}$; $b = -\frac{12.2 + 15.6 \sqrt{-1}}{369.158 - 171.756 \sqrt{-1}} = -0.010401480 - 0.047378420 \sqrt{-1}$, $c = -\frac{369.158 + 171.756 \sqrt{-1}}{165777.7525} = -0.00222682 - 0.00103606 \sqrt{-1}$; $u^2 = 0.00033257346 - 0.00046251108 \sqrt{-1}$, $au^2 = -0.0000504773 - 0.0002007162 \sqrt{-1}$; $u^3 = -0.0000020280 + 0.0000134446 \sqrt{-1}$, $a^2 = -0.0221601 - 0.1301157 \sqrt{-1}$, $b - 2a^2 = 0.0339187 + 0.2128530 \sqrt{-1}$, and $(b - 2a^2)u^3 = -0.0000029304 + 0.0000000247 \sqrt{-1}$.

Hence, by using three terms of (b), we shall get $x = u - au^2 - (b - 2a^2)u^3 = -0.02118619 + 0.01108841 \sqrt{-1}$, and thence $a' + x = 7.77881381 \pm 3.91108841 \sqrt{-1}$ are the imaginary roots, using + for ± for one of them, and − for the other; it may be noticed that these roots are correct to about seven decimal places, both in their real and imaginary parts.

Remarks.—Since the sum and product of the roots found are nearly equal to 15.55762762 and 75.80655683, and that − 19 is the negative of the sum of all the roots of the proposed equation, while 200 equals their product; it follows that if we subtract 15.557, etc., from 19, we shall get 3.44237238 for the sum of the remaining two roots, and that if we divide 200 by 75.80655683, we shall get 2.63829552 for the product of the same roots.

Hence, since the sum and product of the remaining roots are known, we easily find them to be 2.2905598 and 1.1518125, which are correct to about six decimal places.

Thus, we perceive how we may sometimes find the real roots of an equation from its imaginary roots; and at the same time we have a verification of the correctness of the rules which have been given for the treatment of imaginary expressions, when they are to be added, subtracted, multiplied, etc.

It may be added, that if we put 2.29 and 1.15 separately, for a' in the values of u, a, b, etc., we shall, from (b), easily get the remaining roots of the equation to any degree of exactness that may be required.

3. To find the roots of $y^3 - 5y^2 + 8y - 1 = 0$.

Putting $a' + x$ for y, we easily reduce the equation to $x + \frac{3a' - 5}{3a'^2 - 10a' + 8}x^2 + \frac{1}{3a'^2 - 10a' + 8}x^3 = -\frac{a'^3 - 5a'^2 + 8a' - 1}{3a'^2 - 10a' + 8} = u$; consequently, $a = \frac{3a' - 5}{3a'^2 - 10a' + 8}$, $b = \frac{1}{3a'^2 - 10'a + 8}$, $c = 0$, $d = 0$, etc.

Also, putting $a' = a_1 \pm b_1 \sqrt{-1}$, we get $a'^3 - 5a'^2 + 8a' - 1 = a_1^3 - 5a_1^2 - 3a_1b_1^2 + 5b_1^2 - 1 + (3a_1^2b_1 - 10a_1b_1 - b_1^3 + 8b_1)\sqrt{-1}$, and $3a'^2 - 10a' + 8 = 3(a_1^2 - b_1^2) - 10a_1 + 8 + (6a_1b_1 - 10b_1)\sqrt{-1}$. From the forms of these equations, it is clear that a_1 and b_1 can not differ much from 2 and 1; consequently, putting these values for a_1 and b_1 in the equations, we soon find that 2.4 and 1.2 are more correct values of a_1 and b_1; and putting these values of a_1 and b_1 in the equations, they give $a'^3 - 5a'^2 + 8a' - 1 = 0.056 - 0.192\sqrt{-1}$, and $3a'^2 - 10a' + 8 = -3.04 + 5.28\sqrt{-1}$. Hence, we have $u = \frac{0.056 - 0.192\sqrt{-1}}{3.04 - 5.28\sqrt{-1}} = \frac{1.184 - 0.288\sqrt{-1}}{37.12} = 0.03189655 - 0.00775851\sqrt{-1}$, $a = \frac{2.2 + 3.6\sqrt{-1}}{3.04 - 5.28\sqrt{-1}} = \frac{-12.32 + 22.56\sqrt{-1}}{37.12} = -0.3318965 + 0.6077585\sqrt{-1}$, $b = \frac{1}{3.04 - 5.28\sqrt{-1}} = \frac{3.04 + 5.28\sqrt{-1}}{37.12} = 0.081896 + 0.142241\sqrt{-1}$; $u^2 = 0.00095719 - 0.000494945\sqrt{-1}$, and

$au^2 = -0.00001686 + 0.00074601\sqrt{-1}$; $u^3 = 0.000026706 - 0.00002321\sqrt{-1}$, $b - 2a^2 = 0.600326 + 0.949092\sqrt{-1}$, and $(b - 2a^2)u^3 = 0.00003806 + 0.00001141\sqrt{-1}$.

Substituting the preceding values in (b), and taking only three terms of the series, we get $x = u - au^2 - (b - 2a^2)u^3 = 0.03187535 - 0.0085159\sqrt{-1}$, which is correct to four decimal places; consequently, $a' + x$ gives $2.43187535 \pm 1.1914841\sqrt{-1}$ for the expression of the imaginary roots, which are correct to four decimal places, both in their real and imaginary parts.

Since -5 is the negative of the sum of all the roots of the equation, and that 4.8637507 is that of the roots found, we have $5 - 4.8637507 = 0.1362493 =$ the remaining root, correct to four decimal places.

4. To find the roots of "$y^4 - 5y^3 + 13y^2 - 17y + 12 = 0$."

Here, by putting $a' + x$ for y, and $a' = a_1 + b_1\sqrt{-1}$, and proceeding as in the last question, we find that it will answer our purpose to put $a_1 = 1$, and $b_1 = 1.4$, or $a' = 1 + 1.4\sqrt{-1}$; then, from the substitution of $1 + 1.4\sqrt{-1}$ for a', etc., we soon find $a' + x = 0.9999$, etc. $\pm$ 1.4142, etc. $\sqrt{-1}$ for two of the imaginary roots, which indicate that $1 \pm \sqrt{-2}$ are two of the imaginary roots.

Dividing the equation by $(y - 1 + \sqrt{-2}) \times (y - 1 - \sqrt{-2}) = (y - 1)^2 + 2 = y^2 - 2y + 3$, we get $y^2 - 3y + 4$ for the quotient; consequently, the given equation is equivalent to $(y^2 - 2y + 3) \times (y^2 - 3y + 4) = 0$. Hence, if we put the factor $y^2 - 3y + 4$ equal to 0, and solve the resulting quadratic, we shall get $\dfrac{3 \pm \sqrt{-7}}{2}$ for the two remaining roots of the equation.

5. To find the integral roots of $x^3 - x^2 - 41x + 105 = 0$.

If a stands for an integral root, we know (from principles heretofore given) that 105 must be exactly divisible by a, and that the first member of the equation must be exactly divisible by $x - a$. Hence, if we put 1, 0, and -1, successively, for x in the given equation, the results must be severally divisible by $1 - a$, $-a$, and $-1 - a$. If a is positive,

$1 - a$, $-a$, $-1 - a$ is an arithmetical progression, whose terms increase (numerically) by the addition of -1; while if a is negative, the progression becomes $1 + a$, a, and $-1 + a$, whose terms decrease by the addition of -1.

To find the positive roots, we put 1, 0, and -1, successively, for x, and get 64, 105, and 144 for the corresponding results; then taking -3 (the least required divisor of 105) for $-a$, we have -2, -3, -4 for the representatives of $1 - a$, $-a$, and $-1 - a$; and dividing 64, 105, and 144, severally, by -2, -3, and -4, we get -32, -35, -36 for the quotients, which we shall use to find the remaining roots. Taking -5, the least divisor of -35 (as before), for $-a$, we get -4, -5, and -6 for the corresponding values of $1 - a$, $-a$, and $-1 - a$. Dividing -32, -35, and -36, severally, by -4, -5, and -6, we get 8, 7, and 6 for the quotients, which may be taken for the series $1 + a$, a, and $-1 + a$.

Hence, putting 3, 5, -7, separately for x in the equation, it will be found to be satisfied, and of course 3, 5, and -7 are its roots.

Remark.—The elegant process used in the solution of this example is called the *Method of Divisors*, and is due to Newton. (See p. 38 of his Universal Arithmetic, second edition, published in 1728.)

6. To find the integral roots of $x^3 + 9x^2 + 14x - 24 = 0$.

Here we shall have $\sqrt[3]{24} + 1$ for a superior limit to the positive roots, and $-(9 + 1) = -10$ for an inferior limit to the negative roots; and it is easy to perceive that 1 is a positive root, and the only positive root.

To find the negative roots, we shall suppose that $-a$ is a negative root; then, since the first member of the equation must be exactly divisible by $x + a$, if we put 2, 0, and -2 for x in the equation, the results, 48, -24, and -24, must severally have $2 + a$, a, and $-2 + a$ for corresponding divisors.

Hence, we get 8, 6, and 4 for one set of corresponding divisors, and dividing 48, -24, and -24 severally by 8, 6, and 4, we get 6, -4, and -6, which give 6, 4, and 2 for the remaining set of corresponding divisors.

Substituting 1, -4, and -6 for x in the equation, we find that it is satisfied, and of course 1, -4, -6 are its roots.

Remark.—If we had found 6 of the first set of divisors not to be a root of the equation, we should have used 48, -24, and -24 to get the remaining divisors; and we must proceed in like manner in every case, since such divisors do not correspond to a root of the proposed equation.

7. To find the rational roots of the equation $6x^3 - 5x^2 - 22x + 24 = 0$.

If the equation has rational roots, it is clear that it must have rational factors, which may be represented by $ax - b$, such that a and b are integers, a being a divisor of 6, and b of 24; consequently, either a or b, or each of them (neglecting the sign of b), may be 1, if required.

Since the first member of the equation must be exactly divisible by $ax - b$, if we put 1, 0, -1 successively for x, the results must have $a - b$, $-b$, and $-a - b$ for corresponding divisors when the root is positive, and $a + b$, b, and $-a + b$ will be the corresponding divisors when the root is negative.

Hence, putting 1, 0, and -1, successively, for x in the equation, we get 3, 24, and 35 for the corresponding results, which have 3, 2, and 1 for divisors, and putting $a + b = 3$, $b = 2$, $-a + b = 1$, we get $a = 1$, and we find that -2 is a root of the equation.

Dividing 3, 24, and 35 by 3, 2, and 1, we get 1, 12, and 35, from which we are to find the remaining roots; and it is clear that they can not give any negative root.

To get the positive roots, we take 2 (a factor of 6) for a; and adding $-b$ to it, we have $2 - b$ for a divisor of 1; consequently, if we put 3 (a factor of 12) for b, the divisor $2 - b$ becomes -1, and thence we have -1, -3, -5 for corresponding divisors of 1, 12, and 35.

Hence, since the equation is exactly divisible by $2x - 3$, we shall have $x = \frac{3}{2}$ for one of the positive roots of the equation.

Dividing 1, 12, and 35 by -1, -3, -5, we get -1, -4, and -7; consequently, we have $a - b = -1$, $-b = -4$, and $-a - b = -7$, which give $a = 3$; consequently, $3x - 4$

is an exact divisor of the equation, and $x=\frac{4}{3}$ is its remaining root.

Hence, the equation is reduced to $(x+2)(2x-3)(3x-4)=0$, which is satisfied by putting either of its factors equal to 0.

Remarks.—1st. If we multiply the terms of the equation by 6^2, and put y for $6x$, then it will be reduced to $(6x)^3-5(6x)^2-132(6x)+864=0$, or $y^3-5y^2-132y+864=0$; consequently, if we find the rational roots of this equation, as in the 5th and 6th examples, and divide each of them by 6, we shall get the rational roots of the proposed equation; and it is plain that we may proceed in the same way in all analogous cases.

2d. When an equation is reduced to the form $y^n+A_1y^{n-1}+A_2y^{n-2}+\ldots\ldots+A_{n-1}y+A_n=0$, such that n is a positive integer, and A_1, A_2, $A_3\ldots\ldots A_n$ positive or negative integers, according to the nature of the case, then the rational roots of the equation shall be integers.

For since the rational roots must clearly be either integers or rational fractions, if possible, let $x=\frac{b}{a}$ stand for a rational root; then it is clear that we may suppose b and a to be *prime* to each other, or to have 1 for their greatest common divisor.

Because $\frac{b}{a}$ is a root of the equation, it must be exactly divisible by $y-\frac{b}{a}=\frac{ay-b}{a}$; consequently, since $ay-b$ and a are prime to each other, the equation must be exactly divisible by $ay-b$.

Since in this division y is arbitrary, and the term y^n different from any other term of the equation, it is clear that $ay-b$ can not exactly divide the equation, unless ay is an exact divisor of y^n.

Hence, since it is clear that y^n is not exactly divisible by ay, when a is an integer different from 1, it follows that we must have $a=1$; consequently, the rational roots of the equation must be of the form $y=b=$ an integer.

8. To find the rational roots of "$64x^4 - 328x^3 + 574x^2 - 393x + 90 = 0$."

Because the equation is complete, and has no permanence of signs, it follows from the rule of Descartes that it can not have any negative roots.

Let, then, $ax-b$ (as in the preceding example) stand for any simple rational divisor of the first member of the equation, such that a is a factor of 64, while b is a factor of 90.

Putting 1, 0, and -1 successively for x in the equation and in the divisor, we have the results 7, 90, and 1449, whose corresponding divisors are $a-b$, $-b$, and $-a-b$, which increase numerically by the addition of $-a$.

Putting -1, -2, and -3 for the corresponding divisors (which are the simplest), we have $a-b=-1$, $-b=-2$, and $-a-b=-3$, which give $a=1$ and $b=2$; consequently, the divisor $ax-b$ becomes $x-2$; and since this is an exact divisor of the first member of the equation, we have $x=2=$ one of the roots.

Dividing 7, 90, and 1449 by -1, -2, and -3, we have the quotients -7, -45, -483, from which we are to find the remaining roots.

Putting 2 (the simplest divisor of 64, excepting 1, already used) for a, and taking 1, the simplest divisor of 45, for b, we have $a-b=1$, $-b=-1$, and $-a-b=-3$ for corresponding divisors of -7, -45, and -483, and the divisor $ax-b$ becomes $2x-1$.

Because $2x-1$ exactly divides the first member of the equation, it follows that, by putting $2x-1=0$, we shall have $x=\frac{1}{2}$ for another root of the equation; consequently, dividing -7, -45, and -483 by 1, -1, and -3, we have -7, 45, and 161 for the quotients, which will enable us to find the remaining roots.

Putting $a=4$ and $b=3$, we have $a-b=1$, $-b=-3$, $-a-b=-7$ for the divisors of -7, 45, and 161; consequently, since the equation is divisible by $4x-3$, we have $x=\frac{3}{4}$ for another root of the equation.

Dividing -7, 45, and 161 by 1, -3, and -7, we get the quotients -7, -15, and -23; consequently, since all the roots but one have been found, we must have $a-b=-7$, $-$

$b = -15$, $-a-b=-23$, which give $a=8$, $b=15$, and $8x-15$ must be a divisor of the equation, and thence $x = \frac{15}{8}$ = the remaining root.

Remark.—If we multiply the equation by $64 = 8^2$, and put $8x = y$, or $x = \frac{y}{8}$, we have $(8x)^4 - 41(8x)^3 + 574(8x)^2 - 3144(8x) + 5760 = 0$, or $y^4 - 41y^3 + 574y^2 - 3144y + 5760 = 0$.

Solving the equation in y (as in the 5th and 6th examples), we get 4, 6, 15, and 16 for the roots; and dividing each of these by 8, we get the roots of the proposed equation, as before.

9. To resolve $x^4 - 17x^2 - 20x - 6 = 0$ into rational factors, and thence to find its roots.

Since it is clear that the equation has no rational factors of the first degree, we shall try to resolve it into rational factors of the second degree; and it is clear, since the coefficient of x^4 is 1, that $x^2 + ax + b$ is the proper form for a factor of the second degree.

If we put 1, 0, and -1 successively for x in the equation and in the factor, we shall get the results -42, -6, and -2, which have $1+a+b$, b, and $1-a+b$ for corresponding divisors, noticing that $a+b$, b, and $-a+b$ are in arithmetical progression, being formed by the successive additions of $-a$ to the first and second terms.

After a few trials, we find that it will answer our purpose to put $1+a+b=-6$, $b=-3$, and $1-a+b=2$, which give $a=-4$; consequently, putting -4 and -3 for a and b, we have $x^2+ax+b = x^2-4x-3$.

Because the division of the equation by x^2-4x-3 gives the quotient x^2+4x+2, it is reduced to $(x^2-4x-3)(x^2+4x+2)=0$; consequently, since the equation is satisfied by putting either of its factors equal to 0, we get, from $x^2-4x-3=0$ and $x^2+4x+2=0$, $2+\sqrt{7}$, $2-\sqrt{7}$, $-2+\sqrt{2}$, and $-2-\sqrt{2}$ for the roots of the proposed equation.

10. To resolve $6x^4 - 5x^3 - 2x^2 - 29x + 10 = 0$ (if possible) into two rational quadratic factors.

It is clear that we may represent either of the quadratic factors by $mx^2 + ax + b$, on the supposition that m is a factor of 6, and b a factor of 10.

Putting $m = 2$, and proceeding as in the last question, we get the results -20, 10, and 48, having $2 + a + b$, b, and $2 - a + b$ for their divisors.

After a few trials, we find $2 + a + b = 10$, $b = 5$, and $2 - a + b = 4$, which give $a = 3$; consequently, $mx^2 + ax + b$ becomes $2x^2 + 3x + 5$, which divides the given equation, and gives the quotient $3x^2 - 7x + 2$; consequently, the equation is reduced to $(2x^2 + 3x + 5) \times (3x^2 - 7x + 2) = 0$.

Putting the factors equal to 0, we easily get $\frac{-3 + \sqrt{-31}}{4}$, $\frac{-3 - \sqrt{-31}}{4}$, 2, and $\frac{1}{3}$ for the roots of the given equation.

Remarks.—1st. The preceding methods for finding the quadratic factors are essentially the same as those given by Newton, at p. 40, etc., of his Universal Arithmetic.

2d. It is clear that the preceding question can be solved by first finding the simple factors; but we have found the quadratic factor, for the purpose of showing that the quadratic factors may be found without a knowledge of the simple factors.

11. Given, the first term of an increasing geometrical progression equal to 2, and the sum of its first four terms equal to 80, to find the progression.

Let x stand for the ratio, then the conditions give the equation $x^3 + x^2 + x = 39$, whose solution gives $x = 3$; consequently, the progression is 2, 6, 18, 54, etc.

12. Divide $1000 between A and B, so that the cube of A's share shall equal the square of B's share.

Let x^2 and x^3 stand for the shares of A and B, and one of the conditions is satisfied; and to satisfy the other, we must solve the equation $x^3 + x^2 = 1000$.

Developing the root of this equation, by (b) of Section XV., we get $x = 10 - \frac{1}{3} + \frac{1}{90} - \frac{1}{4050} + \frac{1}{3645000} -$ etc. $= 10 - 0.33\dot{3}. + 0.011\dot{1}. - 0.0002469 + 0.00000027 -$

etc., which, by taking five terms, gives $x = 9.677531$, very nearly.

Hence, the shares of A and B are easily found to be \$93.6546 and \$906.3455, very nearly.

13. To find a number, such that the sum of its square, cube, and fourth roots shall equal 88, supposing the square and fourth roots (each) to be taken with the positive sign.

Let x^{12} stand for the number, then the conditions give the equation $x^6 + x^4 + x^3 = 88$, or $x^6 + x^4 + x^3 - 88 = 0$.

Let a stand for the root, then the first member of the last equation is exactly divisible by $x - a$; consequently, putting 1, 0, and -1 successively for x in the equation and divisor, the results -85, -88, -87 have $1 - a$, $-a$, and $-1 - a$ for corresponding divisors.

By putting $1 - a = -1$, $-a = -2$, $-1 - a = -3$, we get $a = 2$, and the divisor becomes $x - 2$; consequently, since the first member of the equation is divisible by $x - 2$, we have $x = 2$ for its root, and $x^{12} = 2^{12} = 4096$ is the required number.

14. To find x from the equation $\sqrt[3]{x^3 - 5} - \sqrt{8 - x^2} = 2$, without freeing the equation from radicals.

Let a stand for a near value of x, and put $x = a + y$; then we shall approximately have $x^2 = a^2 + 2ay$, $x^3 = a^3 + 3a^2y$, and $\sqrt{8 - x^2} = \sqrt{8 - a^2} - \dfrac{ay}{\sqrt{8 - a^2}}$, $\sqrt[3]{x^3 - 5} = \sqrt[3]{a^3 - 5} + \dfrac{a^2y}{\sqrt[3]{(a^3 - 5)^2}}$; consequently, the given equation is reduced to

$$\sqrt[3]{a^3 - 5} - \sqrt{8 - a^2} + \left(\frac{a^2}{\sqrt[3]{(a^3 - 5)^2}} + \frac{a}{\sqrt{(8 - a^2)}}\right)y = 2 \text{ nearly,}$$

which gives $y = (2 + \sqrt{8 - a^2} - \sqrt[3]{(a^3 - 5)}) \div \left(\dfrac{a^2}{\sqrt[3]{(a^3 - 5)^2}} + \dfrac{a}{\sqrt{(8 - a^2)}}\right)$.

After a few trials, we find that it will answer our purpose to put $a = 2.8$; which gives $\sqrt{8 - a^2} = 0.4$, $\sqrt[3]{a^3 - 5} = 2.568859$, and $2 + \sqrt{8 - a^2} - \sqrt[3]{a^3 - 5} = -0.168859$; we also have $\dfrac{a^2}{\sqrt[3]{(a^3 - 5)^2}} + \dfrac{a}{\sqrt{(8 - a^2)}} = 8.1880$.

Hence, we shall have $y = -\dfrac{0.16886}{8.1880} = -0.020$; consequently, $x = a + y = 2.8 - 0.020 = 2.78$ very nearly. By putting 2.78 for a, and repeating the process, we shall get x more nearly, and so on, to any desired degree of exactness.

Remark.—It is easy to perceive, from what is said at p. 494, how we may apply (b) (see p. 494) to find y, by a converging series, and thence to get $x = a + y$ to any degree of exactness.

15. To find x and y from the equations $\sqrt{x} - 2\sqrt[3]{y} = 5$ and $\sqrt[3]{x} + 3\sqrt{y} = 17$, without removing the surds.

Let a and b stand for near values of x and y, and put $x = a + p$, $y = b + q$ for their exact values; then we shall, approximately, have $\sqrt{x} = \sqrt{a} + \frac{p}{2\sqrt{a}}$, $\sqrt{y} = \sqrt{b} + \frac{q}{2\sqrt{b}}$, $\sqrt[3]{x} = \sqrt[3]{a} + \frac{p}{3\sqrt[3]{a^2}}$, and $\sqrt[3]{y} = \sqrt[3]{b} + \frac{q}{3\sqrt[3]{b^2}}$.

Hence, the proposed equations are reduced to $\sqrt{a} + \frac{p}{2\sqrt{a}} - 2\sqrt[3]{b} - \frac{2q}{3\sqrt[3]{b^2}} = 5$, and $\sqrt[3]{a} + \frac{p}{3\sqrt[3]{a^2}} + 3\sqrt{b} + \frac{3q}{2\sqrt{b}} = 17$, which give $\frac{p}{2\sqrt{a}} - \frac{2q}{3\sqrt[3]{b^2}} = 5 - \sqrt{a} + 2\sqrt[3]{b} = \text{A}$, and $\frac{p}{3\sqrt[3]{a^2}} + \frac{3q}{2\sqrt{b}} = 17 - \sqrt[3]{a} - 3\sqrt{b} = \text{B}$.

Solving these equations (regarding p and q as the unknown letters), we get $p = (9\text{A}\sqrt[3]{b^2} + 4\text{B}\sqrt{b}) \div \left(\frac{9\sqrt[3]{b^2}}{2\sqrt{a}} + \frac{4\sqrt{b}}{3\sqrt[3]{a^2}}\right)$, and $q = (3\text{B}\sqrt[3]{a^2} - 2\text{A}\sqrt{a}) \div \left(\frac{9\sqrt[3]{a^2}}{2\sqrt{b}} + \frac{4\sqrt{a}}{3\sqrt[3]{b^2}}\right)$.

After a few trials, we find that it will answer our purpose to put $a = 105$ and $b = 18$, which give $5 - \sqrt{a} + 2\sqrt[3]{b} = \text{A} = -0.005$ very nearly, and $17 - \sqrt[3]{a} - 3\sqrt{b} = \text{B} = -0.445$ nearly.

Hence we deduce $p = -2.403$ and $q = -1.117$; and thence we have $x = a + p = 102.597$, and $y = b + q = 16.883$, for nearer values of x and y.

Substituting these values of x and y in the given equations, we have $\sqrt{x} - 2\sqrt[3]{y} = 4.997$, and $\sqrt[3]{x} + 3\sqrt{y} = 17.004$ very nearly.

If we put 102.597 and 16.883 for a and b in $x = a + p$ and $y = b + q$, and repeat the process, we shall get more correct values of x and y, and so on, to any required degree of exactness.

Remarks.—1st. We have given this and the preceding question for the purpose of showing how, from the proposed equations, we may proceed to find the unknown letters, without any previous reduction of the equations.

2d. It is clear that a similar process can be applied to any number of equations which contain as many unknown letters as there are equations, provided the unknown letters are real.

For if, by trial, we find values of the unknown letters which nearly satisfy the equations, then we can, as in the present question, find the corrections of the assumed values of the unknown letters to any degree of exactness that may be required.

3d. Thus, it is evident (after we have assumed suitable values for the unknown letters) that we may regard the solution of equations as being reduced to the solution of equations of the first degree.

16. To find the approximate values of x, y, and z, from the equations $x^2 + yz = 16$, $y^2 + xz = 17$, $z^2 + xy = 18$.

1. Since the right members of the equations do not differ much from equality, it is clear that we may solve them on the supposition that x, y, and z are nearly equal to each other; consequently, by adding them, and putting $6t^2$ for the sum of the terms in the first member of the resulting equation, we shall get $6t^2 = 51$ or $t^2 = 8.5$, whose positive square root gives $t = 2.9$ for the first approximation to the values of x, y, and z.

Putting $x = 2.9 + p$, $y = 2.9 + q$, $z = 2.9 + r$, and rejecting p^2, q^2, r^2, pq, pr, and qr (on account of their supposed minuteness), we get $x^2 = 8.41 + 5.8p$, $y^2 = 8.41 + 5.8q$, $z^2 = 8.41 + 5.8r$, $xy = 8.41 + 2.9(p + q)$, $xz = 8.41 + 2.9(p + r)$, and $yz = 8.41 + 2.9(q + r)$; consequently (by substituting from these equations), the given equations are easily reduced to $5.8p + 2.9(q + r) = -0.82$, $5.8q + 2.9(p + r) = 0.18$, and

$5.8r + 2.9(p+q) = 1.18$, or $2p + q + r = -0.282$, $2q + p + r = 0.062$, and $2r + p + q = 0.406$.

By adding these equations, we get $p + q + r = 0.046$, which being subtracted from the successive equations gives $p = -0.328$, $q = 0.016$, and $r = 0.360$; consequently, we have $x = 2.9 + p = 2.572$, $y = 2.9 + q = 2.916$, and $z = 2.9 + r = 3.260$, which are correct to one decimal place in each.

If we now put $x = 2.5 + p$, $y = 2.9 + q$, and $z = 3.2 + r$, and proceed as before, we shall get more correct values of x, y, and z, and so on, to any degree of exactness required.

It may be added that the values of x, y, and z, to six decimal places, are $x = 2.525513$, $y = 2.969152$, and $z = 3.240580$.

2. The given equations also admit of three more answers, as is manifest from the following considerations.

Thus, if we at first reject x^3, on the supposition that x is very small (so that x^3 may at first be omitted), the equations will become $yz = 16$, $y^2 + xz = 17$, and $z^2 + xy = 18$. Because these equations are nearly satisfied by putting $y = 4$ and $z = 4$, we shall assume $y = 4 + q$ and $z = 4 + r$; then, rejecting q^2, r^2, qx, rx, and qr (on account of their supposed minuteness), we shall have $y^2 = 16 + 8q$, $z^2 = 16 + 8r$, $xz = 4x$, $xy = 4x$, and $yz = 16 + 4(q + r)$.

Hence, by substitution, the equations will be reduced to $16 + 4(q + r) = 16$, $16 + 8q + 4x = 17$, and $16 + 8r + 4x = 18$, or $q = -r$, $8q + 4x = 1$, $8r + 4x = 2$; consequently, by adding the last two of these (since $q + r = 0$), we shall get $8x = 3$, or $x = \frac{3}{8} = 0.375$ nearly, for which we may clearly put $x = 0.4$.

Substituting 0.4 for x in the equations $8q + 4x = 1$ and $8r + 4x = 2$, they become $8q + 1.6 = 1$ and $8r + 1.6 = 2$, which give $q = -\frac{0.6}{8} = -0.075$ and $r = \frac{0.4}{8} = 0.05$; and thence $y = 4 + q$ and $z = 4 + r$ become $y = 3.925$ and $z = 4.05$.

Hence, we have $x = 0.4$, $y = 3.925$, $z = 4.05$, which are (each) correct to one decimal place.

If we put $x = 0.4 + p$, $y = 3.9 + q$, and $z = 4.0 + r$, in the given equations, we can find x, y, and z, to any required degree of exactness; and by continuing the approximation, we easily get $x = 0.418919$, $y = 3.912226$, $z = 4.044884$, correct to six decimal places.

Similarly, if we reject y^2, and then z^2, on account of their supposed minuteness, as before, we shall get two other sets of answers, expressed by $x = 4.003766$, $y = -0.007099$, $z = 4.245989$, and $x = 4.173264$, $y = 4.287022$, $z = -0.330331$.

Since the equations are not affected by changing the signs of x, y, and z, it follows that, if the signs of the values of x, y, and z in the preceding answers are changed, the results will satisfy the equations.

Remarks.—1st. This example, under the form $x^2 + yz = a = 16$, $y^2 + xz = b = 17$, $z^2 + xy = c = 18$, was proposed by Dr. Pell, near the close of the 17th century, and solved in a general manner by Dr. Wallis, who reduced its solution to that of an equation of the fourth degree, by eliminating y and z from the equations.

2d. When 16, 17, and 18 are put for a, b, and c, it is clear that our solution must be equivalent to that of Wallis, obtained from the solution of his equation of the fourth degree.

Hence, if an equation containing one unknown letter is reduced to equations of lower degrees, containing more than one unknown letter, it is manifest that their solution will be equivalent to that of the reduced equation.

17. To find the roots of $x^3 - 7x + 7 = 0$.

By putting $x^2 - 7 = y$ or $x^2 = 7 + y$, the equation becomes $xy + 7 = 0$, which shows that x and y must have unlike signs; consequently, the solution of the proposed equation is reduced to that of the equations $x^2 = 7 + y$ and $xy + 7 = 0$, such that x and y must have contrary signs.

1. After a few trials, we find that, by putting $y = 2.29$, we shall get $x^2 = 9.29$, whose negative square root gives $x = -3.047$, and thence $xy + 7$ becomes $-6.97763 + 7$, a positive result; and in like manner, $y = 2.30$ gives $x = -3.049$, and thence $xy + 7$ becomes $-7.0127 + 7$, a negative result; consequently, 2.29 and 2.30 are limits to y, and -3.047, -3.049 are the corresponding limits of x.

Putting $x = -3.048 + p$ and $y = 2.295 + q$, and proceeding as in the preceding examples, we get $q = -0.004696 - 6.096p$ and $2.295p - 3.048q + 0.004840 = 0$.

Solving these equations, we have $p = -0.000917$ and $q = 0.000894$, and thence $x = -3.048917$ and $y = 2.295894$.

If we put $x = -3.048917 + p$ and $y = 2.295894 + q$, and proceed as before, and so on, then x can be found to any required degree of exactness.

2. Because $y = -4.1$ gives $x = 1.70$, and $xy + 7 = -6.97 + 7$, and that $y = -4.2$ gives $x = 1.67$ and $xy + 7 = -7.014 + 7$, it follows that the equation has another root, which lies between 1.70 and 1.67; and proceeding as before, this root will be found to be 1.692021 +.

In like manner, since $y = -5.1$ gives $x = 1.378$ and $xy + 7 = -7.0278 + 7$, and that $y = -5.2$ gives $x = 1.341$ and $xy + 7 = -6.9732 + 7$, it follows that the equation has another root, which lies between 1.37 and 1.34; and proceeding as above, this root will be found to be 1.356895 +.

18. To find the real roots of $x^3 - 5x^2 + 8x - 1 = 0$.

By putting $x^2 - 5x + 8 = y$, we have $xy - 1 = 0$, or $xy = 1$; consequently, we have to satisfy the equations $x^2 - 5x + 8 = y$ and $xy = 1$, noticing that, by the rule of Descartes, the equation has no negative roots.

After a few trials, it will be found that $y = 7.4$ in the first equation gives $x = 0.123$ nearly, and the second equation gives $xy = 0.9102$ instead of 1.

In like manner, $y = 7.3$ gives $x = 0.145$, and xy becomes $xy = 1.0585$; consequently, 0.123 and 0.145 are limits to the value of x. Hence, the root is easily found; and since x and y must (each) be positive, it is easy to perceive that the remaining roots of the equation are imaginary.

ANOTHER GENERAL METHOD OF DEVELOPING THE ROOTS OF EQUATIONS.

(1.) Let $Ax^n + A_1x^{n-1} + A_2x^{n-2} + \dots + A_{n-1}x + An = 0$, (1), stand for any equation; then, if $r, r_1, r_2, r_3 \dots r_{n-1}$ represent its roots, and $\frac{y}{z}$ is put for x, (1) is easily reduced to

$$Ay^n + A_1y^{n-1}z + A_2y^{n-2}z^2 + \dots + A_{n-1}yz^{n-1} + A_nz^n = A(y - zr) . (y - zr_1) . (y - zr_2) \times (y - zr_3) . (y - zr_4) \times \dots \times (y - zr_{n-1}) = 0, (2).$$

(**2.**) Supposing any one of the factors in (2), as $y - zr$, is either 0, or very nearly equal to 0, then we shall have (either exactly or very nearly) $y = zr$ or $x = \frac{y}{z} = r$. Hence, if $y + t$ and $z + v$ are (separately) put for y and z in (2), we shall easily get $nAy^{n-1} + (n-1)A_1y^{n-2}z + (n-2)A_2y^{n-3}z^2 + \ldots . + A_{n-1}z^{n-1} = A(y - zr_1) . (y - zr_2) \times (y - zr_3) \times \ldots . \times (y - zr_{n-1})$ nearly (by omitting the terms which involve the factor $y - zr$, on account of its supposed minuteness), for the coefficient of the simple power of t; and in like manner we have $A_1y^{n-1} + 2A_2y^{n-2}z + 3A_3y^{n-3}z^2 + 4A_4y^{n-4}z^3 + \ldots . + nA_nz^{n-1} = -Ar(y - zr_1) . (y - zr_2) . (y - zr_3) \times (y - zr_4) . (y - zr_5) \times \ldots . \times (y - zr_{n-1})$, for the coefficient of the simple power of v, where it is clear that, in these coefficients, we may put $z = 1$ and $y = x$.

(**3.**) Since the coefficient of v divided by that of t gives $-r$ for the quotient, it follows, if we know the value of x, which gives $x = r$ (either exactly or very nearly), that the quotient resulting from the division of the coefficient of v by that of t, will (exactly or very nearly) equal the negative of the root r, which corresponds to $x = r$, and that whether r is positive or negative, real or imaginary.

Hence, if r is real and positive, the coefficients of t and v must have unlike signs; and if r is real and negative, the coefficients of t and v must have like signs.

(**4.**) If we put $z = 1$ and $y = x$ in the coefficients of t and v, and then multiply the coefficient of t by x, and add that of v to the product, and divide the sum by n, it is clear that the result will agree with the first member of (1).

Hence, if x differs but little from r, and we divide the first member of (1) by the coefficient of t, the quotient will evidently equal $x - r$ very nearly; consequently, if we asssume x, and find that the quotient resulting from the division of the coefficient of v by that of t is numerically less than the assumed value of x, it follows that too great a numerical value has been assumed for x, and *vice versa*.

(**5.**) Regarding r as being the least root of (1), r_1 as the next greater root, and so on (considering the numerically greatest negative root as being the least, and so on); then, supposing x to increase from being less than r, and to pass through the roots, it is clear, from an inspection of the coefficients of t and v, that their signs will not be changed by the passage of x through r, and that their signs will be changed by the passage of x through r_1, since r and r_1 will be interchanged, and so on.

Hence, if the signs of the coefficients of t and v do not change, it follows, if (1) is of an even degree, that it has no real root, and if (1) is of an odd degree, that it has only one real root.

Hence, too, if x (during its increase) introduces one change of the signs of the coefficients of t and v, it is clear that (1) will (generally) have two real roots, one of which will be less than the greater value of x; and if x (during its increase) produces another change of the signs of the coefficients of t and v, it is also clear that (1) has (generally) three real roots, two of which will be less than the greater value of x, and so on; noticing that, in what has been said, the roots of (1) are supposed to be unequal.

(6.) If (1) has equal roots, it is clear that if we put the greatest common divisor of the coefficients of t and v equal to 0, and solve the resulting equation, and then increase the number of equal roots of each kind thus found by unity, that we shall get the number of equal roots of each kind contained in (1).

EXAMPLES.

1. To find the roots of $x^2 - 5x + 6 = 0$.

Here $2x - 5$ and $-5x + 12$ are the coefficients of t and v, which are suitable for finding the positive roots; and if we put $-x$ for x, they will become $-2x - 5$ and $5x + 12$, which are adapted to the negative roots; noticing that x must be positive.

Since $-2x - 5$ and $5x + 12$ have unlike signs when x is positive, it follows that the equation has no negative roots, which is in accordance with the rule of Descartes, since the given equation is complete and has no permanence of signs.

To find the positive roots from $2x - 5$ and $-5x + 12$, we suppose x to increase from $x = 0$, and to pass through them; then, as they become -5 and 12 when $x = 0$, it follows that one root must be found from $2x - 5$ and $-5x + 12$, when the value of the first has the sign $-$, and the second the sign $+$. Consequently, putting 1 and 2 successively for x, we find that by putting 2 for x they become -1 and 2, which give $2 \div -1 = -2$, and of course 2 is a positive root of the given equation.

Because $2x - 5$ and $-5x + 12$ must change their signs before the remaining root can be found, we put 3 for x, and thence get $2x - 5 = 1$ and $-5x + 12 = -3$, which give $-3 \div 1 = -3$, and of course 3 is the remaining positive root.

Hence, the given equation is reduced to $x^2 - 5x + 6 = (x - 2) . (x - 3) = 0$, which is clearly as it ought to be.

2. To find the real roots of $x^2 - 2x + 3 = 0$.

Here the coefficients of t and v are $2x - 2$ and $-2x + 6$, which clearly show that the equation has no negative roots. Also, since the coefficients show that the positive roots (if there are any) must be greater than 1, it follows that there can be no positive roots; for if there can be any positive roots greater than 1, then such a positive value can be assigned to x as will change the signs of both coefficients; which is clearly impossible when x is greater than 1.

3. "To find the real roots of $x^3 - 5x^2 - 18x + 72 = 0$."

Here the coefficients of t and v become $3x^2 - 10x - 18$ and $-5x^2 - 36x + 216$, which will enable us to find the positive roots; and changing x into $-x$, they become $3x^2 + 10x - 18$ and $-5x^2 + 36x + 216$, which, by assigning positive values to x, will enable us to find the negative roots.

After a few trials, we find that by putting 4 for x in $3x^2 + 10x - 18$ and $-5x^2 + 36x + 216$, they become 70 and 280, which give $280 \div 70 = 4$; consequently, -4 is a negative root of the given equation, and it is clear that it is the only negative root.

To find the positive roots from $3x^2 - 10x - 18$ and $-5x^2 - 36x + 216$, we observe that when $x = 0$ they become -18 and 216, and that when $x = 7$, they become 59 and -281; consequently, since the coefficients change their signs between $x = 0$ and $x = 7$, it is clear that the positive roots must lie between $x = 0$ and $x = 7$.

After a few trials, we find that by putting $x = 3$, the coefficients become -21 and 63, which give $63 \div -21 = -3$; consequently, 3 is a positive root.

In like manner, by putting $x = 6$, we get 30 and -180 for the coefficients of t and v, which give $-180 \div 30 = -6$, consequently 6 is the remaining positive root.

4. To find the roots of $x^3 - 9x^2 + 27x - 27 = 0$.

Since $3x^2 - 18x + 27$ and $-9x^2 + 54x - 81$ are the coefficients of t and v, and that the coefficient of v, divided by that of t, gives -3 for the quotient, it follows that the equation is equivalent to $(x - 3)^3 = 0$.

5. To find the roots of $x^n - nx^{n-1}a + \frac{n(n-1)}{1\,.\,2}x^{n-2}a^2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}a^3 + \text{etc.} = 0$.

Here we have $nx^{n-1} - n(n-1)x^{n-2}a + \frac{n(n-1)(n-2)}{1\,.\,2}x^{n-3}a^2 - \text{etc.}$, and $-nx^{n-1}a + n(n-1)x^{n-2}a^2 - \frac{n(n-1)(n-2)}{1\,.\,2}x^{n-3}a^3 + \text{etc.}$, for the coefficients of t and v. Hence, by dividing the coefficient of v by that of t, we get $-a$ for the quotient; consequently, the equation is equivalent to $(x - a)^n = 0$, without regard to the nature of n.

Remark.—It is hence clear that, whether n is positive or negative, rational or irrational, real or imaginary, we shall always have $(x - a)^n = x^n - nx^{n-1}a + \frac{n(n-1)}{1\,.\,2}x^{n-2}a^2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}a^3 + \text{etc.}$, for the correct expansion of $(x - a)^n$, according to the descending powers of x and the ascending powers of a.

6. To find the roots of $x^4 - 6x^2 - 8x - 3 = 0$.

Since the coefficients of t and v are $4x^3 - 12x - 8$ and $-12x^2 - 24x - 12$, which have $x^2 + 2x + 1 = (x + 1)^2$ for a divisor, it follows that the given equation has three equal roots, having -1 for their common value. And dividing the equation by $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$, we get $x - 3$ for the quotient, consequently we shall have $x = 3$ for the remaining root of the equation.

7. To find the number of real roots of $x^4 - 4x^3 + 8x - 15 = 0$.

Since the coefficients of t and v are $4x^3 - 8x + 8$ and $-8x^2 + 24x - 60$, by rejecting their common factor 4, they may clearly be reduced to $x^3 - 2x + 2$ and $-2x^2 + 6x - 15$; and putting $-x$ for x, these become $-x^3 + 2x + 2$ and $-2x^2 - 6x - 15$, which will enable us to find the number of negative roots.

From $x^3 - 2x + 2$ and $-2x^2 + 6x - 15$, it is easy to perceive that the equation has one positive root, which lies between 1 and 2, whose first figures are easily found to be 1.9; and since $x^3 - 2x + 2$ and $-2x^2 + 6x - 15$ do not (both) change their signs for values of x that are greater than 1.9, it is clear that the equation has no other positive root.

In like manner, from $-x^3 + 2x + 2$ and $-2x^2 - 6x - 15$, it is easy to perceive that the equation has one negative root, which lies between -2 and -3; and that the equation has no other (real) negative root.

Hence, the given equation has two real and two imaginary roots.

8. To find the number of real roots of $3x^3 - 7x^2 + 2x - 20 = 0$.

The coefficients of t and v being $9x^2 - 14x + 2$ and $-7x^2 + 4x - 60$, it is easy to show that the equation has only one real root, which lies between 2 and 3, the other roots being imaginary, since the coefficients do not change their signs when x is greater than 2.

9. To find the first figures of the roots of "$x^3 - 12x + 8 = 0$."

Here the coefficients of t and v are $3x^2 - 12$ and $-24x + 24$, which will enable us to find the positive roots; and putting $-x$ for x, they become $3x^2 - 12$ and $24x + 24$, which will serve to find the negative roots.

Putting 3 for x in $3x^2 - 12$ and $24x + 24$, they become 13 and 96, and give $96 \div 13 = 7\frac{5}{13}$; which, being greater than 3, it follows that x is greater than 3.

Again, putting 4 for x, the preceding coefficients become 36 and 120, and give $120 \div 36 = 3\frac{1}{3}$, which, being less than 4, it follows that x is less than 4.

Hence, it results that -3 is the principal part of the negative root; and since $3x^2-12$ and $24x+24$ do not both change their signs when x is greater than 3, it is clear that the given equation has no other negative root.

To find the first figures of the positive roots, from $3x^2-12$ and $-24x+24$, we remark that when $x=0$, they become -12 and 24, which have contrary signs, and it is clear that the least positive root must be found from $3x^2-12$ and $-24x+24$, when they have the same signs as -12 and 24; and since $x=1$ reduces $-24x+24$ to 0, it is clear that the least positive root must be less than 1.

Hence, after a few trials, we find that the least positive root lies between 0.6 and 0.7; consequently, its principal part is 0.6.

Because the remaining root must be found from $3x^2-12$ and $-24x+24$, after they have changed their signs, we soon find that 3 is its first figure.

10. To find the first figures of the roots of $x^3-2x-5=0$.

Here we have $3x^2-2$ and $-4x-15$ for the coefficients of t and v, from which to find the positive roots; and $3x^2-2$, $4x-15$ are the coefficients from which to find the negative roots.

It is easy to show, from $3x^2-2$ and $4x-15$, that the equation has no negative root; and from $3x^2-2$ and $-4x-15$, that it has only one positive root, whose first figure is 2.

11. To find the number of real roots of "$x^4-2x^2+6x+10=0$."

Since $4x^3-4x+6$ and $-4x^2+18x+40$ are the coefficients of t and v for the positive roots, and $-4x^3+4x+6$ and $-4x^2-18x+40$ those for the negative roots, it is easy to show that the equation has no real roots.

12. To find the first figures of the roots of "$x^3+11x^2-102x+181=0$."

Here the coefficients of t and v for the positive roots are $3x^2+22x-102$ and $11x^2-204x+543$; and those for the negative roots are $3x^2-22x-102$ and $11x^2+204x+543$.

Because the coefficients for the negative roots must clearly both be positive, it is easy to show that -17 is the principal part of the only negative root which the equation can have.

Also, because the coefficients for the positive roots become -102 and 543, when 0 is put for x, it is clear that one positive root must be found from the coefficients, when the first is negative and the second positive, and that the second positive root must be found from the coefficients after they have changed their signs.

Hence, we easily find that the equation has two positive roots which lie between 3 and 4, whose principal parts are 3.213 and 3.229.

13. To find the first figures of the real roots of "$x^6 - 5x^4 + x^3 - x^2 - 1 = 0$."

Here $6x^5 - 20x^3 + 3x^2 - 2x$ and $-10x^4 + 3x^3 - 4x^2 - 6$, being the coefficients of t and v for the positive roots, and $-6x^5 + 20x^3 + 3x^2 + 2x$ and $-10x^4 - 3x^3 - 4x^2 - 6$ those for the negative roots, it is evident that the equation can have only one positive and one negative root, since the coefficients of v can not change their signs when x is positive.

Because the substitution of 2 and 3 for x in the coefficients for the negative roots gives -16, -206, and -885, -933 for the corresponding coefficients of t and v, it clearly follows that -2 is the first figure of the negative root. And in a similar way we get 2.2 for the first figures of the positive root.

14. To find the number of real roots of "$x^{10} - 10x^8 - x^4 + x - 11 = 0$."

Because $10x^9 - 80x^7 - 4x^3 + 1$ and $-20x^8 - 6x^4 + 9x - 110$ are the coefficients of t and v for the positive roots, and $-10x^9 + 80x^7 + 4x^3 + 1$ and $-20x^8 - 6x^4 - 9x - 110$ those for the negative roots (x being positive), it is easy to perceive that the coefficients of v can not change their signs, and of course the equation can not have more than one positive and negative root; and it is easy to show that 3 and -3 are the first figures of these roots.

15. To find the first figures of the roots of "$8x^3 - 6x - 1 = 0$."

Because $24x^2 - 6$ and $-12x - 3$ are the coefficients of t and v for the positive roots, it is easy to perceive that the equation can have only one positive root, whose first figures are 0.9. And because $24x^2 - 6$ and $12x - 3$ are the coefficients for the negative roots, and that $12x - 3$ changes its sign when x is greater than $\frac{1}{4}$, it is easy to perceive that one of the roots must be found by making x less than $\frac{1}{4}$; consequently, we easily find that -0.1 is the principal part of the root.

Similarly, since $24x^2-6$ changes its sign when x is greater than $\frac{1}{2}$, we must evidently find the remaining negative root, by making x greater than $\frac{1}{2}$; indeed, its principal part is found to be -0.7.

16. To find the roots of "$x^7-9x^5+6x^4+15x^3-12x^2-7x+6=0$."

Here we have $7x^6-45x^4+24x^3+45x^2-24x-7$, and $-18x^5+18x^4+60x^3-60x^2-42x+42$, for the coefficients of t and v; and since the coefficient of v is reducible to $(x-1)^2.(x+1)\times(-18x^2+42)$, it is easy to perceive that the coefficient of t is also reducible to $(x-1)^2.(x+1).(7x^3+7x^2-31x-7)$.

From the coefficients of t and v, it follows that the equation has $(x-1)^3.(x+1)^2$ for a factor, and of course it has three roots whose common value is 1, and two roots whose common value is -1.

To find the unequal roots of the equation, it is clear that we may take $7x^3+7x^2-31x-7$ and $-18x^2+42$ for the coefficients of t and v for the positive roots; and putting $-x$ for x, we may take $-7x^3+7x^2+31x-7$ and $-18x^2+42$ for the coefficients for the negative roots.

After a few trials, we find that by putting 3 for x in the coefficients for the negative roots, they become -40 and -120, which give $-120\div-40=3$; consequently, -3 is a root of the equation.

Similarly, by putting 2 for x in the coefficients for the positive roots, they become 15 and -30, which give $-30\div 15=-2$; consequently, 2 is a root of the equation.

Hence, the proposed equation is reduced to $x^7-9x^5+6x^4+15x^3-12x^2-7x+6=(x-1)^3.(x+1)^2.(x-2).(x+3)=0$.

17. To find limits to the roots of "$x^4 - 2x^3 - 7x^2 + 10x + 10 = 0$."

It is easy to perceive that the coefficients of t and v for the positive and negative roots may be represented by $2x^3 - 3x^2 - 7x + 5$, $-x^3 - 7x^2 + 15x + 20$; and $-2x^3 - 3x^2 + 7x + 5$, $x^3 - 7x^2 - 15x + 20$.

Putting $x = 0$ and $x = 1$, separately, in the coefficients for the negative roots, they become 5 and 20, 7 and -1; consequently, since the signs of the coefficients do not both change between $x = 0$ and $x = 1$, and that $20 \div 5 = 4$ is greater than 0, while $-1 \div 7$ is less than 1, it follows that -1 and 0 are limits to one of the negative roots; and in a similar way the remaining negative root will be found to have -3 and -2 for limits.

In like manner, from the coefficients for the positive roots, we get 2, 2.3, and 2.3, 3 for their limits.

Concluding Remark.—Because we may find the imaginary roots of an equation in much the same way that we find the real roots, it follows that we can by *our method* find all the roots of any *numerical algebraic equation* with great facility.

SECTION XX.

SOLUTION OF EXPONENTIAL EQUATIONS.

(1.) An exponential equation is one in which the unknown letter enters as an exponent.

Thus, $2^x = 9$, $3^y = 50$, $a^x = b$, $x^x = c$, $x^y = d$, are exponential equations.

If we take the exponential equation $A^x = B$, (1); then, since $A = 1 + A - 1$, $B = 1 + B - 1$, if we put $A - 1 = a$ and $B - 1 = b$, (1) will be reduced to $(1 + a)^x = 1 + b$, (2), in which a and b may be positive or negative, according to the nature of the case.

(2.) Putting $x = \frac{y}{z}$ (3), (2) by substituting $\frac{y}{z}$ for x, is easily reduced to $(1+a)^y=(1+b)^z$, or (expanding the members by the binomial theorem) we have $1+ya+\frac{y(y-1)}{1\,.\,2}a^2+\frac{y(y-1)(y-2)}{1\,.\,2\,.\,3}a^3+\text{etc.}=1+zb+\frac{z(z-1}{1\,.\,2}b^2+\frac{z(z-1)(z-2)}{1\,.\,2\,.\,3}b^3+\text{etc.}$, which, by arranging the terms according to the ascending powers of y and z, is easily reduced to the form $\left(a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+\frac{a^5}{5}-\text{etc.}\right)y+A_1y^2+A_2y^3+\text{etc.}=\left(b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\frac{b^5}{5}-\text{etc.}\right)z+B_1z^2+B_2z^3+\text{etc.}$, (4).

Now, since $a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+\frac{a^5}{5}-\text{etc.}$, and $b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\frac{b^5}{5}-\text{etc.}$, are called the *hyperbolic logarithms* of $1+a$ and $1+b$, if we denote them by writing l' before $(1+a)$ and $(1+b)$, we shall have $l'(1+a)=a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+\frac{a^5}{5}-\text{etc.}$, and $l'(1+b)=b-\frac{b^2}{2}+\frac{b^3}{3}-\text{etc.}$, (5); consequently (by substitution from (5) in (4)), we shall have $l'(1+a)\times y+A_1y^2+A_2y^3+\text{etc.}=l'(1+b)\times z+B_1z^2+B_2z^3+\text{etc.}$; and if we divide the terms of this by z $\left(\text{since by (3) } x=\frac{y}{z}\right)$, it will become $l'(1+a)\times x+A_1xy+A_2xy^2+\text{etc.}=l'(1+b)+B_1z+B_2z^2+\text{etc.}$, (6).

Because (6) ought clearly to be satisfied without limiting the values of y and z, it is manifest (from the method of undetermined coefficients, see Sec. XV.) that we must have $l'(1+a)\times x=l'(1+b)$, or $x=\frac{l'(1+b)}{l'(1+a)}$, (7); *consequently, x equals the hyperbolic logarithm of* $1+b$ *divided by that of* $1+a$, *or (which is the same) x is found from* (1) *by dividing the hyperbolic logarithm of* B *by that of* A.

Since $l'(1+a)\times x=l'(1+b)=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\text{etc.}$, if we (for brevity) represent $l'(1+a)\times x$ by v, we shall have $v=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\text{etc.}$; consequently, if we revert this series (see (5) of the Reversion of Series, p. 436), we shall get

$b = v + \frac{v^2}{2} + \frac{v^3}{2.3} + \frac{v^4}{2.3.4} +$ etc., which being put for b in (2), gives $(1 + a)^x = 1 + v + \frac{v^2}{2} + \frac{v^3}{2.3} +$ etc., or restoring the value of v, we shall have $(1 + a)^x = 1 + l'(1 + a)x + \frac{(l'(1 + a)x)^2}{1\,.\,2} + \frac{(l'(1 + a)x)^3}{1\,.\,2\,.\,3} +$ etc., (8), which is called the *Exponential Theorem;* and it is clear that it is only another form of the Binomial Theorem.

Hence, if we represent the hyperbolic logarithm of A by A′, we shall have $A^x = 1 + A'x + \frac{(A'x)^2}{1.2} +$ etc.; consequently, (1) is reduced to $1 + A'x + \frac{A'^2x^2}{1.2} + \frac{A'^3x^3}{1.2.3} +$ etc. $=$ B, (9), which (after having found a near value of x by trial) will clearly enable us (as in the solution of Algebraic Equations) to find x to any required degree of exactness.

(**3.**) If we multiply $1 + a$ by $1 + b$, we shall have $(1 + a).(1 + b) = 1 + a + b + ab$, or putting $a + b + ab = c$ we shall have $(1 + a).(1 + b) = 1 + c$; consequently, if we raise the members of this equation to the power whose exponent is any arbitrary number, as t, we shall have $(1 + a)^t.(1 + b)^t = (1 + c)^t$, which by (8) is reduced to $(1 + l'(1 + a)t +$ etc.)$.(1 + l'(1 + b)t +$ etc.) $= (1 + l'(1 + c)t +$ etc.), or we shall have $1 + [l'(1 + a) + l'(1 + b)]t +$ etc. $= 1 + l'(1 + c)t +$ etc., or $l'(1 + a) + l'(1 + b) = l'(1 + c)$, (10), because t is arbitrary.

It follows, from (10), *that the hyperbolic logarithm of a product equals the sum of the hyperbolic logarithms of its factors; and reciprocally, the hyperbolic* logarithm of a quotient equals the hyperbolic logarithm *of the dividend, minus that of the divisor.*

Hence, too, the hyperbolic logarithm of a power equals that of the root, multiplied by the index of the power; and reciprocally, if the hyperbolic logarithm of a power is divided by the index, the quotient equals the hyperbolic logarithm of the root.

(**4.**) Supposing A in (1) to be invariable, while B and x are variable, or receive different values, then clearly the values B may be regarded as being derived from A, considered as the *root* or *base.*

If we put $m = \frac{1}{l'\text{A}}$, and call m the *modulus*, then we shall (according to what has been shown) have $x = \frac{l'\text{B}}{l'\text{A}} = ml'\text{B}$, (11); consequently, *the exponent of* A *equals the hyperbolic logarithm of the corresponding power divided by the hyperbolic logarithm of* A (*the root*), *or* (*which is the same*) *multiplied by the modulus.*

Because A and (of course) $l'\text{A}$ are invariable, it is clear that the exponents of A may be taken to represent the hyperbolic logarithms of the corresponding powers B; consequently, if we call x the logarithm of B, taken to the base A, and represent the logarithm by writing l before B, we shall have $x = l\text{B}$, and (11) will be reduced to $l\text{B} = \frac{l'\text{B}}{l'\text{A}} = ml'\text{B}$, (12).

If for B in (12) we put A, we shall get $l\text{A} = \frac{l'\text{A}}{l'\text{A}} = 1$; consequently, the logarithm of the root or base is always 1, or unity.

Hence, if e denotes the base of hyperbolic logarithms, since $l'e = 1$, we shall from (8) get $e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{etc.}$, (13); consequently, if in this we put $x = 1$, we shall have $e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{etc.} = 2.7182818.$ = the *root* or *base* of hyperbolic logarithms.

If for B in (12) we put 1, then since $1 = 1 + (1 - 1)$, we shall, from (5), have $l(1) = \frac{l'(1)}{l'\text{A}} = 0$; consequently, any logarithm of 1 equals 0.

If for A we put $1 - a$, and suppose a to be positive and less than 1, A will be positive and less than 1, and from (5) we shall have $l'\text{A} = l'(1 - a) = -\left(a + \frac{a^2}{2} + \frac{a^3}{3} + \text{etc.}\right)$, a negative result; consequently, it is clear from (12), that if the base is less than 1, the logarithms of all positive numbers less than 1 will be positive, while the logarithms of all numbers greater than 1 will be negative; and reciprocally, if the base is greater than 1, the logarithms of all numbers greater than 1 will be positive, while the logarithms of all positive numbers less than 1 will be negative.

If we put $B = 1 - b$, we shall have $l'B = l'(1 - b) = -\left(b + \frac{b^2}{2} + \frac{b^3}{3} + \text{etc.}\right)$; consequently, if for b we put $1 - c$, and suppose c to be an infinitesimal, so that for b we may put 1, we shall get $l'B = l'[1 - (1 - c)] = l'c = l'o = -\left(1 + \frac{1}{2} + \frac{1}{3} + \text{etc.}\right)$, a negative result, which we will now show to be the negative of an unlimitedly great number.

Thus, supposing a to stand for any positive number, we shall have

$$\frac{1}{a} = \frac{1}{1 - (1 - a)} = 1 + \frac{1 - a}{2 - (2 - a)} = 1 + \frac{1 - a}{2} + \frac{(1 - a)\left(1 - \frac{a}{2}\right)}{3 - (3 - a)} = 1 + \frac{1 - a}{2} + \frac{(1 - a)\left(1 - \frac{a}{2}\right)}{3} + \frac{(1 - a)\left(1 - \frac{a}{2}\right)\left(1 - \frac{a}{3}\right)}{4 - (4 - a)} = 1 + \frac{1 - a}{2} + \frac{(1 - a)\left(1 - \frac{a}{2}\right)}{3} + \frac{(1 - a)\left(1 - \frac{a}{2}\right)\left(1 - \frac{a}{3}\right)}{4} + \text{etc.}, \quad (14);$$

which is a series that may be taken as a representative of $\frac{1}{a}$. If a is an infinitesimal, since $\frac{1}{a} = \frac{1}{0} = \infty =$ an unlimitedly great number, it follows that the series in the right member of (14) represents an unlimitedly great number; consequently, since the series in the right member of (14) (when a is an infinitesimal) can not exceed the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}$, it follows that the series $1 + \frac{1}{2} + \frac{1}{3} + \text{etc.}$, represents an unlimitedly great number; hence $l'o =$ the negative of an unlimitedly great positive number.

Hence, when the base is greater than 1 and finite, we shall have $lo = \frac{l'o}{l'A} =$ the negative of an unlimitedly great positive number.

(5.) If for A in (1) and (12) we put 10, they will become $10^x = B$, (15), and $x = lB = \frac{l'B}{l'A}$ (16); and $x = lB$ becomes what is called the *Briggs* or *Common Logarithm* of B.

Thus, since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc., 0, 1, 2, 3, etc., are the common logarithms of 1, 10, 100, 1000, etc., and because $10^{-1} = \frac{1}{10} = 0.1$, $10^{-2} = \frac{1}{100} = 0.01$, $10^{-3} =$ 0.001, etc., -1, -2, -3, etc., are the common logarithms of 0.1, 0.01, 0.001, etc.

If we represent any positive whole number (including 0) by n, it is clear, from what has been done, that the common logarithm of any number which lies between 10^n and 10^{n+1}, must lie between n and $n + 1$; consequently, the logarithm evidently equals $n +$ an interminable decimal; noticing that the integer n is called the *characteristic* of the logarithm. Similarly any number which lies between 10^{-n-1} and 10^{-n} has for its logarithm a number which lies between $-(n + 1)$ and $-n$; consequently, the logarithm equals $-(n + 1) +$ an interminable decimal; noticing that the characteristic $-(n + 1)$ is negative.

Thus 2, which lies between 10^0 and 10^1, has 0.3010300 for its (approximate) logarithm, its characteristic being 0; 89, which lies between 10^1 and 10^2, has 1.9493900 for its logarithm; 625.43, which lies between 10^2 and 10^3, has 2.7961787 for its logarithm; and 0.843, which lies between 10^{-1} and 10^0, has $-1 + 0.9258276 = -1.9258276$ for its logarithm, 0.08934, which lies between 10^{-2} and 10^{-1}, has -2.9510459 for its logarithm, and so on; noticing, that when the characteristic is negative, the sign $-$ is sometimes written over instead of before it; thus, for -2.9510459, we may write $\bar{2}.9510459$, and so on.

Hence, *it follows that the characteristic of the logarithm of any number greater than unity is one less than the number of places of integers in the number; and the characteristic of the logarithm of any decimal fraction is negative, and equal to unity added to the number of naughts which immediately follow the decimal point;* noticing, that the characteristic is sometimes called the *index* of the logarithm.

Again, if N stands for any whole number, then we shall have $l\frac{N}{10^n} = lN - l10^n = lN - n$; consequently, the decimal part of the logarithm of $\frac{N}{10^n}$ is the same as that of N, (since n is supposed to be an integer), while the characteristic of the logarithm of N is diminished by the number of units in n.

Thus, $l37.42 = l\frac{3742}{10^2} = l3742 - 2 = 3.5731038 - 2 = 1.5731038$, and $\log.\ 0.0956 = \log.\ \frac{956}{10^4} = \log.\ 956 - 4 = 2.9804579 - 4 = \bar{2}.9804579$.

Hence, *all numbers which consist of the same figures placed in the same order* (whether they are wholly integral or partly or wholly decimal), are such that the decimal parts of their logarithms are the same.

(6.) For the method of finding the logarithms of numbers from the tables, we shall refer the reader to the directions which generally precede the tables.

We will now proceed to illustrate what has been done by the following

EXAMPLES.

1. To find the value of x in $10^x = 2$, in which $x =$ the common logarithm of 2.

From the equation we get $10 = 2^{\frac{1}{x}}$, and since $8 = 2^3$, we get $\frac{10}{8} = 2^{\frac{1}{x}} \div 2^3$ or $1 + \frac{1}{4} = 2^{\frac{1}{x}-8} = 2^{\frac{1-3x}{x}}$, which gives $\left(1+\frac{1}{4}\right)^{\frac{x}{1-8x}} = 2 = \frac{1}{1-\frac{1}{2}} = \frac{1}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)}$; consequently, putting $y = \frac{x}{1-3x}$, we have, by taking the hyperbolic logarithms (by (5)), $yl'\left(1+\frac{1}{4}\right) = l'\frac{1}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} = -l'\left(1-\frac{1}{3}\right) - l'\left(1-\frac{1}{4}\right)$, or $y\left(\frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^3 - \frac{1}{4}\left(\frac{1}{4}\right)^4 + \text{etc.}\right) = \Big(\frac{1}{3}$

$$+\frac{1}{2}\left(\frac{1}{3}\right)^2+\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{4}\left(\frac{1}{3}\right)^4+\text{etc.}\bigg)+\left(\frac{1}{4}+\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{3}\left(\frac{1}{4}\right)^3+\frac{1}{4}\left(\frac{1}{4}\right)^4+\text{etc.}\right).$$

Because $\frac{1}{4}=0.25$, $\left(\frac{1}{4}\right)^2=0.0625$, $\left(\frac{1}{4}\right)^3=0.015625$, $\left(\frac{1}{4}\right)^4=0.00390625$, and so on, we soon find $\frac{1}{4}-\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{3}\left(\frac{1}{4}\right)^3-$ etc. $=0.25-0.03125+0.0052083333\dot{3}.-0.0009765625+$ etc. $=0.22314355134$, which is correct to ten decimal places; also, $\frac{1}{4}+\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{3}\left(\frac{1}{4}\right)^3+$ etc. $=0.2876820724$, correct to ten decimal places; similarly, $\frac{1}{3}+\frac{1}{2}\left(\frac{1}{3}\right)^2+\frac{1}{3}\left(\frac{1}{3}\right)^3+$ etc. $=0.4054651081$.

Hence, $y=\frac{0.6931471805}{0.2231435513}$, which, by restoring the value of y, gives $x=\frac{0.6931471805}{2.3025850929}=0.3010299956$, = the common logarithm of 2, correct to ten decimal places.

Because $l'2=0.6931471805$, it is manifest that $l'10=2.3025850929$, and of course $m=\frac{1}{l'10}=0.4342944819$, = the modulus of common logarithms; *it being the number by which any hyperbolic logarithm being multiplied, the result will be the corresponding common logarithm, and* vice versa.

Because $-l'\left(1-\frac{1}{3}\right)=l'\frac{3}{2}=0.4054651081$, and $l'2+l'\frac{3}{2}=l'3$, we get $l'3=1.0986122886$, and thence $l3=\frac{l'3}{l'10}=0.4771212547=$ the common logarithm of 3; also, $l'4=2l'2=1.3862943611$, and $l4=2l2=0.6020599913$, $l'5=l'10-l'2=1.6094379124$, $l5=l10-l2=1-l2=0.6989700043$, $l'6=l'3+l'2=1.7917594692$, $l6=l3+l2=0.7781512503$, $l'8=l'2^3=3l'2=2.0794415416$, $l8=3l2=0.9030899869$, $l'9=l'3^2=2.1972245773$, $l9=2l3=0.9542425094$; similarly, since $12=4\times3$, $15=5\times3$, $16=2^4$, $18=9\times2$, $20=10\times2$, etc., it will be easy to find the hyperbolic and common logarithms of 12, 15, 16, etc., from the preceding logarithms.

Remarks.—1st. It is clear, from what has been done, that the calculation of the common logarithm of 2 enables us to find the hyperbolic and common logarithms of an unlimited number of other numbers.

2d. It is also clear that if we calculate the logarithms of the prime numbers, we can thence get the logarithms of all the composite numbers resulting from the products of the primes or their powers, by adding their logarithms.

Thus, since $l14 = l2 + l7$ and $l22 = l2 + l11$, etc., it appears that we get the logarithm of 14 by adding the logs. of 2 and 7, etc.

3d. The method of calculating the logarithms of the primes 7, 11, 13, 17, etc., may be given as follows: thus, since $7 = 8 - 1 = 8\left(1 - \frac{1}{8}\right)$, $11 = 12\left(1 - \frac{1}{12}\right)$, etc, if we take their common logarithms, we shall get $l7 = l8 + ml'\left(1 - \frac{1}{8}\right)$, $l11 = 12 + ml'\left(1 - \frac{1}{12}\right)$, etc.

It may be well to notice that there are other methods, which are often more expeditious than the preceding. Thus, since $7^4 = 2401 = 2400\left(1 + \frac{1}{2400}\right)$, we have $4l7 = l2400 + ml'\left(1 + \frac{1}{2400}\right) = 2 + l24 + m\left[\frac{1}{2400} - \frac{1}{2}\left(\frac{1}{2400}\right)^2 + \frac{1}{3}\left(\frac{1}{2400}\right)^3 - \text{etc.}\right] = 2 + l6 + l4 + 0.4342944819\left[\frac{1}{2400} - \frac{1}{2}\left(\frac{1}{2400}\right)^2 + \text{etc.}\right] = 2 + l6 + l4 + 0.00018095603 - 0.000000037699 + 0.00000000001 - \text{etc.} = 3.38039216002$; consequently, $l7 = \frac{3.38039216002}{4} = 0.8450980400$ (= the common logarithm of 7), correct to ten decimal places. Similarly, the logarithms of 11, 13, 17, 19, etc., can easily be found from $2l11 = l120 + ml'\left(1 + \frac{1}{120}\right)$, $2l13 = l168 + ml'\left(1 + \frac{1}{168}\right)$, $2l17 = l288 + ml'\left(1 + \frac{1}{288}\right)$, $2l19 = l360 + ml'\left(1 + \frac{1}{360}\right)$, $2l23 = l528 + ml'\left(1 + \frac{1}{528}\right)$, and so on.

4th. We will terminate these remarks by showing how to find the number corresponding to any given common logarithm, without using the tables.

Thus, to find the number whose common logarithm is 3.44496555; since the characteristic is 3, it follows that 1000 is a factor of the sought number; consequently, subtracting 3 from the logarithm, we reduce it to 0.44496555, from which we have to find the remaining factors.

Since $0.44496555 \times 2 = 0.88993110$, we get $0.44496555 = \frac{1 - 0.11006890}{2} = \frac{1}{2} - 0.05503445$; consequently, since the number corresponding to the logarithm $\frac{1}{2}$ is $\sqrt{10}$, we have $\sqrt{10}$ for another factor of the sought number; and subtracting $\frac{1}{2}$ from the preceding logarithm, it is reduced to -0.05503445, which, being negative, it results if N stands for the number whose logarithm is 0.05503445, that $\frac{1000\sqrt{10}}{N}$ will be the sought number.

Dividing 0.05503445 by the modulus 0.43429448, we get the quotient 0.1267215 for the hyperbolic logarithm of N.

To find N from its hyperbolic logarithm, we shall take the exponential theorem (8), given at p. 523, which, by putting $x = 1$ and N for A, and using 0.1267215 for $l'A$, gives $N = 1 + 0.1267215 + 0.0080291 + 0.0003391 + 0.0000107 + 0.0000002 + \text{etc.} = 1.1351006$.

Hence, since $\sqrt{10} = 3.162277$, $\frac{1000\sqrt{10}}{N}$ becomes $\frac{3162.277}{1.1351006} = 2785.9002$ for the number corresponding to the given logarithm, which is correct to four decimals.

For another example, we shall find the number having 4.7892346 for its common logarithm.

Here $10000\sqrt{10}$ has 4.5 for its logarithm; and by subtracting 4.5 from the given logarithm, it is reduced to 0.2892346; consequently, if we find the number having 0.2892346 for its logarithm, and multiply it by $10000\sqrt{10}$, the product will be the sought number. Because $\sqrt[4]{10}$ has 0.25 for its logarithm, if from 0.2892346 we subtract 0.25, the given logarithm is reduced to 0.0392346; consequently, if we represent the number corresponding to this logarithm by N, the required number will be expressed by $10000\sqrt{10} \times \sqrt[4]{10} \times N = 10000\sqrt[4]{1000} \times N$.

Because $\frac{0.03923460}{0.43429448} = 0.09034105 = lN$, we get, from the exponential theorem, $N = 1 + 0.09034105 + 0.00408075 + 0.00012289 + 0.00000277 + 0.00000005 + \text{etc.} = 1.09454751$.

Since $10000\sqrt[4]{1000} = 56234.1266$ nearly, we get 61550.92 for the sought number, correct to two decimals.

2. To find x and y from $7^x = 342$ and $23^y = 0.243$.

Taking the logarithms of the members of the equations, we get the more simple forms $xl7 = l342$ and $yl23 = l0.243$. Hence, from the tables of logarithms, we get $x = \frac{2.5340261}{0.8450980} = 2.9984996$ and $y = \frac{\bar{1}.3856063}{1.3617278} = -\frac{0.6143937}{1.3617278} = -0.4511868$.

3. Let P represent any principal at compound interest for t years, a the amount of a unit of P in one year; then, if A is the amount of P, it is proposed to find the equation which connects a, P, A, and t.

Because the amounts of different principals in the same time are clearly as the principals, it is manifest that we shall have $1 : a :: a : a^2 =$ the amount of a unit of P in two years, and in the same way we have $1 : a :: a^2 : a^3 =$ the amount of a unit of P in three years, and so on; *consequently,* $a^t =$ *the amount of a unit of* P *in* t *years.*

We also have the proportion $1 : a^t :: P : A$, which gives the equivalent equation $a^t = \frac{A}{P}$, which, by taking the logarithms, gives the required equation $tla = lA - lP$; *consequently, if any three of the four quantities* a, P, A, *and* t *are known, this equation will enable us to find the fourth.*

Thus, if $a = \$1.06$, $P = \$235.62$, and $A = \$1925.32$, the equation gives $tl1.06 = l1925.32 - l235.62$, or $t =$

$\frac{l1925.32 - l235.62}{l1.06}$; consequently, from the tables, we have t $= \frac{0.9122907}{0.0253059} = 36.0505$ years.

4. Given, $x^x = 32$, to find x.

By taking the logarithms of the members of the equation, we get $xlx = l32$, which clearly gives x a very little greater than 3.08.

Putting $x = 3.08 + y = a + y$, we have $xlx = (a + y)l(a + y) = al(a + y) + yl(a + y) = l32 = b$; and since $l(a + y) = la + ml'\left(1 + \frac{y}{a}\right) = la + \frac{my}{a} +$ etc., we get (by omitting y^2, y^3, etc.,) $y = \frac{b - ala}{m + la}$.

Since $b = l32 = 1.5051500$, $ala = 3.08l3.08 = 1.5047361$, and m (the modulus) $= 0.4342944$, we get $y = \frac{0.0004137}{0.9228451} =$ 0.000448; consequently, we shall have $x = a + y = 3.080448$ very nearly; and it is manifest, if we put 3.080448 for a, and repeat the process, and so on, that we shall get x to any required degree of exactness.

5. Given, $x^{x^x} = 17$, to find x.

By taking the logarithms of the members (since x^x is the exponent of x) of the equation, we have $x^x lx = l17$, and taking the logarithms (of the members of this) we have $xlx + l(lx) = l(l17)$ or $xlx = l(l17) - l(lx)$; and taking the logarithms (of the members of this) we finally get $lx + l(lx) = l[l(l17) - l(lx)]$.

Since it is easy to perceive that $x = 2$ very nearly, we shall put $x = 2 + y = a + y$; consequently, we shall have $lx = l(a + y) = la + \frac{my}{a} +$ etc., and $l(lx) = l(la + \frac{my}{a} +$ etc.$)$ $= l(la) + \frac{m^2y}{ala} +$ etc.

Hence, $lx + l(lx) = l[l(l17) - l(lx)]$ is easily reduced to $la + l(la) + \left(\frac{m}{a} + \frac{m^2}{ala}\right)y +$ etc. $= l[l(l17) - l(la) - \frac{m^2y}{ala} -$ etc.], or (by putting $l(l17) - l(la) = b$, and rejecting y^2, y^3, etc.,) we shall get $la + l(la) + \left(\frac{m}{a} + \frac{m^2}{ala}\right)y = lb - \frac{m^3y}{bala}$, which gives $y = [lb - (la + l(la))] \div \left(\frac{m}{a} + \frac{m^2}{ala} + \frac{m^3}{bala}\right)$, or $y = l\left[\frac{b}{ala}\right] \div \frac{m}{a}\left(1 + \frac{m}{la} + \frac{m^2}{bla}\right)$.

Since $a = 2$ and $b = l(l17) - l(la) = l\left(\frac{l17}{l2}\right) = l(4.0874627)$ $= 0.6114538$, and $\frac{b}{ala} = 1.0156027$, $l\left(\frac{b}{ala}\right) = 0.0067233$, $\frac{m}{a} =$ 0.2171472, $1 + \frac{m}{la} + \frac{m^2}{bla} = 1 + 1.4426947 + 1.0246976 =$ 3.4673923, and $\frac{m}{a}\left(1 + \frac{m}{la} + \frac{m^2}{bla}\right) = 0.75293453$, it is clear that $y = \frac{0.0067233}{0.75293453} = 0.00893$ nearly, and of course $x = 2 + y = 2.00893$ nearly, which (on trial) will be found to satisfy the proposed equation to a great degree of exactness.

6. Supposing equal principals P and P to be put at interest at the same time, the one at 6 per cent. compound interest, and the other at 7 per cent. simple interest; then it is required to find the time, t, when the amounts will be equal.

Since $P(1.06)^t$ is the amount at compound interest, and $P + 0.07Pt$ that at simple interest, we get $(1.06)^t = 1 + 0.07t$ or $tl(1.06) = l(1 + 0.07t)$.

If $a =$ a near value of t, we have (by putting $t = a + y$) $al(1.06) + yl(1.06) = l(1 + 0.07a + 0.07y)$; and by putting $1 + 0.07a = b$, and retaining only the simple power of y, we have $al(1.06) + yl(1.06) = lb + \frac{0.07m}{b}y$.

Hence, we easily get $y\left[l(1.06) - \frac{0.07m}{b}\right] = l\left(\frac{b}{(1.06)^a}\right)$; and putting $l(1.06) - \frac{0.07m}{b} = c$, we finally have $y =$ $l\frac{b}{(1.06)^a} \div c$.

After a few trials, we find that it will answer our purpose to put $a = 6.12$, which gives $b = 1.4284$, $c = 0.0038129$, $l\frac{b}{(1.06)^a} = -0.0000223$.

Hence, we easily get $y = -0.00585$ very nearly, and thence $x = a + y = 6.12 - 0.00585 = 6.11415$ years, very nearly.

7. Find x from the equation "$x^{100} = 100$, or from $x = (10)^{0.02}$."

Since 0.02 equals the common logarithm of x, if we divide this by the modulus 0.43429448, we get 0.04605170 for the hyperbolic logarithm of x; consequently, from the exponential theorem, we get $x = 1.0471285$, which is correct to seven decimal places.

8. To find the 53d root of $5843 = 1.1686 \times 5000$.

Since 3 and 0.6989700043 are the common logarithms of 1000 and 5, we shall have $\frac{3.6989700043}{53} = 0.0697918868$ for the common logarithm of $(5000)^{\frac{1}{53}}$. Dividing this logarithm by 0.4342944819, we get 0.1607017581 for the corresponding hyperbolic logarithm; consequently, from the exponential theorem, we have $(5000)^{\frac{1}{53}} = 1.174334677$.

If we calculate the hyperbolic logarithm of 1.1686, we may in like manner find the value of $(1.1686)^{\frac{1}{53}}$; but it will be more simple to use the binomial theorem, which gives $(1.1686)^{\frac{1}{53}} = 1.002944063$. Hence, we shall have $(5843)^{\frac{1}{53}} = 1.174334677 \times 1.002944063 = 1.177791992$, which is correct to eight decimal places.

Remark.—We have given this and the preceding example for the purpose of showing the use of the exponential and binomial theorems in the extractions of the higher roots of numbers.

SECTION XXI.

RESOLUTION OF WHOLE NUMBERS INTO THEIR FACTORS.

(1.) If N stands for any proposed integer, then if N is even, it will be exactly divisible by 2; consequently, if 2^m is the greatest integral power of 2 which exactly divides N, it is clear that $\frac{N}{2^m}$ must be an odd integer. Hence, the resolution of any proposed integer into factors may be considered as being reduced to the resolution of an odd integer into factors.

(2.) If N stands for an odd integer, and we divide it by 4, the remainder must clearly be either 1 or 3, since 1 and 3 are the only odd integers which are less than 4. Consequently, if n stands for the quotient of the division of N by 4, it follows from the nature of division that N will equal $4n + 1$ or $4n + 3$; and N *is said to be of the form* $4n + 1$ or $4n + 3$.

(3.) If we multiply two integers of the forms $4n + 1$, $4n' + 1$, or $4n + 3$ and $4n' + 3$, by each other, the product will be of the form $4n'' + 1$.

For $(4n + 1) . (4n' + 1) = 16nn' + 4(n + n') + 1$, and $(4n + 3) . (4n' + 3) = 16nn' + 12(n + n') + 9$, products, which are manifestly of the form $4n'' + 1$.

(4.) The product of an integer of the form $4n + 1$, and of an integer of the form $4n' + 3$, will be an integer of the form $4n'' + 3$.

For $(4n + 1) . (4n' + 3) = 16nn' + 4(3n + n') + 3$ is evidently of the form $4n'' + 3$.

(5.) If the odd integer N is of the form $4n + 1$, and has factors, then n must be the product of two factors which are either both of the form $4n' + 1$, $4n'' + 1$, or of the form $4n' + 3$ and $4n'' + 3$.

For, otherwise, N must be the product of two integers, one of which has the form $4n' + 1$, and the other the form $4n'' + 3$; consequently, since the product of these forms is of the form $4n''' + 3$, it follows that N has the form $4n''' + 3$, contrary to the supposition.

(**6.**) If N is of the form $4n + 3$, then if it has factors, it may be shown, in a similar way, that it may be resolved into two factors, one of the form $4n'' + 1$, and the other of the form $4n''' + 3$.

(**7.**) If N is of the form $4n + 1$, and has factors, then it is clear that $a + x$ and $a - x$ may represent two factors into which it can be resolved, such that they must both be of the $4n + 1$ or $4n + 3$; consequently, by addition, we shall have $a + x + a - x = 2a$ of the form $4n + 2$ or $4n + 6$, and of course a must be an odd number. Hence, we shall have $(a + x).(a - x) = a^2 - x^2 = \text{N}$, or $x^2 = a^2 - \text{N}$, (1), in which a must be an odd number; and since $a^2 = \text{N} + x^2$, we have $a = \sqrt{\text{N} + x^2}$; consequently, if N is not an exact square, it is clear that a must be greater than the greatest integer in $\sqrt{\text{N}}$. And since a and N are odd numbers, it follows from (1) that x must be an even number.

In like manner, if N is of the form $4n + 3$, we shall have $x^2 = a^2 - \text{N}$, (2), in which a must be an even number greater than $\sqrt{\text{N}}$ (if N is not an exact square), and x must be an odd number.

(**8.**) Because the squares of whole numbers terminate with an even number of naughts, 25, 1, 4, 6, or 9, it follows that those whole numbers which terminate with an odd number of naughts, 2, 3, 7, 8, and 5, without terminating with 25, can not be exact squares.

(**9.**) Supposing n to stand for 0 or a positive integer, then it is plain that any whole number will be of one of the forms $3n$, $3n + 1$, $3n + 2$, or (which is the same) of one of the forms $3n$, $3n \pm 1$; consequently (by squaring) any integral square must be of one of the forms $9n^2$, $9n^2 \pm 6n + 1$.

Hence, it is clear that if we divide any integral square by 3, the remainder must be 0 or 1; consequently, any integer which, divided by 3, gives 2 for the remainder, can not be an exact square.

It may be shown, in like manner, that any integral square, when divided by 4, must give 0 or 1 for the remainder; consequently, any integer whose division by 4 gives 2 or 3 for the remainder, can not be an exact square.

It is clear that we may in the same way find the forms of the squares which correspond to the divisors 5, 6, 7, etc.

(**10.**) To perceive the use of what has been done, take the following

EXAMPLES.

1. To resolve 15229, which is of the form $4n + 1$, into its factors.

By extracting the square root of the number, we get 123 for the root of the greatest integral square contained in it; consequently, since the proposed number is of the form $4n + 1$, we must use an odd number for a in (1), which is greater than 123, so that we assume $a = 125 + 2n$, on the supposition that n is either 0 or a positive integer.

Hence, since $N = 15229$, (1) gives $x^2 = (125 + 2n)^2 - 15229 = 396 + 500n + 4n^2 = 4(99 + 125n + n^2)$, which, by putting 1 for n, gives $x^2 = 900 = 30^2$, or $x = 30$; consequently, from $a = 125 + 2n = 127$ and $x = 30$, we get $a + x = 157$ and $a - x = 97$, which give $15229 = 157 \times 97$, and the number has been resolved, as required, since 157 and 97 are prime numbers.

2. To resolve 16891, which is of the form $4n + 3$, into its factors.

Because the root of the greatest square contained in the number is 129, and that the number is of the form $4n + 3$ (since a in (2) must be an even number, and greater than 129), we put $a = 130 + 2n$; and since $N = 16891$, (2) gives $x^2 = (130 + 2n)^2 - 16891 = 9 + 520n + 4n^2$, which, by putting $n = 0$, gives $x^2 = 9$, or $x = 3$.

Hence, we have $a = 130$ and $x = 3$, which give $a + x = 133$ and $a - x = 127$ for factors of the proposed number. Hence, since $133 = 19 \times 7$, and that 127, 19, and 7 are prime numbers, it follows that $16891 = 127 \times 19 \times 7$, as required.

3. To resolve 11051, which is of the form $4n + 3$, into its factors.

Since 105 is the root of the greatest square contained in the number, and that the number is of the form $4n + 3$, we put $a = 106 + 2n$, and thence, from (2), get $x^2 = (106 + 2n)^2 - 11051 = 185 + 424n + 4n^2$.

Since 185, divided by 3, gives 2 for the remainder, and that 424 and 4, divided by 3, each give 1 for the remainder, it is easy to see that n must be of the form $3m + 1$, since it can not be of either of the forms $3m$, $3m + 2$.

Hence, by putting $3m + 1$ for n in the preceding equation, we get $x^2 = 613 + 1296m + 36m^2$, in which m must be a positive integer.

Putting 1, 2, 3, 4, etc., successively for m, we find that $m = 7$ gives $x^2 = 107^2$, which gives $x = 107$. Because $a = 106 + 2n = 106 + (3m + 1) \times 2 = 150$, and $x = 107$, we have $a + x = 257$ and $a - x = 43$; consequently, we have $11051 = 257 \times 43$, as required.

Remarks.—1st. It will be perceived that the reason why this is more difficult than the preceding questions, is that the factors are much farther from a ratio of equality.

2d. We have extracted the question from p. 207 of Barlow's Theory of Numbers. Barlow has shown that the number is not a prime, but has not given its factors.

4. To resolve 1082401, which is of the form $4n + 1$, into its factors.

If we multiply the number by 3, it becomes 3247203, of the form $4n+3$; consequently, if we can resolve this number into two factors which are different from the proposed number and 3, we shall of course resolve the given number into factors, as required.

Because 3247203 is of the form $4n + 3$, and that 1801 is the root of the greatest square contained in it, we put $a = 1802 + 2n$, and thence, from (2), we get $x^2 = 1 + 7208n + 4n^2$; consequently, by putting $n = 0$, we get $x = 1$.

Hence, because $a = 1802$ and $x = 1$, we have $a + x = 1803 = 601 \times 3$ and $a - x = 1801$; consequently, we have $1082401 = 1801 \times 601$, as required.

Remarks.—1st. If we multiply 11051 by 5, and proceed in like manner, we immediately get $11051 = 257 \times 43$, as in the preceding example.

2d. The reason of the preceding process clearly consists in bringing the sought factors to as near a ratio of equality as possible.

3d. We have taken the example from p. 286 of the Theorie Des Nombres, of Legendre, who has found the factors by a much more operose process.

5. To resolve 997331, of the form $4n + 3$, into its factors.

Since $997331 = 998^2 + 1327$, and is of the form $4n + 3$, we put $a = 1000 + 2n$, and thence (from (2)), get $x^2 = (1000 + 2n)^2 - 997331 = 2669 + 4000n + 4n^2$.

Because 2669, 4000, and 4, when divided by 3, severally give 2, 1, and 1 for the remainders, it is easy to perceive that n must be of the form $3m + 1$; consequently, putting $n = 3m + 1$, we shall get $x^2 = 6673 + 12024m + 36m^2$.

Also, because 6673, 12024, and 36, when divided by 5, severally give 3, 4, and 1 for the remainders, it is easy to perceive that m must be of one of the forms $5m' + 2$, $5m' + 3$, $5m' + 4$.

Hence, putting 0, 1, 2, 3, 4, 5, etc., successively, for m' in the preceding forms, we soon find that $m' = 99$ gives $m = 5m' + 3 = 498$, which, being put for m, gives $x^2 = 6673 + 12024m + 36m^2 = 14922769 = 3863^2$, or $x = 3863$. Hence, $a + x = 1000 + 2n + x = 1000 + 2(3m + 1) + x = 3990 + 3863 = 7853$, and $a - x = 3990 - 3863 = 127$; consequently, we shall have $997331 = 7853 \times 127$, as required.

6. To resolve 5428681, of the form $4n + 1$, into its factors.

Multiplying the number by 41, it becomes $222575921 = 14918^2 + 29197$; consequently, from (1) we get $x^2 = (14919 + 2n)^2 - 222575921 = 640 + 59676n + 4n^2 = 4(160 + 14919n + n^2)$. Because $160 + 14919n + n^2$ must be an exact square, and that 160, 14919, and 1, divided by 3, give 1, 0, and 1 for the remainders, it follows that n must be of the form $3m$; consequently, putting $3m$ for n, we get $x^2 = 4(160 + 44757m + 9m^2)$; and in much the same way, it appears that m must be of one of the forms $5m'$, $5m' + 1$, $5m' + 2$.

Hence, putting 0, 1, 2, 3, 4, etc., for m', successively, in these forms, we soon get $m' = 7$, which gives $m = 5m' + 1$

$= 36$, such that $x^2 = 4(160 + 1611252 + 11664) = 2548^2$, or $x = 2548$. Hence, from $a = 14919 + 2n = 14919 + 6m = 15135$ and $x = 2548$, we get $a + x = 17683$ and $a - x = 12587 = 307 \times 41$; consequently, 17683 and 307 are the factors of the proposed number.

Remarks.—1st. Multiplying the number given in the preceding example by 61, and that in this by 57, and then proceeding as in Ex. 4, we immediately get their factors.

2d. This and the preceding example have been taken from p. 417 of Gauss' Recherches Arithmetiques.

(**11.**) Supposing the odd integer N to have A and B for its factors, and that A is much less than B, and that $N = AB$ is multiplied by the odd integer m, then we shall clearly have $\left(\frac{mA + B}{2}\right)^2 - \left(\frac{mA - B}{2}\right)^2 = mAB$, (3); consequently, a and x in (1) and (2) may be expressed by $\frac{mA + B}{2}$ and $\frac{mA - B}{2}$

If we divide B by A, it is easy to perceive that such an integer may be found for m, that the corresponding value of $\frac{mA - B}{2}$ shall be less than $\pm \frac{A}{2}$, and numerically less than any other value of $\frac{mA - B}{2}$.

Hence, *we may suppose the determination of the factors* A *and* B *of* N *to be reduced to that of finding m, such that* $\frac{mA - B}{2}$ *shall be a minimum,* when the sign is neglected.

Supposing $2m'$ to be an integer less than m, and that m is changed to $m \pm 2m'$, (3) becomes $\left(\frac{mA + B}{2} \pm m'A\right)^2 - \left(\frac{mA - B}{2} \pm m'A\right)^2 = (m \pm 2m')AB$, (4), in which the difference of the successive values of $\frac{mA + B}{2} \pm m'A$ and $\frac{mA - B}{2} \pm m'A$ that result from $m' = 1, 2, 3$, etc., successively, is numerically equal to A.

And if p is the root of the greatest integral square contained in mAB, and q the integer to be added to p (according to the directions given for using (1) and (2)), so as to give $a = p + q = \frac{mA + B}{2}$, and representing $\frac{mA - B}{2}$ by c, then from (1) and (2) we shall get $\left(\frac{mA + B}{2}\right)^2 - \left(\frac{mA - B}{2}\right)^2 = (p + q)^2 - c^2 = mAB$, (5). Hence, putting $p^2 + r = mAB$, (5) gives $r = 2pq + q^2 - c^2$ or $\frac{r}{p} = 2q + \frac{q^2 - c^2}{p}$, (6); and from $r = 2pq + q^2 - c^2$ we get $c = \pm\sqrt{2pq + q^2 - r}$, (7), which gives c when p, q, and r are known.

If 1 or 2 is put for q, it is clear from (6) that when $\frac{c^2}{p}$ has its least value, $\frac{r}{p}$ must have nearly its greatest value; consequently, *in finding the factors* A *and* B *of* N, *it will be necessary to pay particular attention to those values of* m *which give the greatest values of* $\frac{r}{p}$.

Supposing (by approximation) that the least value of $\frac{mA - B}{2}$ is expressed by $\mp \frac{A}{2}$, then we shall get $B = mA \pm A$ and $mAB = m^2A^2 \pm mA^2$, or $\sqrt{mAB} = A\sqrt{m^2 \pm m}$; and since p is not much less than $\sqrt{mAB}$, we shall (approximately) have $p = A\sqrt{m^2 \pm m}$, which (if m is much greater than unity) may be reduced to $p = mA$ nearly.

Dividing $\left(\frac{mA - B}{2}\right)^2 = \frac{A^2}{4}$ by $p = mA$, we shall get $\frac{c^2}{p} = \frac{A}{4m}$ nearly; consequently, (6) becomes $\frac{r}{p} = 2q + \frac{q^2}{mA} - \frac{A}{4m}$ nearly, which, if q equals 1 or 2, may be reduced to $\frac{r}{p} = 2q - \frac{A}{4m}$ nearly, (8).

If A is very small in comparison to B, and 1 or 2 is put for q, then, since m must be a large number, and that $\frac{c^2}{p}$ is less than $\frac{A}{4m}$, it is clear from (8) that the greatest values of $\frac{r}{p}$ will generally be greater than unity.

Reciprocally, if, in finding the factors A and B of N, m is a large number, and the values of $\frac{r}{p}$ that correspond to $m-2$, m, and $m+2$ are greater than unity, or if two of them that are successive are greater than unity, while the remaining value does not differ much from it, then it is clear that A is small in comparison to B, and since the value of c that corresponds to the greatest of the values of r will generally be less than $\frac{A}{2}$ neglecting the sign, it is also manifest that A and B can (generally) be found immediately from (1) and (2).

To illustrate what has been done, take the following

EXAMPLES.

1. To resolve 997331, of the form $4n+3$, into its factors.

After a few trials, we find that by representing $m-2$, m, and $m+2$ by 59, 61, and 63, we shall get $\frac{13629}{7670}$, $\frac{12790}{7799}$, and $\frac{10377}{7926}$ for the corresponding values of $\frac{r}{p}$; and since $7799-7670=129$ and $7926-7799=127$, it follows from (4) (if 59, 61, and 63 are the correct values of $m-2$, m, and $m+2$) that A ought not to differ much from 129 or 127. Indeed, since 997331×61 is of the form $4n+3$, we shall get $a=p+q=7799+1$; consequently, from $p=7799$, $q=1$, $r=12790$, and $c=\pm\sqrt{2pq+q^2-r}$, we get $c=\pm 53$; hence, from $a \pm c$ we have 7853 and $7747=127\times 61$ for the factors of 997331×61, and of course 127 and 7853 are the factors of 997331.

2. To resolve 5428681, of the form $4n+1$, into its factors.

Putting 55, 57, and 59 for $m-2$, m, and $m+2$, we get $\frac{13614}{17279}$, $\frac{26717}{17590}$, and $\frac{25363}{17896}$ for the corresponding values of $\frac{r}{p}$; consequently, if 55, 57, and 59 are the correct values of $m-2$, m, and $m+2$, since $17590-17279=311$ and 17896

$- 17590 = 306$, we ought to have A nearly equal to 311 or 306.

Because 5428681×57 is of the form $4n + 1$, we put $a = p + q = 17590 + 1$, and thence from $p = 17590$, $q = 1$, $r = 26717$, and $c = \pm \sqrt{2pq + q^2 - r}$, we get $c = \pm 92$.

Hence, from $a \pm c$ we get 17683 and $17499 = 307 \times 57$ for the factors of 5428681×57, and of course 307 and 17683 are the factors of 5428681.

3. To resolve 262657, of the form $4n + 1$, into its factors.

Multiplying the number by 1, 3, 5, 7, 9, 11, etc., successively, we get $\frac{513}{512}$, $\frac{1202}{887}$, $\frac{2260}{1145}$, $\frac{2574}{1355}$, $\frac{1544}{1537}$, $\frac{2626}{1699}$, $\frac{2432}{1847}$, $\frac{3599}{1984}$, $\frac{400}{2113}$, $\frac{4194}{2233}$, $\frac{2693}{2348}$, $\frac{4262}{2457}$, $\frac{2581}{2562}$, and so on, for the corresponding values of $\frac{r}{p}$.

Because so many of the first values of $\frac{r}{p}$ are greater than unity, it clearly follows from what has been done, that the factor A must be extremely small in comparison to the remaining factor, B, of the given number; indeed, it is easily shown from what has been done, that the given number is a *prime*, or has no other factors than unity and itself.

4. To resolve 333667, of the form $4n + 3$, into its factors.

Multiplying by 1, 3, 5, 7, etc., successively, we get $\frac{738}{577}$, $\frac{1001}{1000}$, $\frac{1654}{1291}$, $\frac{885}{1528}$, $\frac{3179}{1732}$, $\frac{3012}{1915}$, and so on, for the corresponding values of $\frac{r}{p}$. Hence, it is easy to infer (as in the last question) that the given number is a prime.

Remarks.—This and the preceding example have been taken from p. 288 and p. 287 of Legendre's Théorie Des Nombres, where the numbers are shown to be primes.

SECTION XXII.

PERMUTATIONS AND COMBINATIONS.

(1.) The different arrangements that may be made of quantities or things, when any number of them are taken at a time, as to their order, are called their *Permutations.* Thus, if we take the letters a, b, c, etc., singly; since each letter can have one position, there will of course be as many positions or permutations as there are letters; and if we take the letters a and b together, since they can be written in the orders ab and ba, there will be two permutations.

Supposing the letters $a, b, c, d, e, \ldots . t$, to represent n quantities or things; then if m is a whole number not greater than n, we propose to find the number of permutations that can be made of the n letters, by taking m of them at a time.

If we find the permutations that may be made of the $n-1$ letters $b, c, d, e, \ldots . t$, when $m-1$ of them are taken at a time, and write a before each of them, we shall have all the permutations that can be made when *a stands first.* Similarly, the $n-1$ letters $a, c, d, e, \ldots . t$, when $m-1$ of them are taken at a time, and b written before each of them will give the same number of permutations (as before) in which *b stands first;* and so on, for the letters $c, d, e, \ldots . t$.

Hence, adding the number of permutations in which $a, b, c, \ldots . t$, severally *stand first,* it is clear that the sum equals n times the number of permutations that may be made of the $n-1$ letters $a, b, c, \ldots . s$, when taken $m-1$ at a time.

If we take $n-1$ and $m-1$ for n and m, it follows, from what has been shown, that the number of permutations of the $n-1$ letters $a, b, c, \ldots . s$, when taken $m-1$ at a time, equals $n-1$ times the number of permutations that may be made of the $n-2$ letters $a, b, c, d, e, \ldots . r$, taken $m-2$ at a time, and the number of permutations of the $n-2$ letters taken $m-2$ at a time, equals $n-2$ times the number of permutations of the $n-3$ letters $a, b, c, d, e, \ldots . q$, taken

$m - 3$ at a time, and so on, to the number of permutations that may be made of $n - (m - 1) = n - m + 1$ letters a, b, c, etc., taken $m - (m - 1) = 1$ at a time, which clearly make $n - m + 1$ permutations.

Hence, it follows that the number of permutations that can be made of the n letters $a, b, c, d, e, \ldots . t$, when taken m at a time, is truly expressed by the product $n(n - 1)(n - 2)(n - 3) \times \ldots . \times (n - m + 1)$, (1); consequently, if $m = n$ or if all the letters are taken at a time, the number of permutations is expressed by the product $1 . 2 . 3 . 4 . 5 \times \ldots . \times (n - 1)n$, (2).

EXAMPLES.

1. To find the number of permutations of the 5 letters a, b, c, d, e, taken 3 at a time, and when they are all taken at a time.

Because $n = 5$, $m = 3$, and $n - m + 1 = 3$, (1) gives $5 . 4 . 3 = 60$ = the number of permutations when 3 letters are taken at a time; and (2) gives $1 . 2 . 3 . 4 . 5 = 120$ = the number of permutations when the letters are all taken at a time.

2. To find the number of permutations that may be made of 7 different things taken 5 at a time, and when they are all taken at a time.

Ans. $7 . 6 . 5 = 210$, and $1 . 2 . 3 . 4 . 5 . 6 . 7 = 5040$.

3. To find the number of different permutations that may be made of the letters in the word *good*, when the letters are all taken together.

If the letters were all different from each other, we should have $1 . 2 . 3 . 4 = 24$ for the answer; but since the letter o is repeated, and that interchanging o and o in any permutation does not affect it, it is clear that we must divide the preceding result by $1 . 2$ = the number of permutations between two different things, when both are taken together; consequently, $\frac{1 . 2 . 3 . 4}{1 . 2} = 12$ = the required number of permutations.

4. To find the number of different permutations among the letters in *aaabbcd*, when the letters are all taken together.

From the reasoning in the last question, it is plain that the answer is $\frac{1.2.3.4.5.6.7}{1.2.3 \times 1.2} = 420$.

Remark.—If we take $a^p b^q c^r$, etc., in which p, q, r, etc., are positive integers, and represent $p + q + r$, etc., by N, and the products $1.2.3.4 \times \ldots . \times p$, $1.2.3.4 \times \ldots . \times q$, and so on, severally, by P, Q, etc.; then it is easy to perceive from what has been done, that the different permutations of the letters in the given expression (when the letters are all taken together) will be expressed by

$$\frac{1.2.3.4 \times \ldots . \times (N-1)N}{P \times Q \times R \times \text{etc.}}.$$

5. To find the different permutations that can be made of the letters in *aaabbcd*, when three of the letters are taken together.

The different groups that can be made of every three letters of the given expression are *aaa*, *aab*, *aac*, *aad*, *abb*, *abc*, *abd*, *acd*, *bbc*, *bbd*, *bcd*; consequently, the sum of the different permutations among the three letters of each of these groups will clearly equal the required number of permutations. Hence, by adding the permutations of the letters of each of the groups, in the order in which they are given, we get $1 + 3 + 3 + 3 + 3 + 6 + 6 + 6 + 3 + 3 + 6 = 43 =$ the required number of permutations.

(2.) The different collections of quantities or things, when any number of them are taken at a time, without regarding their order, are called their *combinations*.

Thus, the letters a, b, c, etc., taken singly, give as many combinations as there are letters, and a, b, taken together, give one combination, since ab and ba are equivalent.

Universally, since (2) gives $1.2.3 \ldots . m$ permutations among m different things taken together, or one combination of them (when they are taken together), it follows that if the expression for the permutations in (1) is divided by $1.2.3 \ldots . m$, the quotient $\frac{n(n-1)(n-2) \times \ldots . \times (n-m+1)}{1\ .\ 2\ .\ 3\ .\ .\ .\ .\ .\ .\ .\ m}$, (3), is the correct expression for the number of combinations that may be made of n different things taken m at a time.

EXAMPLES.

1. To find the number of combinations of 5 different things, taken 3 at a time.

Since $n = 5$, $m = 3$, and $n - m + 1 = 3$, (3) gives $\frac{5 \,.\, 4 \,.\, 3}{1 \,.\, 2 \,.\, 3}$ $= 10 =$ the required number.

2. To find the number of combinations of 12 different things, taken 7 at a time.

Ans. $\frac{12 \,.\, 11 \,.\, 10 \,.\, 9 \,.\, 8 \,.\, 7 \,.\, 6}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7} = 11 \,.\, 9 \,.\, 8 = 792.$

3. To find the number of combinations of 20 different things, taken 10 at a time. *Ans.* 184756.

(**3.**) Again, if several things of one sort are to be combined with several things of another sort, etc , by taking a certain number of them at a time, then we may proceed to find the combinations as in the following simple

EXAMPLES.

1. To find the different combinations that can be made of the letters in a^3b^2c, by taking 3 letters at a time.

Ans. a^3, a^2b, a^2c, ab^2, abc, b^2c, which make 6 different combinations.

2. To find the different combinations that can be made of the letters in $a^4b^3c^2d$, taken 5 at a time.

Ans. a^4b, a^4c, a^4d, a^3b^2, a^3bc, a^3bd, a^3c^2, a^3cd, a^2b^3, a^2b^2c, a^2b^2d, a^2bc^2, a^2bcd, a^2c^2d, ab^3c, ab^3d, ab^2c^2, ab^2cd, abc^2d, b^3c^2, b^3cd, b^2c^2d, which make 22 combinations.

3. Representing n different quantities or things by a, b, c, d, e, ... t; then supposing each letter may be repeated, it is required to find the number of different combinations that can be made by taking m letters at a time.

Combining a with each of the letters, we get the n different combinations a^2, ab, ac, ad, ae, ... at, and combining b with b, and each of the following letters, we have the $n-1$ different combinations b^2, bc, bd, be, ... bt, and so on, to the single combination t^2; consequently, when $m = 2$, we have $n + (n - 1) + (n - 2) + (n - 3) + \ldots . + 1 = \frac{n(n + 1)}{1 \,.\, 2}$ $=$ the required number of combinations.

Again, combining a with each of the preceding combinations, we have $\frac{n(n+1)}{1\,.\,2}$ combinations, in which three letters are taken at a time; and by putting $n-1$ for n, we have $\frac{(n-1)n}{1\,.\,2}$ combinations, with which combining b we have $\frac{(n-1)n}{1\,.\,2}$ combinations, in which three letters are taken at a time; and putting $n-1$ for n in $\frac{(n-1)n}{1\,.\,2}$, we have $\frac{(n-2)(n-1)}{1\,.\,2}$ combinations, which, combined with c, give $\frac{(n-2)(n-1)}{1\,.\,2}$ combinations, each consisting of three letters; and we may proceed in a similar way to the last combination $\frac{1\,.\,2}{1\,.\,2}$.

Hence, when 3 is put for m, we shall have $\frac{n(n+1)}{1\,.\,2}+\frac{(n-1)n}{1\,.\,2}+\frac{(n-2)(n-1)}{1\,.\,2}+\ldots\ldots+\frac{1\,.\,2}{1\,.\,2}=\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$ combinations, each containing 3 letters.

Similarly, combining a with each of the preceding combinations, and b with the combinations whose number is $\frac{(n-1)(n)(n+1)}{1\,.\,2\,.\,3}$, and so on, we get $\frac{n(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3\,.\,4}$ for the number of different combinations when $m=4$.

It is hence easy to perceive that, when each combination consists of m letters, the number of different combinations is truly expressed by $\frac{n(n+1)(n+2)\times\ldots\ldots\times(n+m-1)}{1\,.\,2\,.\,3\times\ldots\ldots\times m}$, (4).

Remark.—It is easy to perceive, from what has been done, that (4) gives the number of different terms in the expansion of the positive integral power $(a+b+c+\ldots\ldots+t)^m$; thus, if $n=3$ and $m=2$, (4) gives $\frac{3\,.\,4}{1\,.\,2}=6$ for the number of different terms in the expansion $(a+b+c)^2=a^2+2ab+2ac+b^2+2bc+c^2$.

(4.) Supposing that a and b are to be combined with c and d, so that each combination shall consist of two different letters, and that the resulting combinations are to be combined with e and f, so that each combination shall consist of three different letters, and so on; then it is clear that the products in the expansions $(a + b)(c + d) = ac + bc + ad + bd$, $(a + b)(c + d) \times (e + f) = ace + bce + ade + bde + acf + bcf + adf + bdf$, and so on, (5) will represent the required combinations.

If a and b are put for c and d, for e and f, and so on, (5) gives $(a + b)(a + b)(a + b) \times \ldots . \times (a + b)$ to n factors $=$ $(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \ldots . + nab^{n-1} + b^n$, (6); which shows that the combination a^n occurs once, the combination $a^{n-1}b$ occurs n times, the combination $a^{n-2}b^2$ occurs $\frac{n(n-1)}{1 \cdot 2}$ times, and so on, for the remaining combinations. If n is an even number, the middle combination in (6) will occur a greater number of times than any other; and if n is an odd number, the two middle combinations will occur an equal number of times, each of these combinations occurring a greater number of times than any other combination in (6).

(**5.**) If all the different combinations that can be made with the n letters $a, b, c, d, \ldots . t$, when taken one at a time, two at a time, three at a time, and so on, and such that the letters in each combination shall be different from each other, are to be found; then it is clear that the products in the expansion $(1 + a)(1 + b)(1 + c) \times \ldots . \times (1 + t) = 1 + a + b + c + \ldots . + t + (ab + ac + ad + \ldots . + at + bc + bd + be + \ldots . + bt +$ etc.$) + (abc + abd + abe + \ldots . + abt + acd + ace + \ldots . + act +$ etc.$) +$ etc., (7), after 1 the first term is rejected) will represent the required combinations.

By putting 1, 1, 1, etc., for a, b, c, etc., in (7), and subtracting 1 from the product, it is clear that $2^n - 1 =$ the number of all the combinations in (7).

If $-a$, $-b$, $-c$, etc., are put for a, b, c, etc., in (7), it becomes $(1 - a)(1 - b)(1 - c) \times \ldots . \times (1 - t) = 1 - (a + b + c + \ldots . + t) + (ab + ac + ad +$ etc.$) - (abc + abd + abe +$ etc.$) +$ etc.; consequently, adding this to (7), and dividing the sum by 2, the result gives $[(1 + a)(1 + b) \times (1 + c) \times \ldots . \times (1 + t) + (1 - a)(1 - b)(1 - c) \times \ldots . (1 - t)] \div 2$

$= 1 + (ab + ac + ad + \text{etc.}) + (abcd + abce + \text{etc.}) + \text{etc.}$, (8), which gives all the combinations that consist of an even number of letters.

Putting 1, 1, 1, etc., for a, b, c, etc., in (8), it reduces to $2^{n-1} - 1 =$ the number of all the combinations that consist of an even number of letters.

Because $2^n - 1$ equals all the combinations in (7), it follows that $2^n - 1 - (2^{n-1} - 1) = 2^{n-1}$ equals the number of all the combinations in (7) that consist of an odd number of letters; consequently, since $2^{n-1} - (2^{n-1} - 1) = 1$, it follows that the number of combinations which consist of an odd number of letters exceeds the number of combinations that consist of an even number of letters by 1, or unity.

Remark.—If $a, b, c \dots . t$ represent prime numbers, it is manifest that the combinations in (7) (including 1) will give all the unequal divisors of the number $abc \times \dots . \times t = \text{N}$, when 1 and N are considered as divisors.

Thus, if $a = 3$, $b = 5$, $c = 7$, $d = 0$, $e = 0$, etc., we have $abc = \text{N} = 105$; and from (7), $(1 + 3)(1 + 5)(1 + 7) = 1 + 3 + 5 + 7 + 3 \times 5 + 3 \times 7 + 5 \times 7 + 3 \times 5 \times 7 = 1 + 3 + 5 + 7 + 15 + 21 + 35 + 105$, which are all the separate divisors of 105, noticing that the number of the divisors 3, 5, 7, $3 \times 5 \times 7$ exceeds that of the divisors 3×5, 3×7, 5×7, by unity.

(6.) Supposing the terms of any number of geometrical progressions, as $a, a^2, a^3 \dots . a^m$, $b, b^2, b^3 \dots . b^n$, c, c^2, c^3, c^p, and so on to any extent, are to be combined with each other; then it is evident that the development of the product $(1 + a + a^2 + a^3 + \dots . + a^m) \times (1 + b + b^2 + b^3 + \dots . + b^n) \times (1 + c + c^2 + c^3 + \dots . + c^p) \times \text{etc.}$, (9) (after 1 is rejected) will give all the different combinations, as required. If a, b, c, etc., are prime numbers, it is clear, from the nature of whole numbers (as heretofore shown), that the combinations in (9) (when 1 is included among them) will give all the integral divisors of the whole number, which is expressed by $a^m \times b^n \times c^p \times \text{etc.} = \text{N}$, (10).

Thus, if $a = 2$, $b = 3$, $c = 5$, $d = 7$, $e = 11$, $f = 13$, and $m = 3$, $n = 2$, $p = 1$, $q = 1$, $r = 1$, $s = 1$; then, from (10), we get $a^3b^2cdef = \text{N} = 360360$, which has 192 divisors, as is clear from (9), since the number of different combinations is clearly expressed by the product $(1 + m) \times (1 + n) \times (1 + p) \times$ etc., (11), where it is plain that all the natural numbers 1, 2, 3, 4, 5 .... to 15 inclusive, are divisors.

If the terms of the progressions are supposed to be continued to infinity, it results that $\frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$, etc., are their generating functions, because $\frac{1}{1-a}, \frac{1}{1-b}$, etc., when expanded according to the ascending powers of a, b, c, etc., give the progressions.

If $(1 - ax)(1 - bx)(1 - cx) \times$ etc., is represented by $f(x)$, it is easy to perceive that the expansion of $\frac{1}{f(x)}$, according to the ascending powers of x, may be expressed by $1 + (a + b + c + \text{etc.})x + (a + b + c + \text{etc.})^2x^2 + (a + b + c + \text{etc})^3x^3 +$ etc., provided that unity is put for each coefficient in the expansions of the powers $(a + b + c + \text{etc.})^2$, $(a + b + c + \text{etc.})^3$, etc.

Hence, we get $\frac{1}{f(x)} = 1 + (a + b + c + \text{etc})x + (a^2 + ab + ac + ad + \text{etc.} + b^2 + bc + bd + \text{etc.})x^2 + (a^3 + a^2b + a^2c + a^2d + \text{etc.} + abc + abd + abe + \text{etc.} + b^3 + b^2c + b^2d + \text{etc.})x^3 +$ etc., (12), whose law of continuation is evident.

If 1 is put for x in (12), the resulting terms will express the corresponding combinations in (9); and if in these combinations the successive prime numbers 2, 3, 5, 7, 11, 13, etc., are put for a, b, c, d, etc., we shall get the series of natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and so on, to infinity.

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